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COEFFICIENT BOUNDS FOR p-VALENTLY CLOSE-TO-CONVEX FUNCTIONS ASSOCIATED WITH VERTICAL STRIP DOMAIN

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Abstract. By considering a certain univalent function that maps the unit disk $\mathbb{U}$ onto a strip domain, we introduce new subclasses of analytic and p-valent functions and determine the coefficient bounds for functions belonging to these new classes. Relevant connections of some of the results obtained with those in earlier works are also provided.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C} := \mathbb{C}^* \cup \{0\}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers.

Assume that $\mathcal{H}$ is the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

and let the class $\mathcal{P}$ be defined by

$$\mathcal{P} = \{p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \ (z \in \mathbb{U})\}.$$

For two functions $f, g \in \mathcal{H}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \ (z \in \mathbb{U})\},$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

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Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N},\ z \in \mathbb{U}) \tag{1}$$

which are analytic in the open unit disk $\mathbb{U}$. In particular, we set $\mathcal{A}_1 := \mathcal{A}$ for the class of analytic functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \qquad (z \in \mathbb{U}). \tag{2}$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}_p$ is said to be p-valently starlike of order $\alpha\,(0 \leq \alpha < 1)$ with complex order b $(b \in \mathbb{C}^*)$, if it satisfies the inequality

$$\Re\left\{1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)\right\} > \alpha \qquad (z \in \mathbb{U}).$$

We denote the class which consists of all functions $f \in \mathcal{A}_p$ satisfying the above condition by $\mathcal{S}_p^*(b,\alpha)$. In particular, we get the class

(*i*) $\mathcal{S}_p^*(b,0) = \mathcal{S}_p^*(b)$ of p-valently starlike functions of complex order b,
(*ii*) $\mathcal{S}_p^*(1,\alpha) = \mathcal{S}_p^*(\alpha)$ of p-valently starlike functions of order α,
(*iii*) $\mathcal{S}_p^*(1,0) = \mathcal{S}_p^*$ of p-valently starlike functions,
(*iv*) $\mathcal{S}_1^*(b,\alpha) = \mathcal{S}^*(b,\alpha)$ of starlike functions of complex order b,
(*v*) $\mathcal{S}_1^*(1,\alpha) = \mathcal{S}^*(\alpha)$ of starlike functions of order α,
(*vi*) $\mathcal{S}_1^*(1,0) = \mathcal{S}^*$ of starlike functions.

A function $f \in \mathcal{A}_p$ is said to be p-valently convex of order $\alpha\,(0 \leq \alpha < 1)$ with complex order b $(b \in \mathbb{C}^*)$, if it satisfies the inequality

$$\Re\left\{1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \qquad (z \in \mathbb{U}).$$

We denote the class which consists of all functions $f \in \mathcal{A}_p$ satisfying the above condition by $\mathcal{K}_p(b,\alpha)$. In particular, we get the class

(*i*) $\mathcal{K}_p(b,0) = \mathcal{K}_p(b)$ of p-valently convex functions of complex order b,
(*ii*) $\mathcal{K}_p(1,\alpha) = \mathcal{K}_p(\alpha)$ of p-valently convex functions of order α,
(*iii*) $\mathcal{K}_p(1,0) = \mathcal{K}_p$ of p-valently convex functions,
(*iv*) $\mathcal{K}_1(b,\alpha) = \mathcal{K}(b,\alpha)$ of convex functions of complex order b,
(*v*) $\mathcal{K}_1(1,\alpha) = \mathcal{K}(\alpha)$ of convex functions of order α,
(*vi*) $\mathcal{K}_1(1,0) = \mathcal{K}$ of convex functions.

It is clear that

$$f \in \mathcal{K}_p(b,\alpha) \Leftrightarrow \frac{1}{p}zf' \in \mathcal{S}_p^*(b,\alpha).$$

DEFINITION 1.1. Let $0 \leq \alpha, \delta < 1$ and $b, \gamma \in \mathbb{C}^*$. A function $f \in \mathcal{A}_p$ is said to be p-valently close-to-convex of order α with complex order b and type δ (or *Libera type* *p-valently close-to-convex of complex order b*) if there exists a function $g \in \mathcal{S}_p^*(\gamma, \delta)$ such that the inequality

$$\Re\left\{1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{g(z)} - 1\right)\right\} > \alpha \qquad (z \in \mathbb{U})$$

holds. We denote the class which consists of all functions $f \in \mathcal{A}_p$ satisfying the above condition by $\mathcal{C}_p^{\gamma,\delta}(b, \alpha)$.

