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PARA-KENMOTSU METRIC AS A η-RICCI SOLITON

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Abstract. The purpose of the paper is to study of Para-Kenmotsu metric as a η-Ricci soliton. The paper is organized as follows:

• If an η-Einstein para-Kenmotsu metric represents an η-Ricci soliton with flow vector field V, then it is Einstein with constant scalar curvature $r = -2n(2n + 1)$.

• If a para-Kenmotsu metric g represents an η-Ricci soliton with the flow vector field V being an infinitesimal paracontact transformation, then V is strict and the manifold is an Einstein manifold with constant scalar curvature $r = -2n(2n + 1)$.

• If a para-Kenmotsu metric g represents an η-Ricci soliton with non-zero flow vector field V being collinear with ξ, then the manifold is an Einstein manifold with constant scalar curvature $r = -2n(2n + 1)$.

Finally, we cited few examples to illustrate the results obtained.

1. Introduction

On a Riemannian manifold (M, g) Ricci soliton is defined by the partial differential equation

$$\pounds_V g + 2S + 2\lambda g = 0,$$

where $\pounds_V$ denotes the Lie-derivative in the direction of the flow vector field V, S is the Ricci tensor of g and λ being constant. Ricci soliton is treated as a natural generalization of Einstein metric (i.e., the Ricci tensor is a constant multiple of the Riemannian metric g). A Ricci soliton is trivial if V is either zero or killing on M. A Ricci soliton is said to be shrinking, steady, and expanding, as λ is negative, zero, and positive, respectively (and there are many examples of each of them [11], [4]). Otherwise, it will be called indefinite. Many authors have studied Ricci solitons in many contexts viz on Kähler manifolds [5], on contact manifolds [10], on Sasakian [8], α-Sasakian [12] and K-contact manifolds [17], on Kenmotsu [13], [9] and f-Kenmotsu manifolds [7] etc.

In para-contact geometry, Ricci solitons appeared first in the paper of G. Calvaruso and D. Perrone [2]. Recently, many authors made a rigorous study of Ricci Solitons in the framework of paracontact manifolds [1], [14] and etc.

Cho and Kimura in [6] made a rigorous study of real hypersurfaces in a complex space form and generalized the notion of Ricci soliton to η-Ricci soliton, defined on

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(M, g) by

$$S + \frac{1}{2}\pounds_V g + \lambda g + \mu\eta \otimes \eta = 0, \tag{1.1}$$

where λ and μ being constants. Later on Călin and Crasmareanu studied η-Ricci soliton in the framework of complex space forms [3].

In the present paper, our goal is to study η-Ricci soliton in the context of paracontact geometry, precisely on a para-Kenmotsu manifold. Our paper is organised as follows: First, if an η-Einstein para-Kenmotsu metric represents an η-Ricci soliton with flow vector field V, then it is Einstein with constant scalar curvature. Secondly, if a para-Kenmotsu metric g represents an η-Ricci soliton with the flow vector field V being an infinitesimal paracontact transformation, then V is strict and the manifold is an Einstein manifold with constant scalar curvature and the final result is the study of a para-Kenmotsu metric g admitting an η-Ricci soliton with non-zero flow vector field V being collinear with ξ. The concluding section of the paper contains an example of para-Kenmotsu metric verifying the results obtained in previous section.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M^{2n+1} has an almost paracontact structure (ϕ, ξ, η) if there exists a $(1,1)$-type tensor field ϕ, a characteristic (Reeb) vector field ξ and a 1-form η satisfying the following conditions:

$$\phi(\xi) = 0, \quad \eta \circ \phi = 0 \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi^2 = I - \eta \otimes \xi. \tag{2.2}$$

The distribution $\mathbb{D} : p \in M \to D_p \subset T_p(M) : D_p = \ker \eta = \{X \in T_p(M) : \eta(X) = 0\}$ is called paracontact distribution generated by η. Moreover, $\mathbb{D}$ is a $2n$-dimensional almost paracomplex distribution. Since g is non-degenerate metric on M and ξ is non-isotropic, the paracontact distribution $\mathbb{D}$ is non-degenerate.

