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SYMMETRIC BI-DERIVATIONS OF SUBTRACTION ALGEBRAS

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Abstract. In this paper, we introduce the notion of symmetric bi-derivations on subtraction algebra and investigated some related properties. We prove that a map $D : X \times X \to X$ is a symmetric bi-derivation on X if and only if D is a symmetric map and it satisfies $D(x - y, z) = D(x, z) - y$ for all $x, y, z \in X$.

1. Introduction

B. M. Schein [4] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]. He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [6] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In this paper, we introduced the notion of symmetric bi-derivations on subtraction algebra and investigated some related properties. We prove that a map $D : X \times X \to X$ is a symmetric bi-derivation on X if and only if D is a symmetric map and it satisfies $D(x - y, z) = D(x, z) - y$ for all $x, y, z \in X$.

2. Preliminaries

We first recall some basic concepts which are used to present the paper.
By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x-y)-z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice

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with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true for every $x, y, z \in X$ (see [4]):

(p1) $(x - y) - y = x - y$.
(p2) $x - 0 = x$ and $0 - x = 0$.
(p3) $(x - y) - x = 0$.
(p4) $x - (x - y) \leq y$.
(p5) $(x - y) - (y - x) = x - y$.
(p6) $x - (x - (x - y)) = x - y$.
(p7) $(x - y) - (z - y) \leq x - z$.
(p8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(p9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(p10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(p11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.
(p12) $(x - y) - z = (x - z) - (y - z)$.

A mapping d from a subtraction algebra X to a subtraction algebra Y is called a *morphism* if $f(x - y) = f(x) - f(y)$ for all $x, y \in X$. A self map d of a subtraction algebra X which is a morphism is called an *endomorphism*.

LEMMA 2.1. *Let X be a subtraction algebra. Then the following properties hold:*
(1) $x \wedge y = y \wedge x$, for every $x, y \in X$.
(2) $x - y \leq x$ for all $x, y \in X$.

LEMMA 2.2. *Every subtraction algebra X satisfies the following property*

$$(x - y) - (x - z) \leq z - y$$

for all $x, y, z \in X$.

DEFINITION 2.3. Let X be a subtraction algebra and Y a non-empty set of X. Then Y is called a *subalgebra* if $x - y \in Y$ whenever $x, y \in Y$.

DEFINITION 2.4. A nonempty subset I of a subtraction algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in I$,
(I2) for any $x, y \in X$, $y \in I$ and $x - y \in I$ implies $x \in I$.

For an ideal I of a subtraction algebra X, it is clear that $x \leq y$ and $y \in I$ imply $x \in I$ for any $x, y \in X$.

DEFINITION 2.5. Let X be a subtraction algebra. A mapping $D(.,.) : X \times X \to X$ is called *symmetric* if $D(x, y) = D(y, x)$ holds for all $x, y \in X$.

DEFINITION 2.6. Let X be a subtraction algebra and $x \in X$. A mapping $d(x) = D(x, x)$ is called a *trace* of $D(.,.)$, where $D(.,.) : X \times X \to X$ is a symmetric mapping.

DEFINITION 2.7. Let X be a subtraction algebra. By a *derivation* of X, a self-map f of X satisfying the identity $f(x - y) = (f(x) - y) \wedge (x - f(y))$ for all $x, y \in X$ is meant.

3. Symmetric bi-derivations of subtraction algebras

In what follows, let X denote a subtraction algebra unless otherwise specified.

Definition 3.1. Let X be a subtraction algebra and $D : X \times X \to X$ be a symmetric mapping. We call D a symmetric bi-derivation on X if it satisfies the following condition

$$D(x - y, z) = (D(x, z) - y) \wedge (x - D(y, z))$$

for all $x, y, z \in X$.

Example 3.2. Let $X = \{0, a, b\}$ be a subtraction algebra with the following Cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a map $D : X \times X \to X$ by

$$D(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), (0, a), (a, 0), (0, b), (b, 0) \\ a & \text{if } (x, y) = (a, a), (a, b), (b, a) \\ b & \text{if } (x, y) = (b, b) \end{cases}$$

Then it is easily checked that D is a symmetric bi-derivation of subtraction algebra X.

