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GENERALIZED SEQUENTIAL CONVOLUTION PRODUCT FOR THE GENERALIZED SEQUENTIAL FOURIER-FEYNMAN TRANSFORM

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Abstract. This paper is a further development of the recent results by the authors on the generalized sequential Fourier-Feynman transform for functionals in a Banach algebra $\hat{\mathcal{S}}$ and some related functionals. We investigate various relationships between the generalized sequential Fourier-Feynman transform and the generalized sequential convolution product of functionals. Parseval's relation for the generalized sequential Fourier-Feynman transform is also given.

1. Introduction

In [16], the authors introduced the concept of generalized sequential Feynman integral to define the generalized sequential Fourier-Feynman transform for functionals in the Banach algebra $\hat{\mathcal{S}}$ which was introduced by Cameron and Storvick [1]. Their results are generalization of the previous results on the sequential Feynman integral and sequential Fourier-Feynman transform introduced and studied by Cameron and Storvick [1–3].

In the present paper we introduce the concept of generalized sequential convolution product for the generalized sequential Fourier-Feynman transform. Existence of the generalized sequential convolution product for functionals in the Banach algebra $\hat{\mathcal{S}}$ and some related functionals are established in Theorems 3.3, 3.4, 3.5 and 3.6.

In Section 4, we show that the generalized sequential Fourier-Feynman transform of the generalized sequential convolution product of functionals is a product of the generalized sequential Fourier-Feynman transforms of these functionals. We also obtain Parseval's relation for the generalized sequential Fourier-Feynman transform for some functionals.

Results in [6] on the sequential Fourier-Feynman transform and the sequential convolution product can be obtained as corollaries of our results here.

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Let $C_0[0,T]$ be the space of continuous functions $x(t)$ on $[0,T]$ such that $x(0)=0$. Let a subdivision σ of $[0,T]$ be given:

$$\sigma : 0 = \tau_0 < \tau_1 < \cdots < \tau_m = T,$$

and let $X = X(t)$ be a polygonal curve in $C_0[0,T]$ based on a subdivision σ and the real numbers $\vec{\xi} = \{\xi_k\}$, that is,

$$X(t) = X(t, \sigma, \vec{\xi})$$

where

$$X(t,\sigma,\vec{\xi}) = \frac{\xi_{k-1}(\tau_k - t) + \xi_k(t - \tau_{k-1})}{\tau_k - \tau_{k-1}}$$

when $\tau_{k-1} \le t \le \tau_k$, $k = 1, 2, \ldots, m$ and $\xi_0 = 0$. If there is a sequence of subdivisions $\{\sigma_n\}$, then σ, m and τ_k will be replaced by σ_n, m_n and $\tau_{n,k}$.

Let Z_h be the Gaussian process

$$Z_h(x,t) = \int_0^t h(s)\,dx(s),$$

where $h(\neq 0$ a.e.$)$ is in $L_2[0,T]$ and the integral $\int_0^t h(s)\,dx(s)$ denotes the Paley-Wiener-Zygmund (PWZ) integral [5, 8, 9, 13].

Note that Z_h is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0,T]} Z_h(x,s) Z_h(x,t)\,dm(x) = \int_0^{\min\{s,t\}} h^2(u)\,du,$$

where the integral on the left-hand side of the last expression denotes the Wiener integral. Of course if $h \equiv 1$ on $[0,T]$, then $Z_h(x,t) = x(t)$ is the standard Wiener process.

The standard Wiener process is stationary in time, while the Gaussian process Z_h is non-stationary in time, unless h is equal to the constant function 1.

Let $q \neq 0$ be a given real number and let $F(x)$ be a functional defined on a subset of $C_0[0,T]$ containing all the polygonal curves in $C_0[0,T]$. Let $\{\sigma_n\}$ be a sequence of subdivisions such that the norm $\|\sigma_n\| \to 0$ and let $\{\lambda_n\}$ be a sequence of complex numbers with $\operatorname{Re}\lambda_n > 0$ such that $\lambda_n \to -iq$. Then if the integral in the right hand side of (1.1) exists for all n and if the following limit exists and is independent of the choice of the sequences $\{\sigma_n\}$ and $\{\lambda_n\}$, we say that the generalized sequential Feynman integral with parameter q exists and it is denoted by

$$\int^{\mathrm{sf}_q} F(Z_h(x,\cdot))\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^{m_n}} W_{\lambda_n}(\sigma_n, \vec{\xi}) F(Z_h(X(\cdot,\sigma_n,\vec{\xi}),\cdot))\,d\vec{\xi}, \tag{1.1}$$