The definition of the almost paracontact structure implies that the endomorphism ϕ has rank $2n$, $\phi\xi = 0$ and $\eta \circ \phi = 0$. If a manifold M^{2n+1} with (ϕ, ξ, η)-structure admits a semi-Riemannian metric g satisfying $g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y)\ \forall\ X, Y \in \mathfrak{X}(M)$, then M^{2n+1} has an almost paracontact metric structure and g is called compatible metric. Any compatible metric g with a given almost paracontact structure is of signature $(n+1, n)$. Any almost paracontact structure always admits a compatible metric.

Moreover, if $g(X, \phi Y) = d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta[X, Y])\ \forall\ X, Y \in \mathfrak{X}(M)$, then η is paracontact form and the almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta)$ is said to be paracontact metric manifold.

A paracontact metric manifold for which ξ is Killing is called a K-paracontact manifold. A paracontact structure on M^{2n+1} naturally gives rise to an almost paracomplex structure on the product $M^{2n+1} \times \mathbb{R}$. If this almost paracomplex structure is integrable, then the given paracontact metric manifold is said to be a para-Sasakian. Equivalently, (see [19]) a paracontact metric manifold is a para-Sasakian if and only if

$$(\nabla_X \phi)Y = -g(X, Y) + \eta(X)\eta(Y),\ \forall\ X, Y \in \mathfrak{X}(M).$$

If for every $X, Y \in \mathfrak{X}(M)$, $(\nabla_X \phi)Y = \eta(Y)\phi(X) + g(X, \phi Y)\xi$, then the manifold M^{2n+1} is said to be a para-Kenmotsu manifold.

For all $X, Y \in \mathfrak{X}(M)$, the following properties are true for para-Kenmotsu manifold [19]:

$$\nabla_X \xi = -X + \eta(X)\xi \tag{2.3}$$

$$(\nabla_X \eta)Y = -g(X, Y) + \eta(X)\eta(Y) \tag{2.4}$$

$$(\pounds_\xi g)(X, Y) = 2\big(- g(X, Y) + \eta(X)\eta(Y)\big) \tag{2.5}$$

$$\pounds_\xi \phi = 0, \quad \pounds_\xi \eta = 0. \tag{2.6}$$

Moreover, denoting by R the curvature tensor of g, we have the following [19]:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \tag{2.7}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.8}$$

$$S(X, \xi) = -2n\eta(X) \tag{2.9}$$

$$Q\xi = -2n\xi, \tag{2.10}$$

where $X, Y \in \mathfrak{X}(M)$, S is the Ricci tensor and Q being the Ricci operator defined as

$$S(X, Y) = g(QX, Y). \tag{2.11}$$

Furthermore, for a para-Kenmotsu metric manifold, the following relation is true [15]:

$$(\pounds_\xi Q)X = (\nabla_\xi Q)X = 2QX + 4nX, \tag{2.12}$$

for any $X \in \mathfrak{X}(M)$.

A vector field W on a semi-Riemannian manifold (M^{2n+1}, g) is said to be paracontact or infinitesimal paracontact transformation if it preserves the contact form η, i.e., there exists a smooth function $\rho : M^{2n+1} \to \mathbb{R}$ satisfying

$$\pounds_W \eta = \rho\eta. \tag{2.13}$$

If the vector field W is strict, then $\rho = 0$. By virtue of parallelism of the semi-Riemannian metric g, the commutation formulae (see page 23 of [18]):

$$(\pounds_W \nabla_X g - \nabla_W \pounds_X g - \nabla_{[W,X]} g)(Y, Z) = -g((\pounds_X \nabla)(X, Y), Z) - g((\pounds_W \nabla)(X, Z), Y),$$

reduces to

$$(\nabla_W \pounds_X g)(Y, Z) = g((\pounds_W \nabla)(X, Y), Z) + g((\pounds_W \nabla)(X, Z), Y), \tag{2.14}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. The following formulae are also known (see [18]):

$$(\pounds_W R)(X, Y)Z = (\nabla_W \pounds_X \nabla)(Y, Z) - (\nabla_Y \pounds_W \nabla)(X, Z), \tag{2.15}$$

$$(\pounds_W \nabla)(X, Y) = \pounds_W \nabla_X Y - \nabla_X \pounds_W Y - \nabla_{[W,X]} Y, \tag{2.16}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