In particular, we get the class $\mathcal{C}_p^{1,\delta}(1, \alpha) = \mathcal{C}_p(\alpha, \delta)$ of Libera type p-valently close-to-convex functions, and $\mathcal{C}_1^{1,\delta}(1, \alpha) = \mathcal{C}(\alpha, \delta)$ of Libera type close-to-convex functions [7].

Definition 1.2. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$ and $b \in \mathbb{C}^*$. Then the function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{S}_{b,p}(\alpha, \beta)$ if it satisfies the inequalities

$$\alpha < \Re\left\{1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right)\right\} < \beta \qquad (z \in \mathbb{U}).$$

In particular, we get the classes $\mathcal{S}_{1,p}(\alpha, \beta) = \mathcal{S}_p(\alpha, \beta)$, $\mathcal{S}_{b,1}(\alpha, \beta) = \mathcal{S}_b(\alpha, \beta)$ introduced by Kargar-Ebadian-Sokol [5] and $\mathcal{S}_{1,1}(\alpha, \beta) = \mathcal{S}(\alpha, \beta)$ introduced by Kuroki and Owa [6].

Remark 1.3. If we let $\beta \to \infty$ in Definition 1.2, then the class $\mathcal{S}_{b,p}(\alpha, \beta)$ reduces to the class $\mathcal{S}_p^*(b, \alpha)$.

Definition 1.4. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$ and $b \in \mathbb{C}^*$. Then the function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{K}_{b,p}(\alpha, \beta)$ if it satisfies the inequalities

$$\alpha < \Re\left\{1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} < \beta \qquad (z \in \mathbb{U}).$$

It is clear that

$$f \in \mathcal{K}_{b,p}(\alpha, \beta) \Leftrightarrow \frac{1}{p}zf' \in \mathcal{S}_{b,p}(\alpha, \beta).$$

For $p = 1$, the class $\mathcal{K}_{b,p}(\alpha, \beta)$ reduces to the class $\mathcal{K}_b(\alpha, \beta)$ introduced by Kargar-Ebadian-Sokol [5].

Remark 1.5. If we let $\beta \to \infty$ in Definition 1.4, then the class $\mathcal{K}_{b,p}(\alpha, \beta)$ reduces to the class $\mathcal{K}_p(b, \alpha)$.

Definition 1.6. Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$ and $b \in \mathbb{C}^*$. We denote by $\mathcal{C}_{b,p}^{\gamma,\delta}(\alpha, \beta)$ the class of functions $f \in \mathcal{A}_p$ satisfying

$$\alpha < \Re\left\{1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{g(z)} - 1\right)\right\} < \beta \qquad (z \in \mathbb{U}),$$

where $g \in \mathcal{S}_{\gamma,p}(\delta, \beta)$ with $0 \leq \delta < 1 < \beta$ and $\gamma \in \mathbb{C}^*$.

In particular, we get the class $\mathcal{C}_{1,1}^{1,\delta}(\alpha, \beta) = \mathcal{S}_g(\alpha, \beta)$ introduced by Bulut [2].

Remark 1.7. If we let $\beta \to \infty$ in Definition 1.6, then the class $\mathcal{C}_{b,p}^{\gamma,\delta}(\alpha, \beta)$ reduces to the class $\mathcal{C}_p^{\gamma,\delta}(b, \alpha)$.

It is worthy to note that for given $0 \leq \alpha < 1 < \beta$ and $b \in \mathbb{C}^*$, $f \in \mathcal{S}_{b,p}(\alpha, \beta)$ if and only if the following two subordination equations are satisfied:

$$1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)}-1\right) \prec \frac{1+(1-2\alpha)z}{1-z} \qquad \text{and} \qquad 1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)}-1\right) \prec \frac{1-(1-2\beta)z}{1+z}.$$

Let us consider the analytic function $f_{\alpha,\beta} : \mathbb{U} \to \mathbb{C}$ defined by

$$f_{\alpha,\beta}(z) = 1 + \frac{\beta-\alpha}{\pi} i \log\left(\frac{1-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right) \qquad (0 \leq \alpha < 1 < \beta) \tag{3}$$

with $f_{\alpha,\beta}(0) = 1$. Kuroki and Owa [6] proved that the function $f_{\alpha,\beta}$ maps the unit disk $\mathbb{U}$ onto the vertical strip domain

$$\Omega_{\alpha,\beta} = \{w \in \mathbb{C} : \alpha < \Re(w) < \beta\} \tag{4}$$

conformally and the function $f_{\alpha,\beta}$ is a convex univalent function in $\mathbb{U}$ having the form

$$f_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{5}$$

where

$$B_n = \frac{\beta-\alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}}\right) \qquad (n \in \mathbb{N}). \tag{6}$$