Proposition 3.3. *Let D be a symmetric bi-derivation of subtraction algebra X and d the trace of symmetric bi-derivation D on X. Then the following identities hold:*
(1) $D(0, 0) = 0$.
(2) $D(0, x) = D(x, 0) = 0$ for all $x \in X$.
(3) $d(x) \leq x$ for all $x \in X$.

Proof. (1) Since $D(0, 0) = D(0 - 0, 0)$, we have

$$\begin{aligned} D(0, 0) &= D(0 - 0, 0) = (D(0, 0) - 0) \wedge (0 - D(0, 0)) \\ &= D(0, 0) \wedge 0 = D(0, 0) - (D(0, 0) - 0) \\ &= D(0, 0) - D(0, 0) = 0. \end{aligned}$$

(2) For all $x \in X$, we get

$$\begin{aligned} D(0, x) =& D(0 - 0, x) = (D(0, x) - 0) \wedge (0 - D(0, x)) \\ &= D(0, x) \wedge 0 = D(0, x) - (D(0, x) - 0) \\ &= D(0, x) - D(0, x) = 0. \end{aligned}$$

(3) Since $d(x) = D(x, x)$, we obtain

$$\begin{aligned} d(x) &= D(x, x) = D(x - 0, x) = (D(x, x) - 0) \wedge (x - D(0, x)) \\ &= D(x, x) \wedge x = D(x, x) - (D(x, x) - x) \\ &= x - (x - D(x, x)) \quad \text{(by (S2))} \\ &\leq x \quad \text{(by Lemma 2.1 (2))} \end{aligned}$$

□

Proposition 3.4. *Let X be a subtraction algebra and d the trace of symmetric bi-derivation D on X. Then $d(0) = 0$.*

Proof. Let $x \in X$. Then we have

$$\begin{aligned} d(0) &= D(0,0) = D(0-x,0) = (D(0,0)-x) \wedge (0 - D(x,0)) \\ &= (0-x) \wedge 0 - 0 = (0 \wedge 0) = 0. \end{aligned}$$

This completes the proof. □

Proposition 3.5. *Let X be a subtraction algebra and d a trace of symmetric bi-derivation D on X. Then the following identities hold.*
(1) *$D(x,y) = D(x,y) \wedge x$ for every $x, y \in X$.*
(2) *$d(x) = d(x) \wedge x$ for every $x, y \in X$.*

Proof. (1) Let $x, y \in X$. Then we have

$$\begin{aligned} D(x,y) &= D(x-0,y) \\ &= (D(x,y)-0) \wedge (x - D(0,y)) \\ &= D(x,y) \wedge (x-0) = D(x,y) \wedge x. \end{aligned}$$

(2) Let $x \in X$. Then we obtain

$$\begin{aligned} d(x) &= D(x,x) = D(x-0,x) \\ &= (D(x,x)-0) \wedge (x - D(0,x)) \\ &= D(x,x) \wedge x = d(x) \wedge x. \end{aligned}$$

□

Proposition 3.6. *Let X be a subtraction algebra and d a trace of symmetric bi-derivation D on X. Then the following identities hold.*
(1) *$D(d(x)-x,x) = 0$ for every $x \in X$.*
(2) *$d(x-d(x)) = 0$ for every $x \in X$.*

Proof. (1) Let $x \in X$. Then we have

$$\begin{aligned} (D(d(x)-x,x) &= D(d(x),x) - x) \wedge (d(x) - D(x,x)) \\ &= (D(d(x),x)-x) \wedge 0 \\ &= (D(d(x),x)-x) - (D(d(x),x)-x) = 0. \end{aligned}$$

(2) Let $x \in X$. Then we obtain

$$\begin{aligned} d(x-d(x)) &= D(x-d(x), x-d(x)) \\ &= (D(x, x-d(x)) - d(x)) \wedge (x - D(d(x), x-d(x))) \\ &= ((D(x-d(x),x)-d(x)) - d(x)) \wedge (x - D(x-d(x), d(x))) \\ &= ((D(x,x)-d(x) \wedge (x - D(d(x),x)) - d(x)) \wedge (x - D(x-d(x),d(x))) \\ &= 0 \wedge (x - D(x-d(x),d(x))) \\ &= 0 \end{aligned}$$