An almost paracontact pseudo-Riemannian manifold is called η-Einstein, if its Ricci tensor has the following form:

$$S(X, Y) = \alpha_1 g(X, Y) + \alpha_2 \eta(X)\eta(Y), \tag{2.17}$$

where α_1 and α_2 are smooth functions on M and $X, Y \in \mathfrak{X}(M)$. If a para-Sasakian manifold is η-Einstein and $n > 1$, then α_1 and α_2 are constants (see [19]), but for para-Kenmotsu this fails to hold [19].

A vector field W on an semi-Riemannian manifold (M^{2n+1}, g) is said to be conformal vector field, if

$$\pounds_W g = 2\tilde{\rho} g, \tag{2.18}$$

where $\tilde{\rho}$ is called the conformal coefficient. If conformal coefficient is zero, then the conformal vector field is Killing vector field.

An infnitesimal automorphism is a vector field such that Lie derivatives along it of all objects of some tensor structure vanish. For an almost paracontact, metric structure, the condition that a vector field W is an infinitesimal automorphism is as follows:

$$\pounds_W \eta = \pounds_W \xi = \pounds_W g = 0. \tag{2.19}$$

3. Main Results

Before we proceed to state and prove the main results of the paper, let us begin with the following lemma.

LEMMA 3.1. *If a para-Kenmotsu metric g represents an η-Ricci soliton with the flow vector field V, then $(\pounds_V R)(X, \xi)\xi = 0$ holds for all $X \in \mathfrak{X}(M)$.*

Proof. Covariantly differentiating (1.1) along an arbitrary vector field W and thereby using (2.3), we obtain

$$(\nabla_W \pounds_V g)(X, Y) = -2(\nabla_W S)(X, Y) + 2\mu\{g(X, W)\eta(Y) + g(Y, W)\eta(X) - 2\eta(X)\eta(Y)\eta(W)\}, \tag{3.1}$$

for all $X, Y \in \mathfrak{X}(M)$. From (2.14) and (3.1), we get

$$g((\pounds_V \nabla)(W, X), Y) + g((\pounds_V \nabla)(W, Y), X) = -2(\nabla_W S)(X, Y) + 2\mu\{g(X, W)\eta(Y) + g(Y, W)\eta(X) - 2\eta(X)\eta(Y)\eta(W)\}. \tag{3.2}$$

By combinatorial combination, we find

$$g((\pounds_V \nabla)(X, Y), W) + g((\pounds_V \nabla)(X, W), Y) = -2(\nabla_X S)(Y, W) + 2\mu\{g(Y, X)\eta(W) + g(W, X)\eta(Y) - 2\eta(X)\eta(Y)\eta(W)\}. \tag{3.3}$$

$$g((\pounds_V \nabla)(Y, W), X) + g((\pounds_V \nabla)(Y, X), W) = -2(\nabla_Y S)(W, X) + 2\mu\{g(W, Y)\eta(X) + g(X, Y)\eta(W) - 2\eta(X)\eta(Y)\eta(W)\}. \tag{3.4}$$

Adding the last two equations and thereby subtracting (3.2) from the resulting one, we get

$$g((\pounds_V \nabla)(X, Y), W) = -(\nabla_X S)(Y, W) - (\nabla_Y S)(W, X) + (\nabla_W S)(X, Y) + 2\mu(g(X, Y)\eta(W) - \eta(X)\eta(Y)\eta(W)). \tag{3.5}$$