LEMMA 1.8. *A function $f \in \mathcal{A}_p$ given by* (1) *belongs to the class $\mathcal{S}_{b,p}(\alpha, \beta)$ if and only if there exists an analytic function q, $q(0) = 1$ and $q(z) \prec f_{\alpha,\beta}(z)$ such that*

$$f(z) = z^p \exp\left\{bp\int_0^z \frac{q(t)-1}{t}dt\right\} \qquad (z \in \mathbb{U}). \tag{7}$$

Proof. Assume that $f \in \mathcal{S}_{b,p}(\alpha, \beta)$ and

$$q(z) = 1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right).$$

Then $q(z) \prec f_{\alpha,\beta}(z)$ and integrating the above equality we get (7). Conversely, if the function f is given by (7), with an analytic function q, $q(0) = 1$ and $q(z) \prec f_{\alpha,\beta}(z)$, then we obtain $1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) = q(z)$. Therefore we have $1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) \prec f_{\alpha,\beta}(z)$ which implies $f \in \mathcal{S}_{b,p}(\alpha, \beta)$. □

Letting $q = f_{\alpha,\beta}$ in Lemma 1.8, we obtain the function

$$\tilde{f}(z) = z^p \exp\left\{bp\int_0^z \frac{f_{\alpha,\beta}(t)-1}{t}dt\right\}$$

and hence

$$\tilde{f}(z) = z^p \exp\left\{\frac{bp(\beta-\alpha)}{\pi} i \int_0^z \frac{1}{t}\log\left(\frac{1-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} t}{1-t}\right)dt\right\}$$

belongs to the class $\mathcal{S}_{b,p}(\alpha, \beta)$. This means that the class $\mathcal{S}_{b,p}(\alpha, \beta)$ is non-empty.

As a consequence of the principle of subordination and (4), we have the following results.

Lemma 1.9. *Let* $f \in \mathcal{A}_p$ *and* $0 \le \alpha < 1 < \beta$; $b \in \mathbb{C}^*$. *Then* $f \in \mathcal{S}_{b,p}(\alpha,\beta)$ *if and only if*

$$1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)}-1\right) \prec 1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right) \qquad (z\in\mathbb{U}).$$

Lemma 1.10. *Let* $f \in \mathcal{A}_p$ *and* $0 \le \alpha < 1 < \beta$; $b \in \mathbb{C}^*$. *Then* $f \in \mathcal{K}_{b,p}(\alpha,\beta)$ *if and only if*

$$1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right) \qquad (z\in\mathbb{U}).$$

Lemma 1.11. *Let* $f \in \mathcal{A}_p$ *and* $0 \le \alpha,\delta < 1 < \beta$; $b,\gamma \in \mathbb{C}^*$. *Then* $f \in \mathcal{C}_{b,p}^{\gamma,\delta}(\alpha,\beta)$ *if and only if*

$$1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{g(z)}-1\right) \prec 1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right) \qquad (z\in\mathbb{U}).$$

The coefficient problem for close-to-convex functions are studied by many authors in recent years, (see, for example [1,3,4,10,12–15]). Upon inspiration from the recent work of Bulut [2] the aim of this paper is to obtain coefficient bounds for the Taylor-Maclaurin coefficients for functions in the function classes $\mathcal{S}_{b,p}(\alpha,\beta)$, $\mathcal{K}_{b,p}(\alpha,\beta)$ and $\mathcal{C}_{b,p}^{\gamma,\delta}(\alpha,\beta)$ of analytic functions which we have introduced here. Also we investigate Fekete-Szegö problem for functions belong to the function classes $\mathcal{S}_{b,p}(\alpha,\beta)$ and $\mathcal{K}_{b,p}(\alpha,\beta)$.

In order to prove our main results, we first recall the following lemmas.