□

Proposition 3.7. *Let X be a subtraction algebra and D a symmetric bi-derivation on X. Then $D(x,y) \leq x$ and $D(x,y) \leq y$ for all $x, y \in X$.*

Proof. For all $x \in X$, we have $D(x,y) = D(x-0,y) = (D(x,y)-0)\wedge(x-D(0,y)) = D(x,y) \wedge x = D(x,y) - (D(x,y) - x) = x - (x - D(x,y)) \leq x$. Hence $D(x,y) \leq x$. Similarly, we have $D(x,y) \leq y$. □

Corollary 3.8. *Let X be a subtraction algebra and D a symmetric bi-derivation on X. Then $D(x,y) - y \leq x - D(x,y)$ for every $x, y \in X$.*

Proof. For all $x, y \in X$, we have $D(x,y) - y \leq x - y$ and $x - y \leq x - D(x,y)$ from (p9) and Proposition 3.7. Hence we obtain $D(x,y) - y \leq x - D(x,y)$. This completes the proof. □

Theorem 3.9. *Let X be a subtraction algebra and $D : X \times X \to X$ be a symmetric map defined by $D(x-y,z) = D(x,z) - y$ for every $x, y \in X$. Then D is a symmetric bi-derivation on X.*

Proof. For any $y \in X$, we have $D(0,y) = D(0-D(0,y),y) = D(0,y) - D(0,y) = 0$. Hence it follows that

$$D(x,y) - x = D(x-x,y) = D(0,y) = 0$$

for all $x, y \in X$. Since $D(x,z) \leq x$ and $D(y,z) \leq y$, we have

$$D(x,z) - y \leq x - y \leq x - D(y,z)$$

for all $x, y, z \in X$. Hence $D(x-y,z) = (D(x,z)-y) \wedge (x - D(y,z)) = D(x,z) - y$ for all $x, y, z \in X$, which implies that D is a symmetric bi-derivation on X. □

Theorem 3.10. *Let X be a subtraction algebra and $D : X \times X \to X$ be a symmetric bi-derivation on X. Then D satisfies $D(x-y,z) = D(x,z) - y$ for all $x, y, z \in X$.*

Proof. Let D be a symmetric bi-derivation and $x, y, z \in X$. Since $D(x,z) \leq x$ and $D(y,z) \leq y$ by Proposition 3.7, we have

$$D(x,z) - y \leq x - y \leq x - D(y,z)$$

for all $x, y, z \in X$. Hence $D(x-y,z) = (D(x,z)-y) \wedge (x - D(y,z)) = D(x,z) - y$ for all $x, y, z \in X$. □

As a consequence of Proposition 3.9 and 3.10, we get the following theorem.

Theorem 3.11. *Let X be a subtraction algebra. A map $D : X \times X \to X$ is a symmetric bi-derivation on X if and only if D is a symmetric map and it satisfies $D(x-y,z) = D(x,z) - y$ for all $x, y, z \in X$.*

Proposition 3.12. *Let X be a subtraction algebra and d be a trace of symmetric bi-derivation D on X. Then $d(x-y) = d(x) - y$ for all $x, y \in X$.*

Proof. Let d be a trace of symmetric bi-derivation D on X. From (p1), we have

$$\begin{aligned} d(x-y) &= D(x-y, x-y) = D(x, x-y) - y \\ &= D(x-y, x) - y = (D(x,x) - y) - y \\ &= (d(x) - y) - y = d(x) - y \end{aligned}$$

for all $x, y \in X$. □

Proposition 3.13. *Let X be a subtraction algebra and d a trace of D. Then $d(x \wedge y) = d(x) - (x - y)$ for all $x, y \in X$.*

Proof. Let $x, y \in X$. From (p1), we have

$$\begin{aligned} d(x \wedge y) &= D(x \wedge y, x \wedge y) \\ &= D(x - (x - y), x - (x - y)) = D(x, x - (x - y)) - (x - y) \\ &= D(x - (x - y), x) - (x - y) \\ &= (D(x, x) - (x - y)) - (x - y) \\ &= d(x) - (x - y). \end{aligned}$$