Differentiating covariantly (2.10) along an arbitrary vector field W and using (2.3), we deduce

$$(\nabla_W Q)\xi = QW + 2nW. \tag{3.6}$$

Substituting ξ for Y in (3.5), we obtain

$$(\pounds_V \nabla)(X,\xi) = -2(QX + 2nX), \quad \text{by (3.6) and (2.12).} \tag{3.7}$$

Differentiating covariantly, the foregoing equation, in the direction of an arbitrary vector field Y, we obtain

$$(\nabla_Y(\pounds_V \nabla))(X,\xi) = (\pounds_V \nabla)(X,Y) - 2(\nabla_Y Q)(X) + 2\eta(Y)(QX + 2nX).$$

Similarly,

$$(\nabla_X(\pounds_V \nabla))(Y,\xi) = (\pounds_V \nabla)(Y,X) - 2(\nabla_X Q)(Y) + 2\eta(X)(QY + 2nY).$$

From last two equations, we get

$$\begin{aligned}(\pounds_V R)(X,Y)\xi = -2((\nabla_X Q)Y - (\nabla_Y Q)X) + 2(\eta(X)QY - \eta(Y)QX)\\ - 4n(\eta(X)Y - \eta(Y)X).\end{aligned} \tag{3.8}$$

Replacing $Y = \xi$ in the foregoing equation yields,

$$(\pounds_V R)(X,\xi)\xi = 0, \text{ by (2.10), (2.12), (3.6).} \tag{3.9}$$

This finishes the proof. □

Lemma 3.2. *If a para-Kenmotsu metric g represents an η-Ricci soliton with the flow vector field V, then the relation between the constants λ and μ is given by $\lambda + \mu = 2n$.*

Proof. Putting $Y = \xi$ in (2.7) and then taking its Lie derivative along the flow vector field V, we obtain

$$(\pounds_V g)(X,\xi)\xi + 2\eta(\pounds_V \xi)X = 0, \text{ by (2.8).} \tag{3.10}$$

Taking Lie derivative of $g(\xi,\xi) = 1$ in the direction of V and then using (1.1), we get

$$\eta(\pounds_V \xi) = \lambda + \mu - 2n. \tag{3.11}$$

Furthermore, an appeal to (1.1) yields

$$(\pounds_V g)(X,\xi) = 2(2n - \lambda - \mu)\eta(X). \tag{3.12}$$

Feeding (3.11) and (3.12) in (3.10) and thereby contracting the resultant, we obtain

$$\lambda + \mu = 2n. \tag{3.13}$$

This completes the proof. □

Now we are going to focus on the main results of the paper.

Theorem 3.1. *If an η-Einstein para-Kenmotsu metric represents an η-Ricci soliton with flow vector field V, then it is Einstein with constant scalar curvature $r = -2n(2n + 1)$.*

Proof. From (2.17), we find

$$r = (2n + 1)\alpha_1 + \alpha_2. \tag{3.14}$$

Again, putting $X = Y = \xi$ in (2.17) we obtain

$$-2n = \alpha_1 + \alpha_2. \tag{3.15}$$

Finding α and β from (3.14) and (3.15) and using (2.17), we get

$$QX = \left(1 + \frac{r}{2n}\right)X - \left(2n + 1 + \frac{r}{2n}\right)\eta(X)\xi. \tag{3.16}$$

Differentiating covariantly, the foregoing equation, in the direction of an arbitrary vector field Y, we obtain