Lemma 1.12. *[11] Let the function* $\mathfrak{g}$ *given by*

$$\mathfrak{g}(z)=\sum_{k=1}^{\infty}\mathfrak{b}_k z^k \qquad (z\in\mathbb{U})$$

be convex in $\mathbb{U}$. *Also let the function* $\mathfrak{f}$ *given by*

$$\mathfrak{f}(z)=\sum_{k=1}^{\infty}\mathfrak{a}_k z^k \qquad (z\in\mathbb{U})$$

be analytic in $\mathbb{U}$. *If*

$$\mathfrak{f}(z)\prec\mathfrak{g}(z) \qquad (z\in\mathbb{U}),$$

then

$$|\mathfrak{a}_k|\le|\mathfrak{b}_1| \qquad (k=1,2,\ldots).$$

Lemma 1.13. *[8] Let* $p\in\mathcal{P}$ *with* $p(z)=1+c_1z+c_2z^2+\cdots$. *Then for any complex number* ν

$$\left|c_2-\nu c_1^2\right|\le 2\max\{1,\,|2\nu-1|\},$$

and the result is sharp for the functions given by

$$p(z)=\frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z)=\frac{1+z}{1-z}.$$

2. Coefficient inequalities for the classes $\mathcal{S}_{b,p}(\alpha,\beta)$ and $\mathcal{K}_{b,p}(\alpha,\beta)$

THEOREM 2.1. *Let α and β be real numbers such that $0 \leq \alpha < 1 < \beta$; $b \in \mathbb{C}^*$ and let the function $f \in \mathcal{A}_p$ be defined by (1). If $f \in \mathcal{S}_{b,p}(\alpha,\beta)$, then*

$$|a_{p+n}| \leq \frac{\prod_{k=2}^{n+1}\left(k-2+\frac{2|b|p(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{n!} \qquad (p,n \in \mathbb{N}).$$

Proof. Let the function $f \in \mathcal{S}_{b,p}(\alpha,\beta)$ be of the form (1). Let us define the function $q(z)$ by

$$q(z) = 1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) \qquad (z \in \mathbb{U}). \tag{8}$$

Then according to the assertion of Lemma 1.9, we get

$$q(z) \prec f_{\alpha,\beta}(z) \qquad (z \in \mathbb{U}), \tag{9}$$

where $f_{\alpha,\beta}(z)$ is defined by (3). Hence, using Lemma 1.12, we obtain

$$\left|\frac{q^{(m)}(0)}{m!}\right| = |c_m| \leq |B_1| \qquad (m \in \mathbb{N}), \tag{10}$$

where

$$q(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}) \tag{11}$$

and (by (6))

$$|B_1| = \left|\frac{\beta-\alpha}{\pi} i\left(1 - e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right)\right| = \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}. \tag{12}$$

Also from (8), we find

$$zf'(z) = p\left\{b\left[q(z)-1\right]+1\right\} f(z) \qquad (z \in \mathbb{U}). \tag{13}$$

Since $a_p = 1$, in view of (13), we obtain

$$na_{p+n} = bp\left[c_n + c_{n-1}a_{p+1} + \cdots + c_1 a_{p+n-1}\right] = bp\sum_{j=1}^{n} c_j a_{p+n-j}. \tag{14}$$

Applying (10) into (14), we get

$$n|a_{p+n}| \leq p|bB_1|\sum_{j=1}^{n}|a_{p+n-j}| \qquad (p,n \in \mathbb{N}).$$

For $n = 1,2,3$, we have

$$\begin{aligned}
|a_{p+1}| &\leq p|bB_1|,\\
|a_{p+2}| &\leq \frac{p|bB_1|}{2}(1+|a_{p+1}|) \leq \frac{p|bB_1|}{2}(1+p|bB_1|),\\
|a_{p+3}| &\leq \frac{p|bB_1|}{3}(1+|a_{p+1}|+|a_{p+2}|) \leq \frac{p|bB_1|\,(1+p|bB_1|)\,(2+p|bB_1|)}{6},
\end{aligned}$$

respectively. Using the principle of mathematical induction and the equality (12), we obtain

$$|a_{p+n}| \leq \frac{\prod_{k=2}^{n+1}(k-2+p|bB_1|)}{n!} = \frac{\prod_{k=2}^{n+1}\left(k-2+p|b|\frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{n!} \qquad (n\in\mathbb{N}).$$

This evidently completes the proof of Theorem 2.1. □

Letting $b=1$ in Theorem 2.1, we have the following result.

Corollary 2.2. *Let α and β be real numbers such that $0\leq\alpha<1<\beta$ and let the function $f\in\mathcal{A}_p$ be defined by* (1)*. If $f\in\mathcal{S}_p(\alpha,\beta)$, then*

$$|a_{p+n}| \leq \frac{\prod_{k=2}^{n+1}\left(k-2+\frac{2p(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{n!} \qquad (p,n\in\mathbb{N}).$$

Letting $p=1$ in Theorem 2.1, we have the following result.