where

$$\begin{aligned} W_\lambda(\sigma,\vec{\xi}) &= \gamma_{\sigma,\lambda} \exp\Big\{-\frac{\lambda}{2}\int_0^T \Big|\frac{dX}{dt}(t,\sigma_n,\vec{\xi})\Big|^2 dt\Big\} \\ &= \Big(\frac{\lambda}{2\pi}\Big)^{m/2}\Big[\prod_{k=1}^m (\tau_k - \tau_{k-1})^{-1/2}\Big] \exp\Big\{-\frac{\lambda}{2}\sum_{k=1}^m \frac{(\xi_k - \xi_{k-1})^2}{\tau_k - \tau_{k-1}}\Big\} \end{aligned} \tag{1.2}$$

and

$$\gamma_{\sigma,\lambda} = \Big(\frac{\lambda}{2\pi}\Big)^{m/2} \prod_{k=1}^m (\tau_k - \tau_{k-1})^{-1/2}. \tag{1.3}$$

When $h \equiv 1$ on $[0,T]$, the generalized sequential Feynman integral is reduced to the sequential Feynman integral $\int^{\mathrm{sf}_q} F(x)\,dx$ defined and studied in [1–3, 6].

2. Generalized sequential Fourier-Feynman transform

Let $D[0,T]$ be the class of elements $x \in C_0[0,T]$ such that x is absolutely continuous on $[0,T]$ and its derivative $x' \in L_2[0,T]$. For $u, v \in L_2[0,T]$ and $x \in C_0[0,T]$, we let

$$\langle u, v\rangle = \int_0^T u(t)v(t)\,dt,$$

and for a subdivision σ of $[0,T]$, we let

$$\langle u, v\rangle_k = \int_{\tau_{k-1}}^{\tau_k} u(t)v(t)\,dt$$

for $k = 1, \ldots, m$. If there is a sequence of subdivision $\{\sigma_n\}$, then $\langle u, v\rangle_k$ will be replaced by $\langle u, v\rangle_{n,k}$.

Let $\mathcal{M} = \mathcal{M}(L_2[0,T])$ be the class of complex measures of finite variation defined on $\mathcal{B}(L_2[0,T])$, the Borel measurable subsets of $L_2[0,T]$. A functional F defined on a subset of $C_0[0,T]$ that contains $D[0,T]$ is said to be an element of $\hat{\mathcal{S}} = \hat{\mathcal{S}}(L_2[0,T])$ if there exists a measure $f \in \mathcal{M}$ such that for $x \in D[0,T]$,

$$F(x) = \int_{L_2[0,T]} \exp\{i\langle v, x'\rangle\}\,df(v). \tag{2.1}$$

Note that $\hat{\mathcal{S}}$ with the norm $\|F\| = \|f\|$ is a Banach algebra [1].

Next we consider one more class of functionals which is different from but are closely related with the expression (2.1).

Let $\mathcal{T}$ be the set of functions Ψ defined on $\mathbb{R}$ by

$$\Psi(r) = \int_{\mathbb{R}} \exp\{irs\}\,d\rho(s) \tag{2.2}$$

where ρ is a complex Borel measure of bounded variation on $\mathbb{R}$.

For $s \in \mathbb{R}$, let $\phi(s)$ be the function $v \in L_2[0,T]$ such that $v(t) = s$ for $0 \le t \le T$; thus $\phi : \mathbb{R} \to L_2[0,T]$ is continuous. For $E \in \mathcal{B}(L_2[0,T])$, let

$$\psi(E) = \rho(\phi^{-1}(E)). \tag{2.3}$$

Thus $\psi \in \mathcal{M}$. Transforming the right hand member of (2.2), we have for $x \in D[0,T]$,

$$\Psi(x(T)) = \int_{L_2[0,T]} \exp\{i\langle u, x'\rangle\}\,d\psi(u), \tag{2.4}$$

and $\Psi(x(T))$, considered as a functional of x, is an element of $\hat{\mathcal{S}}$. These functionals were studied in [3, 4, 6, 16].

In the application of the Feynman integral to quantum theory, the function Ψ in (2.2) corresponds to the initial condition associated with Schrödinger equation.