This completes the proof. □

Corollary 3.14. *Let X be a subtraction algebra and d a trace of D. Then $d(0 \wedge x) = 0$ for every $x \in X$.*

Proof. Since $0 \leq x$ for all $x \in X$, we have $d(0 \wedge x) = d(0) - (0 - x) = 0 - 0 = 0$. This completes the proof. □

Definition 3.15. Let X be a subtraction algebra and D a symmetric bi-derivation of X. For a fixed element $a \in X$, let us define a map $d_a : X \to X$ such that $d_a(x) = D(x, a)$ for every $x \in X$.

Theorem 3.16. *Let X be a subtraction algebra and D a symmetric bi-derivation of X. For each $a \in X$, the map d_a defined above is a derivation of X.*

Proof. For a fixed element $a \in X$, let us define a map $d_a : X \to X$ such that $d_a(x) = D(x, a)$ for every $x \in X$. Now for every $x, y \in X$, we have

$$\begin{aligned} d_a(x - y) &= D(x - y, a) \\ &= (D(x, a) - y) \wedge (x - D(y, a)) \\ &= (d_a(x) - y) \wedge (x - d_a(y)). \end{aligned}$$

This completes the proof. □

Theorem 3.17. *Let X be a subtraction algebra and D a symmetric bi-derivation of X. Then d_a is an isotone derivation of X.*

Proof. Let $x, y \in X$ be such that $x \leq y$. Then by (p8), we obtain $x = y - w$ for some $w \in X$. Hence

$$\begin{aligned} d_a(x) &= d_a(y - w) = D(y - w, a) \\ &= D(y, a) - w \leq D(y, a) = d_a(y) \end{aligned}$$

by Lemma 2.1 (2) and Theorem 3.10. □

Proposition 3.18. *Let X be a subtraction algebra and D a symmetric bi-derivation on X. If there exist $a \in X$ such that $a - D(x, z) = 0$, for all $x, z \in X$, we have $a = 0$.*

Proof. Let X be a subtraction algebra and D a symmetric bi-derivation on X. Assume that there exist $a \in X$ such that $a - D(x, z) = 0$, for all $x, z \in X$. Since D is a symmetric bi-derivation, we get

$$\begin{aligned} 0 &= a - D(a - x, z) = a - ((D(a, z) - x) \wedge a - (D(x, z))) \\ &= a - (D(a, z) - x \wedge 0) = a - 0 = a. \end{aligned}$$

This completes the proof. □

Definition 3.19. Let X be a subtraction algebra and D a symmetric bi-derivation on X. If $x \leq w$ implies $D(x,y) \leq D(w,y)$, D is called an *isotone symmetric bi-derivation* on X.

Theorem 3.20. *Let X be a subtraction algebra and D a symmetric bi-derivation on X. Then D is an isotone symmetric bi-derivation on X.*

Proof. Let $x, w \in X$ be such that $x \leq w$. Then $x = w - v$ from (p8). Hence we have

$$\begin{aligned} D(x,y) &= D(w-v,y) = (D(w,y)-v) \wedge (w - D(v,y)) \\ &= (D(w,y)-v) - ((D(w,y)-v) - (w - D(v,y))) \\ &\leq D(w,y) - v \quad \text{(by Lemma 2.1(2))} \\ &\leq D(w,y). \end{aligned}$$

□

Proposition 3.21. *Let D be a symmetric bi-derivation on X. Then the following identities hold.*

(1) $D(x \wedge y, z) \leq D(x,z)$ for all $x, y, z \in X$.

(2) $D(x \wedge y, z) \leq D(y,z)$ for all $x, y, z \in X$.

Proof. (1) Since $x \wedge y = x - (x - y) \leq x$ from (p4), by Proposition 3.20, we have $D(x \wedge y, z) \leq D(x,z)$ for all $x, y, z \in X$.

(2) Similarly, $x \wedge y = x - (x-y) = y - (y-x) \leq y$ from (p4), we have $D(x \wedge y, z) \leq D(y,z)$ for all $x, y, z \in X$. □

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