Corollary 2.3. *[5] Let α and β be real numbers such that $0\leq\alpha<1<\beta$; $b\in\mathbb{C}^*$ and let the function $f\in\mathcal{A}$ be defined by* (2)*. If $f\in\mathcal{S}_b(\alpha,\beta)$, then*

$$|a_{n+1}| \leq \frac{\prod_{k=2}^{n+1}\left(k-2+\frac{2|b|(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{n!} \qquad (n\in\mathbb{N}).$$

Letting $b=1$ and $p=1$ in Theorem 2.1, we have the following result.

Corollary 2.4. *[6] Let α and β be real numbers such that $0\leq\alpha<1<\beta$ and let the function $f\in\mathcal{A}$ be defined by* (2)*. If $f\in\mathcal{S}(\alpha,\beta)$, then*

$$|a_{n+1}| \leq \frac{\prod_{k=2}^{n+1}\left(k-2+\frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{n!} \qquad (n\in\mathbb{N}).$$

Letting $\beta\to\infty$ in Theorem 2.1, we have the following result.

Corollary 2.5. *Let α be a real number such that $0\leq\alpha<1$; $b\in\mathbb{C}^*$ and let the function $f\in\mathcal{A}_p$ be defined by* (1)*. If $f\in\mathcal{S}_p^*(b,\alpha)$, then*

$$|a_{p+n}| \leq \frac{\prod_{k=2}^{n+1}(k-2+2|b|p(1-\alpha))}{n!} \qquad (p,n\in\mathbb{N}).$$

Theorem 2.6. *Let α and β be real numbers such that $0\leq\alpha<1<\beta$; $b\in\mathbb{C}^*$ and let the function $f\in\mathcal{A}_p$ be defined by* (1)*. If $f\in\mathcal{S}_{b,p}(\alpha,\beta)$, then for any $\mu\in\mathbb{C}$*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{|b|p(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\max\left\{1,\left|\frac{B_2}{B_1}+bpB_1(1-2\mu)\right|\right\},$$

where

$$B_1=\frac{\beta-\alpha}{\pi}i\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right) \quad \text{and} \quad B_2=\frac{\beta-\alpha}{2\pi}i\left(1-e^{4\pi i\frac{1-\alpha}{\beta-\alpha}}\right). \tag{15}$$

The result is sharp.

Proof. If $f \in \mathcal{S}_{b,p}(\alpha,\beta)$, then we have

$$q(z) \prec f_{\alpha,\beta}(z) \qquad (z \in \mathbb{U}),$$

where

$$q(z) = 1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}) \tag{16}$$

and

$$f_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n = 1 + \sum_{n=1}^{\infty} \frac{\beta-\alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}}\right) z^n \qquad (z \in \mathbb{U}).$$

As explained in the proof of Theorem 2.1, from (14) we get

$$c_1 = \frac{1}{bp} a_{p+1}, \qquad c_2 = \frac{2}{bp} a_{p+2} - \frac{1}{bp} a_{p+1}^2. \tag{17}$$

Since $f_{\alpha,\beta}(z)$ is univalent and $q(z) \prec f_{\alpha,\beta}(z)$, the function

$$h(z) = \frac{1 + f_{\alpha,\beta}^{-1}(q(z))}{1 - f_{\alpha,\beta}^{-1}(q(z))} = 1 + h_1 z + h_2 z^2 + \cdots \qquad (z \in \mathbb{U})$$

is analytic and has a positive real part in $\mathbb{U}$. Also we have

$$q(z) = f_{\alpha,\beta}\left(\frac{h(z)-1}{h(z)+1}\right) = 1 + \frac{B_1 h_1}{2} z + \left[\frac{B_1}{2}\left(h_2 - \frac{h_1^2}{2}\right) + \frac{B_2}{4} h_1^2\right] z^2 + \cdots. \tag{18}$$

Thus by (16)-(18) we get

$$a_{p+1} = \frac{bpB_1}{2} h_1, \tag{19}$$

$$a_{p+2} = \frac{bpB_1}{4}\left[h_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - bpB_1\right) h_1^2\right]. \tag{20}$$

Taking into account (19) and (20), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{bpB_1}{4}\left(h_2 - \lambda h_1^2\right), \tag{21}$$

where

$$\lambda = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - bpB_1(1-2\mu)\right]. \tag{22}$$

Our result now follows by an application of Lemma 1.13. The result is sharp for the functions

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) = f_{\alpha,\beta}\left(z^2\right) \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) = f_{\alpha,\beta}(z).$$

This completes the proof of Theorem 2.6. □

Letting $b = 1$ in Theorem 2.6, we have the following result.