(3.17)

$$(\nabla_Y Q)X = \frac{Yr}{2n}\phi^2(X) + (2n+1+\frac{r}{2n})(g(X,Y)\xi + \eta(X)(Y - 2\eta(Y)\xi)), \text{ by (2.2), (2.4),}$$

for any $X \in \mathfrak{X}(M)$. From (3.8) and (3.17), we find

$$(\pounds_V R)(X,Y)\xi = \frac{1}{n}\{Y(r)\phi^2(X) - X(r)\phi^2(Y)\},$$

for any $X, Y \in \mathfrak{X}(M)$. Putting $Y = \xi$ and using (3.9), we get $\xi(r) = 0$. On suitable contraction of (3.6), we find

$$r = -2n(2n+1).$$

Putting the value of r in (3.16), we obtain the desired result. □

By [19], in a 3-dimensional pseudo-Riemannian manifold, we have

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \\ & - \frac{r}{2}\big(g(Y,Z)X - g(X,Z)Y\big), \end{aligned} \tag{3.18}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Substituting Y, Z with ξ in the foregoing equation and using (2.10), we obtain

$$QX = \big(1 + \frac{r}{2}\big)X - \big(3 + \frac{r}{2}\big)\eta(X)\xi, \tag{3.19}$$

for all $X \in \mathfrak{X}(M)$. Thus, proceeding in the same way as in Theorem 3.1 and applying Lemma 3.1 we conclude that $r = -6$. Hence, from (3.19) we have $QX = -2X$ for any $X \in \mathfrak{X}(M)$. Plugging this in (3.18) implies that (M, g) is of constant negative curvature -1. Thus, from Theorem 3.1 we obtain the following corollary:

Corollary 3.1. *If a 3-dimensional para-Kenmotsu metric g represents an η-Ricci soliton with the flow vector field V, then it is of constant negative curvature -1.*

Theorem 3.2. *If a para-Kenmotsu metric g represents an η-Ricci soliton with the flow vector field V being an infinitesimal paracontact transformation, then V is strict and the manifold is Einstein with constant scalar curvature $r = -2n(2n+1)$.*

Proof. Putting $Y = \xi$ in (2.16), we get

$$(\pounds_V \nabla)(X,\xi) = (\pounds_V \eta)(X)\xi + \eta(X)(\pounds_V \xi) - \nabla_X \pounds_V \xi, \tag{3.20}$$

for any $X \in \mathfrak{X}(M)$. Taking Lie derivative of $\eta(X) = g(X,\xi)$ in the direction of V and using (3.12),(3.13) and (2.13), we find

$$g(X, \pounds_V \xi) = \rho\eta(X) \Rightarrow \pounds_V \xi = \rho\xi, \quad \text{by (2.13).} \tag{3.21}$$

Moreover, taking $X = \xi$ in the foregoing equation and using (3.11) and (3.13), we deduce $\rho = 0$, which proves V is strict. Hence by (2.13), we find $\pounds_V \eta = 0$. This proves V is also an infinitesimal automorphism (refer to (2.19)). Finally an appeal to (3.20) yields $(\pounds_V \nabla)(X,\xi) = 0$ for any $X \in \mathfrak{X}(M)$, which further implies the manifold is Einstein with $r = -2n(2n+1)$ (see (3.7)). □

Theorem 3.3. *If a para-Kenmotsu metric g represents an η-Ricci soliton with non-zero flow vector field V being collinear with ξ, then the manifold is an Einstein manifold with constant scalar curvature $r = -2n(2n+1)$.*

Proof. By hypothesis, there exists a smooth function f on M^{2n+1} such that $V = f\xi$. On covariant differentiation along $X(\in \mathfrak{X}(M))$ gives

$$\nabla_X V = (Xf)\xi - f(X - \eta(X)\xi), \text{ by (2.3)}. \tag{3.22}$$

Combining (1.1) and (3.22), we get

$$2S(X,Y)+(Xf)\eta(Y)+(Yf)\eta(X)-2(f-\lambda)g(X,Y)+2(f+\mu)\eta(X)\eta(Y) = 0. \tag{3.23}$$