Cameron and Storvick defined the sequential Fourier-Feynman transform for the first time in [2]. And in [6] the authors and coworkers modified the definition so as to be useful to study the sequential Fourier-Feynman transform in conjunction with the convolution product. For discussions on the differences between the two definitions of sequential Fourier-Feynman transform, see Remarks 1.4 and 3.5 in [6] and Remark

3.2 in [16]. We introduce the definition of the generalized sequential Fourier-Feynman transform from [16].

Definition 2.1. Let q be a nonzero real number. For $y \in D[0,T]$, we define a generalized sequential Fourier-Feynman transform $\Gamma_{q,h}(F)$ of F by the formula

$$\Gamma_{q,h}(F)(y) = \int^{\mathrm{sf}_q} F(Z_h(x,\cdot)+y)\,dx \tag{2.5}$$

if it exists.

For the convenience of the readers, we introduce some results from [16] on the existences and explicit expressions for the generalized sequential Fourier-Feynman transforms of functionals that we work with in this paper.

Theorem 2.2 (Theorem 3.4 in [16]). *Let $F \in \hat{\mathcal{S}}$ be given by (2.1) and q be a nonzero real number. Then the generalized sequential Fourier-Feynman transform $\Gamma_{q,h}(F)(y)$ exists and is given by the formula*

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\Big\{ i\langle v, y'\rangle - \frac{i}{2q}\|vh\|_2^2 \Big\}\,df(v) \tag{2.6}$$

for $y \in D[0,T]$. Furthermore, as a function of y, $\Gamma_{q,h}(F)$ is an element of $\hat{\mathcal{S}}$. In fact,

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\{ i\langle v, y'\rangle\}\,df^t_{q,h}(v) \tag{2.7}$$

for $y \in D[0,T]$, where $f^t_{q,h}$ is the measure in $\mathcal{M}$ defined by

$$f^t_{q,h}(E) = \int_E \exp\Big\{ -\frac{i}{2q}\|vh\|_2^2 \Big\}\,df(v) \tag{2.8}$$

for $E \in \mathcal{B}(L_2[0,T])$.

Theorem 2.3 (Theorem 3.7 in [16]). *For $x \in D[0,T]$, let $F(x) = G(x)\Psi(x(T))$ where $G \in \hat{\mathcal{S}}$ and $\Psi \in \mathcal{T}$ are given by (2.1) with corresponding measure g in $\mathcal{M}$ and (2.2), respectively. Then the generalized sequential Fourier-Feynman transform $\Gamma_{q,h}(F)(y)$ exists and is given by the formula*

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \int_{\mathbb{R}} \exp\Big\{ i\langle v+s, y'\rangle - \frac{i}{2q}\|(v+s)h\|_2^2 \Big\}\,d\rho(s)\,dg(v) \tag{2.9}$$

for $y \in D[0,T]$. Furthermore, as a function of y, $\Gamma_{q,h}(F)$ belongs to $\hat{\mathcal{S}}$. In fact,

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\{ i\langle v, y'\rangle\}\,d\hat{f}^t_{q,h}(v) \tag{2.10}$$

for $y \in D[0,T]$, where $\hat{f}^t_{q,h}$ is the measure given by (2.8) with corresponding measure $\hat{f}$ defined by $\hat{f}(E) = \int_{L_2[0,T]} g(E-u)\,d\psi(u)$ for $E \in \mathcal{B}(L_2[0,T])$.

Theorem 2.4 (Theorem 3.8 in [16]). *Let Φ be a bounded measurable functional defined on $L_2[0,T]$, and let*

$$F(x) = \int_{L_2[0,T]} \exp\{ i\langle v, x'\rangle\}\Phi(v)\,df(v). \tag{2.11}$$

Then the generalized sequential Fourier-Feynman transform $\Gamma_{q,h}(F)(y)$ *exists and is given by the formula*

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\Big\{ i\langle v, y'\rangle - \frac{i}{2q}\|vh\|_2^2 \Big\} \Phi(v)\, df(v) \tag{2.12}$$

for $y \in D[0,T]$. *Furthermore, as a function of* y, $\Gamma_{q,h}(F)$ *belongs to* $\hat{\mathcal{S}}$. *In fact,*

$$\Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\{i\langle v, y'\rangle\}\, d\tilde{f}^t_{q,h}(v) \tag{2.13}$$

for $y \in D[0,T]$, *where* $\tilde{f}^t_{q,h}$ *is the measure given by* (2.8) *with corresponding measure* $\tilde{f}$ *defined by* $\tilde{f}(E) = \int_E \Phi(v)\, df(v)$ *for* $E \in \mathcal{B}(L_2[0,T])$.