Corollary 2.7. *Let α and β be real numbers such that $0 \le \alpha < 1 < \beta$ and let the function $f \in \mathcal{A}_p$ be defined by (1). If $f \in \mathcal{S}_p(\alpha,\beta)$, then for any $\mu \in \mathbb{C}$*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{p(\beta-\alpha)}{\pi} \sin\frac{\pi(1-\alpha)}{\beta-\alpha} \max\left\{1, \left|\frac{B_2}{B_1} + pB_1(1-2\mu)\right|\right\},$$

where B_1 and B_2 are given by (15). The result is sharp.

Letting $p = 1$ in Theorem 2.6, we have the following result.

Corollary 2.8. *Let α and β be real numbers such that $0 \le \alpha < 1 < \beta$; $b \in \mathbb{C}^*$ and let the function $f \in \mathcal{A}$ be defined by* (2)*. If $f \in \mathcal{S}_b(\alpha, \beta)$, then for any $\mu \in \mathbb{C}$*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|b|(\beta - \alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha} \max\left\{1, \left|\frac{B_2}{B_1} + bB_1(1 - 2\mu)\right|\right\},$$

where B_1 and B_2 are given by (15)*. The result is sharp.*

Letting $b = 1$ and $p = 1$ in Theorem 2.6, we have the following result.

Corollary 2.9. *Let α and β be real numbers such that $0 \le \alpha < 1 < \beta$ and let the function $f \in \mathcal{A}$ be defined by* (2)*. If $f \in \mathcal{S}(\alpha, \beta)$, then for any $\mu \in \mathbb{C}$*

$$\left|a_3 - \mu a_2^2\right| \le \frac{\beta - \alpha}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha} \max\left\{1, \left|\frac{B_2}{B_1} + B_1(1 - 2\mu)\right|\right\},$$

where B_1 and B_2 are given by (15)*. The result is sharp.*

Letting $\mu = 1/2$ and $\mu = 1$ in Theorem 2.6, we have the following result.

Corollary 2.10. *Let α and β be real numbers such that $0 \le \alpha < 1 < \beta$; $b \in \mathbb{C}^*$ and let the function $f \in \mathcal{A}_p$ be defined by* (1)*. If $f \in \mathcal{S}_{b,p}(\alpha, \beta)$, then*

$$\left|a_{p+2} - \frac{1}{2}a_{p+1}^2\right| \le \frac{|b|\,p(\beta - \alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha}$$

and

$$\left|a_{p+2} - a_{p+1}^2\right| \le \frac{|b|\,p(\beta - \alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha} \max\left\{1, \left|\frac{B_2}{B_1} - bpB_1\right|\right\},$$

where B_1 and B_2 are given by (15)*. The result is sharp.*

Theorem 2.11. *Let α and β be real numbers such that $0 \le \alpha < 1 < \beta$; $b \in \mathbb{C}^*$ and let the function $f \in \mathcal{A}_p$ be defined by* (1)*. If $f \in \mathcal{K}_{b,p}(\alpha, \beta)$, then*

$$|a_{p+n}| \le \frac{p \prod\limits_{k=2}^{n+1} \left(k - 2 + \frac{2|b|p(\beta-\alpha)}{\pi} \sin \frac{\pi(1-\alpha)}{\beta-\alpha}\right)}{n!\,(p+n)} \qquad (p, n \in \mathbb{N}).$$

Letting $\beta \to \infty$ in Theorem 2.11, we have the following result.