With $X = Y = \xi$ and using (2.10) in the foregoing equation, we find $\xi(f) = 0$. Now for $Y = \xi$ in (3.23), we get $X(f) = 0$ (refer to (2.9) and (3.13)). This proves the smooth function f reduces to a constant function. Moreover, (3.23) gives

$$S(X,Y) = (f - \lambda)g(X,Y) - (f + \mu)\eta(X)\eta(Y). \tag{3.24}$$

This proves the manifold is η-Einstein and hence taking advantage of the first theorem we can say that it is Einstein with constant scalar curvature $r = -2n(2n + 1)$. □

Remark 3.1. If a para-Kenmotsu metric g represents an η-Ricci soliton with flow vector field V being conformal, then V is killing and the manifold is Einstein with constant scalar curvature $r = -2n(2n + 1)$.

4. Example

Example 4.1. We will give an example of a 3-dimensional para-Kenmotsu manifold, which exhibits Theorem 3.3.

Let L be a 3-dimensional real connected Lie group and $\mathfrak{g}$ be its Lie algebra with a basis $\{e_1, e_2, e_3\}$ of left invariant vector fields by the following commutators [19]:

$$[e_1, e_2] = 0,\ [e_1, e_3] = e_1 + \alpha e_2,\ [e_2, e_3] = \alpha e_1 + e_2,\ \alpha \text{ being a constant}.$$

Let g be the Pseudo-Riemannian metric defined by

$$\begin{cases} g(e_1, e_1) = 1,\ g(e_3, e_3) = 1,\ -g(e_2, e_2) = 1, \\ g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0. \end{cases} \tag{4.1}$$

Let $\xi = e_3$ be the vector field associated with the 1-form η. The $(1,1)$-tensor field ϕ be defined by,

$$\phi(e_1) = e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0. \tag{4.2}$$

Then (L, ϕ, ξ, η, g) is a 3-dimensional almost paracontact metric manifold. Since the metric g is left invariant the Koszul equality becomes (refer to [16])

$$\begin{aligned} &\nabla_{e_1} e_3 = e_1, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_1 = -e_3, \\ &\nabla_{e_2} e_3 = e_2, && \nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = e_3, \\ &\nabla_{e_3} e_1 = -\alpha e_2, && \nabla_{e_3} e_2 = -\alpha e_1, && \nabla_{e_3} e_3 = 0. \end{aligned}$$

Also, the Riemannian curvature tensor R is given by,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Then,

$$\begin{aligned} &R(e_1, e_2)e_2 = e_1, && R(e_1, e_3)e_3 = -e_1, && R(e_2, e_1)e_1 = -e_2, \\ &R(e_2, e_3)e_3 = (\alpha - 1)e_1 - e_2, && R(e_3, e_1)e_1 = -e_3, && R(e_3, e_2)e_2 = e_3, \\ &R(e_1, e_2)e_3 = 0, && R(e_2, e_3)e_1 = 0, && R(e_3, e_1)e_2 = 0. \end{aligned}$$

Then, the Ricci tensor S is given by

$$S(e_1, e_1) = -2, \qquad S(e_2, e_2) = 2, \qquad S(e_3, e_3) = -2,$$
$$S(e_1, e_2) = 0, \qquad S(e_1, e_3) = 0, \qquad S(e_2, e_3) = 0.$$

Thus it is easy to see that

$$S(X, Y) = -2g(X, Y), \tag{4.3}$$

which proves the manifold is Einstein. The scalar curvature of the manifold is $r = -6$. If we consider the flow vector field $V = f\xi$, for some constant f, then from the well known formulae $(\pounds_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V)$ we get

$$\pounds_V g = -2fg + 2f\eta \otimes \eta. \tag{4.4}$$

So feeding (4.3) and (4.4) in (1.1), we see that the soliton equation is satisfied for $\lambda = f + 2$ and $\mu = -f$ *i.e.* the metric g admits η-Ricci soliton with the flow vector field V for the constants $\lambda = f + 2$ and $\mu = -f$. Moreover, we obtain $\lambda + \mu = 2$, which proves (3.13), for $n = 1$. Furthermore, Theorem 3.3 is also verified.

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