3. Generalized sequential convolution product

In this section we define and study the generalized sequential convolution product for the generalized sequential Fourier-Feynman transform. We investigate the existence of the generalized sequential convolution product for functionals in Section 2. Also we show that the generalized sequential Fourier-Feynman transform of the generalized sequential convolution product is a product of the generalized sequential Fourier-Feynman transforms for these functionals.

Definition 3.1. Let q be a nonzero real number. For $y \in D[0,T]$, we define the generalized sequential convolution $(F * G)_q$ of F and G by the formula

$$(F * G)_{q,h}(y) = \int^{\text{sf}_q} F\Big(\frac{y + Z_h(x,\cdot)}{\sqrt{2}}\Big) G\Big(\frac{y - Z_h(x,\cdot)}{\sqrt{2}}\Big)\, dx \tag{3.1}$$

if it exists.

Remark 3.2. 1. When $h \equiv 1$ on $[0,T]$, the generalized sequential convolution product $(F * G)_{q,h}$ is reduced to the convolution product $(F * G)_q$ for the sequential Fourier-Feynman transform which was defined and studied on [6].

2. Hence some of the results in [6] can be obtained as corollaries of the results in this section. For example, Theorems 3.1, 3.2, 3,3 and 3.8 in [6] follow from Theorems 3.3, 3.9, 3.10 and 3.4 below, respectively.
3. Convolution product for the analytic Fourier-Feynman transform was introduced and studied in many literatures including [5, 7, 10–12, 14, 15].
4. For $\vec{\eta} = -\vec{\xi}$, we know that $|\frac{\partial(\vec{\xi})}{\partial(\vec{\eta})}| = 1$, $W_{\lambda_n}(\sigma_n, \vec{\xi}) = W_{\lambda_n}(\sigma_n, \vec{\eta})$ and $Z_h(X(\cdot, \sigma_n, \vec{\xi}), \cdot) = -Z_h(X(\cdot, \sigma_n, \vec{\eta}), \cdot)$. So we conclude that the generalized (sequential) convolution product, in common with the convolution product for the analytic Fourier-Feynman transform [7, 10], is commutative.

Theorem 3.3. *Let* $F_j \in \hat{\mathcal{S}}$ *be given by* (2.1) *with corresponding measures* f_j *in* $\mathcal{M}$ *for* $j = 1, 2$. *Then for each nonzero real number* q, *the generalized convolution product* $(F_1 * F_2)_{q,h}$ *exists, belongs to* $\hat{\mathcal{S}}$ *and is given by*

$$\begin{aligned}(F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0,T]} &\exp\Big\{ \frac{i}{\sqrt{2}}\langle v_1 + v_2, y'\rangle - \frac{i}{4q}\|(v_1 - v_2)h\|_2^2 \Big\} \\ &\times df_1(v_1)\, df_2(v_2)\end{aligned} \tag{3.2}$$

for $y \in D[0,T]$. Furthermore, as a function of $y \in D[0,T]$, $(F_1 * F_2)_{q,h}(y)$ is an element of $\hat{\mathcal{S}}$. In fact

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2[0,T]} \exp\{i\langle w, y'\rangle\}\, df^c_{q,h}(w), \tag{3.3}$$

where $f^c_{q,h} = f_{q,h} \circ \phi^{-1}$ and

$$f_{q,h}(E) = \int_E \exp\Big\{-\frac{i}{4q}\|(v_1 - v_2)h\|_2^2\Big\}\, df_1(v_1)\, df_2(v_2) \tag{3.4}$$

for $E \in \mathcal{B}(L_2^2[0,T])$ and $\phi : L_2^2[0,T] \to L_2[0,T]$ is a function defined by $\phi(v_1, v_2) = \frac{v_1+v_2}{\sqrt{2}}$.