Corollary 2.12. *Let α be a real number such that $0 \le \alpha < 1$; $b \in \mathbb{C}^*$ and let the function $f \in \mathcal{A}_p$ be defined by* (1)*. If $f \in \mathcal{K}_p(b, \alpha)$, then*

$$|a_{p+n}| \le \frac{p \prod\limits_{k=2}^{n+1} (k - 2 + 2|b|\,p(1-\alpha))}{n!\,(p+n)} \qquad (p, n \in \mathbb{N}).$$

Theorem 2.13. *Let α and β be real numbers such that $0 \le \alpha < 1 < \beta$; $b \in \mathbb{C}^*$ and let the function $f \in \mathcal{A}_p$ be defined by* (1)*. If $f \in \mathcal{K}_{b,p}(\alpha, \beta)$, then for any $\mu \in \mathbb{C}$*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{|b|\,p^2(\beta - \alpha)}{(p+2)\pi} \sin \frac{\pi(1-\alpha)}{\beta - \alpha} \max\left\{1, \left|\frac{B_2}{B_1} + bpB_1\left(1 - \frac{2p(p+2)}{(p+1)^2}\mu\right)\right|\right\},$$

where B_1 and B_2 are given by (15)*. The result is sharp.*

3. Coefficient inequalities for the class $\mathcal{C}_{b,p}^{\gamma,\delta}(\alpha,\beta)$

THEOREM 3.1. *Let α,β and δ be real numbers such that $0 \le \alpha,\delta < 1 < \beta$; $b,\gamma \in \mathbb{C}^*$ and let the function $f \in \mathcal{A}_p$ be defined by (1). If $f \in \mathcal{C}_{b,p}^{\gamma,\delta}(\alpha,\beta)$, then*

$$|a_{p+1}| \le \frac{2|\gamma| p^2(\beta-\delta)}{(p+1)\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta} + \frac{2|b| p(\beta-\alpha)}{(p+1)\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$

and for $n = 2,3,\ldots$

$$|a_{p+n}| \le \frac{p}{n!\,(p+n)}\prod_{k=2}^{n+1}\left(k-2+\frac{2|\gamma| p(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)$$

$$+\frac{2|b| p(\beta-\alpha)}{(n-1)!\,(p+n)\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}\prod_{k=1}^{n-1}\left(k+\frac{2|\gamma| p(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right) \quad (p\in\mathbb{N}).$$

Proof. Let the function $f \in \mathcal{C}_{b,p}^{\gamma,\delta}(\alpha,\beta)$ be of the form (1). Therefore, there exists a function

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \in \mathcal{S}_{\gamma,p}(\delta,\beta) \tag{23}$$

so that

$$\alpha < \Re\left\{1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{g(z)}-1\right)\right\} < \beta. \tag{24}$$

Note that by Theorem 2.1, we have

$$|b_{p+n}| \le \frac{\prod_{k=2}^{n+1}\left(k-2+\frac{2|\gamma| p(\beta-\delta)}{\pi}\sin\frac{\pi(1-\delta)}{\beta-\delta}\right)}{n!} \qquad (p,n\in\mathbb{N}). \tag{25}$$

Let us define the function $\hat{q}$ by

$$\hat{q}(z) = 1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{g(z)}-1\right) \qquad (z\in\mathbb{U}). \tag{26}$$

Then according to the assertion of Lemma 1.11, we get

$$\hat{q}(z) \prec f_{\alpha,\beta}(z) \qquad (z\in\mathbb{U}), \tag{27}$$

where $f_{\alpha,\beta}(z)$ is defined by (3). Hence, using Lemma 1.12, we obtain

$$\left|\frac{\hat{q}^{(m)}(0)}{m!}\right| = |d_m| \le |B_1| \qquad (m\in\mathbb{N}), \tag{28}$$

where

$$\hat{q}(z) = 1 + d_1 z + d_2 z^2 + \cdots \qquad (z\in\mathbb{U}) \tag{29}$$

and (by (6))

$$|B_1| = \left|\frac{\beta-\alpha}{\pi} i\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right)\right| = \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}. \tag{30}$$

Also from (26), we find

$$zf'(z) = p\{b[\hat{q}(z)-1]+1\}g(z). \tag{31}$$

Since $a_p = b_p = 1$, in view of (31), we obtain

$$(32) \qquad (p+n)\, a_{p+n} - p b_{p+n} = bp\left[d_n + d_{n-1} b_{p+1} + \cdots + d_1 b_{p+n-1}\right] = bp \sum_{j=1}^{n} d_j b_{p+n-j}.$$

Now we get from (28) and (32) ,

$$|a_{p+n}| \leq \frac{p}{p+n}\, |b_{p+n}| + \frac{p\,|bB_1|}{p+n} \sum_{j=1}^{n} |b_{p+n-j}| \qquad (p, n \in \mathbb{N}).$$

Using the fact that

$$\sum_{j=1}^{n} |b_{p+n-j}| = 1 + |b_{p+1}| + |b_{p+2}| + \cdots + |b_{p+n-1}| \leq \frac{\prod_{k=1}^{n-1} \left(k + \frac{2|\gamma| p(\beta-\delta)}{\pi} \sin \frac{\pi(1-\delta)}{\beta-\delta}\right)}{(n-1)!},$$

the proof of Theorem 3.1 is completed. □

Letting $\beta \to \infty$ in Theorem 3.1, we have the following result.