Proof. Let $\sigma : 0 = \tau_0 < \tau_1 < \cdots < \tau_m = T$ be a subdivision of $[0,T]$. Then

$$\begin{aligned} &F_1\Big(\frac{y + Z_h(X(\cdot,\sigma,\vec{\xi}),\cdot)}{\sqrt{2}}\Big) F_2\Big(\frac{y - Z_h(X(\cdot,\sigma,\vec{\xi}),\cdot)}{\sqrt{2}}\Big) \\ &= \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle v_1 + v_2, y'\rangle + \frac{i}{\sqrt{2}}\sum_{k=1}^m \frac{\xi_k - \xi_{k-1}}{\tau_k - \tau_{k-1}}\langle v_1 - v_2, h\rangle_k\Big\} \\ &\times df_1(v_1)\, df_2(v_2). \end{aligned}$$

Let λ be a complex number with $\operatorname{Re}\lambda > 0$, and let

$$\begin{aligned} I_{\sigma,\lambda}(F_1, F_2) = \int_{\mathbb{R}^m} W_\lambda(\sigma,\vec{\xi}) F_1\Big(\frac{y + Z_h(X(\cdot,\sigma,\vec{\xi}),\cdot)}{\sqrt{2}}\Big) \\ \times F_2\Big(\frac{y - Z_h(X(\cdot,\sigma,\vec{\xi}),\cdot)}{\sqrt{2}}\Big)\, d\vec{\xi}. \end{aligned}$$

By the Fubini theorem, we have for $y \in D[0,T]$

$$\begin{aligned} I_{\sigma,\lambda}(F_1, F_2) = \int_{L_2^2[0,T]} J_{\sigma,\lambda}\Big(\frac{v_1 - v_2}{\sqrt{2}}\Big) \exp\Big\{\frac{i}{\sqrt{2}}\langle v_1 + v_2, y'\rangle\Big\} \\ \times df_1(v_1)\, df_2(v_2), \end{aligned}$$

where

$$\begin{aligned} J_{\sigma,\lambda}\Big(\frac{v_1 - v_2}{\sqrt{2}}\Big) = \gamma_{\sigma,\lambda} \int_{\mathbb{R}^m} \exp\Big\{-\frac{\lambda}{2}\sum_{k=1}^m \frac{(\xi_k - \xi_{k-1})^2}{\tau_k - \tau_{k-1}} \\ + \frac{i}{\sqrt{2}}\sum_{k=1}^m \frac{\xi_k - \xi_{k-1}}{\tau_k - \tau_{k-1}}\langle v_1 - v_2, h\rangle_k\Big\}\, d\vec{\xi}. \end{aligned}$$

Letting $\eta_k = \xi_k - \xi_{k-1}$ for $k = 1, 2, \ldots, m$ and using the integration formula $\int_{\mathbb{R}} e^{-a\eta^2 + ib\eta}\, d\eta = (\frac{\pi}{a})^{1/2} \exp\{-\frac{b^2}{4a}\}$ for $\operatorname{Re} a > 0$ and $b \in \mathbb{R}$, we have

$$J_{\sigma,\lambda}\Big(\frac{v_1 - v_2}{\sqrt{2}}\Big) = \exp\Big\{-\frac{1}{4\lambda}\sum_{k=1}^m \frac{\langle v_1 - v_2, h\rangle_k^2}{\tau_k - \tau_{k-1}}\Big\}.$$

Let $\{\sigma_n\}$ be a sequence of subdivisions of $[0,T]$ such that $\|\sigma_n\| \to 0$, and let $\{\lambda_n\}$ be a sequence of complex numbers such that $\operatorname{Re}\lambda_n > 0$ and $\lambda_n \to -iq$ as $n \to \infty$. Let

$$v_{\sigma_n,h}(t) = \begin{cases} \frac{\langle \frac{v_1-v_2}{\sqrt{2}},h\rangle_{n,k}}{\tau_{n,k}-\tau_{n,k-1}}, & \text{if } \tau_{n,k-1} \le t < \tau_{n,k}, \quad k=1,\dots,m \\ 0, & \text{if } t = T. \end{cases}$$

Then since

$$\|v_{\sigma_n,h}\|_2^2 = \sum_{k=1}^{m_n} \int_{\tau_{n,k-1}}^{\tau_{n,k}} (v_{\sigma_n,h}(t))^2\,dt = \frac{1}{2}\sum_{k=1}^{m_n} \frac{\langle v_1-v_2,h\rangle_{n,k}^2}{\tau_{n,k}-\tau_{n,k-1}},$$

we have by Lemma 2.1 of [16],

$$\lim_{n\to\infty} J_{\sigma_n,\lambda_n}\Big(\frac{v_1-v_2}{\sqrt{2}}\Big) = \exp\Big\{-\frac{i}{4q}\|v_1-v_2\|_2^2\Big\}.$$