Corollary 3.2. *Let α and δ be real numbers such that $0 \leq \alpha, \delta < 1$; $b, \gamma \in \mathbb{C}^*$ and let the function $f \in \mathcal{A}_p$ be defined by* (1). *If $f \in \mathcal{C}_p^{\gamma,\delta}(b, \alpha)$, then*

$$|a_{p+1}| \leq \frac{2\,|\gamma|\, p^2 (1-\delta)}{p+1} + \frac{2\,|b|\, p\,(1-\alpha)}{p+1}$$

and for $n = 2, 3, \ldots$

$$\begin{aligned} |a_{p+n}| \;&\leq\; \frac{p}{n!\;(p+n)} \prod_{k=2}^{n+1} \left(k - 2 + 2\,|\gamma|\, p\,(1-\delta)\right) \\ &+ \frac{2\,|b|\, p\,(1-\alpha)}{(n-1)!\;(p+n)} \prod_{k=1}^{n-1} \left(k + 2\,|\gamma|\, p\,(1-\delta)\right) \quad (p \in \mathbb{N}). \end{aligned}$$

Letting $b = \gamma = 1$ and $p = 1$ in Theorem 3.1, we have the following result.

Corollary 3.3. *[2] Let α, β and δ be real numbers such that $0 \leq \alpha, \delta < 1 < \beta$, and let the function $f \in \mathcal{A}$ be defined by* (2). *If $f \in \mathcal{S}_g(\alpha, \beta)$, then*

$$|a_2| \leq \frac{\beta-\delta}{\pi} \sin \frac{\pi\,(1-\delta)}{\beta-\delta} + \frac{\beta-\alpha}{\pi} \sin \frac{\pi\,(1-\alpha)}{\beta-\alpha}$$

and for $n = 2, 3, \ldots$

$$\begin{aligned} |a_{p+n}| \;&\leq\; \frac{1}{(n+1)!} \prod_{k=2}^{n+1} \left(k - 2 + \frac{2\,(\beta-\delta)}{\pi} \sin \frac{\pi\,(1-\delta)}{\beta-\delta}\right) \\ &+ \frac{2\,(\beta-\alpha)}{(n-1)!\;(n+1)\,\pi} \sin \frac{\pi\,(1-\alpha)}{\beta-\alpha} \prod_{k=1}^{n-1} \left(k + \frac{2\,(\beta-\delta)}{\pi} \sin \frac{\pi\,(1-\delta)}{\beta-\delta}\right). \end{aligned}$$

Letting $b = \gamma = 1$, $p = 1$ and $\beta \to \infty$ in Theorem 3.1, we have the coefficient bounds for close-to-convex functions of order α and type δ.

Corollary 3.4. *[7] Let α and δ be real numbers such that $0 \leq \alpha, \delta < 1$ and let the function $f \in \mathcal{A}$ be defined by (2). If $f \in \mathcal{C}(\alpha, \delta)$, then*

$$|a_n| \leq \frac{2(3-2\delta)(4-2\delta)\cdots(n-2\delta)}{n!}\left[n(1-\alpha)+(\alpha-\delta)\right] \qquad (n = 2, 3, \ldots).$$

Letting $b = \gamma = 1$, $p = 1$, $\delta = 0$, $\beta \to \infty$ in Theorem 3.1, we have the following coefficient bounds for close-to-convex functions of order α.

Corollary 3.5. *Let α be a real number such that $0 \leq \alpha < 1$ and let the function $f \in \mathcal{A}$ be defined by (2). If $f \in \mathcal{C}(\alpha)$, then*

$$|a_n| \leq n(1-\alpha)+\alpha \qquad (n = 2, 3, \ldots).$$

Letting $b = \gamma = 1$, $p = 1$, $\alpha = \delta = 0$, $\beta \to \infty$ in Theorem 3.1, we have the well-known coefficient bounds for close-to-convex functions.

Corollary 3.6. *[9] Let the function $f \in \mathcal{A}$ be defined by (2). If $f \in \mathcal{C}$, then*

$$|a_n| \leq n \qquad (n = 2, 3, \ldots).$$

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