Now applying the bounded convergence theorem, we obtain

$$\begin{aligned}(F_1 * F_2)_{q,h}(y) &= \lim_{n\to\infty} I_{\sigma_n,\lambda_n}(F_1,F_2) \\ &= \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle v_1+v_2,y'\rangle - \frac{i}{4q}\|v_1-v_2\|_2^2\Big\} \\ &\quad\times df_1(v_1)\,df_2(v_2)\end{aligned}$$

as we wished. Furthermore since $f_{q,h}$ in (3.4) belongs to $\mathcal{M}$ and ϕ is a Borel measurable function, we see that $f_{q,h}^c \in \mathcal{M}$ and $(F_1 * F_2)_{q,h}(y)$ can be expressed as (3.3) which completes the proof. □

Theorem 3.4. *For $x \in D[0,T]$, let $F_j(x) = G_j(x)\Psi_j(x(T))$ where $G_j \in \hat{\mathcal{S}}$ and $\Psi_j \in \mathcal{T}$ are given by (2.1) with corresponding measures g_j in $\mathcal{M}$ and (2.2), respectively for $j=1,2$. Then for each nonzero real number q, the generalized convolution product $(F_1 * F_2)_{q,h}$ exists, belongs to $\hat{\mathcal{S}}$ and is given by*

$$\begin{aligned}(F_1 * F_2)_{q,h}(y) &= \int_{L_2^2[0,T]}\int_{\mathbb{R}^2} \exp\Big\{\frac{i}{\sqrt{2}}\langle v_1+v_2+s_1+s_2,y'\rangle \\ &\quad - \frac{i}{4q}\|(v_1-v_2+s_1-s_2)h\|_2^2\Big\} \\ &\quad\times d\rho_1(s_1)\,d\rho_2(s_2)\,dg_1(v_1)\,dg_2(v_2)\end{aligned} \tag{3.5}$$

for $y \in D[0,T]$. Furthermore, as a function of $y \in D[0,T]$, $(F_1 * F_2)_{q,h}(y)$ is an element of $\hat{\mathcal{S}}$. In fact

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2[0,T]} \exp\{i\langle w,y'\rangle\}\,d\hat{f}_{q,h}^c(w), \tag{3.6}$$

for $y \in D[0,T]$, where $\hat{f}_{q,h}^c = \hat{f}_{q,h} \circ \phi^{-1}$ is the measure in $\mathcal{M}$ with $\hat{f}_{q,h}$ defined by (3.4) replaced f_1 and f_2 with $\hat{f}_1$ and $\hat{f}_2$, respectively, and $\hat{f}_1$ and $\hat{f}_2$ are the measures defined in the proof below.

Proof. We know from the proof of Theorem 2.3 of [16], that F_j belongs to $\hat{\mathcal{S}}$ and can be expressed as

$$F_j(x) = \int_{L_2[0,T]} \exp\{i\langle w_j,y'\rangle\}\,d\hat{f}_j(v_j)$$

for $y \in D[0,T]$, where $\hat{f}_j \in \mathcal{M}$ is given by $\hat{f}_j(E) = \int_{L_2[0,T]} g_j(E-u_j)\,d\psi_j(u_j)$ for $E \in \mathcal{B}(L_2[0,T])$ and $j=1,2$. Now we apply Theorem 3.3 to obatin

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle w_1+w_2, y'\rangle - \frac{i}{4q}\|(w_1-w_2)h\|_2^2\Big\} \times d\hat{f}_1(w_1)\,d\hat{f}_2(w_2)$$

for $y \in D[0,T]$. By the unsymmetric Fubini theorem and the transformation $v_j = w_j - u_j$, $j=1,2$, we have

$$\begin{aligned}(F_1 * F_2)_{q,h}(y) = \int_{L_2^4[0,T]} &\exp\Big\{\frac{i}{\sqrt{2}}\langle v_1+v_2+u_1+u_2, y'\rangle \\ &- \frac{i}{4q}\|(v_1-v_2+u_1-u_2)h\|_2^2\Big\} \\ &\times dg_1(v_1)\,d\psi_1(u_1)\,dg_2(v_2)\,d\psi_2(u_2)\end{aligned}$$

for $y \in D[0,T]$. By the definition of Ψ_j and ψ_j, $j=1,2$, and the Fubini theorem, we obtain (3.5). Moreover it is obvious to see that $(F_1 * F_2)_{q,h}(y)$ can be expressed as (3.6) and that $(F_1 * F_2)_{q,h}$ belongs to $\hat{\mathcal{S}}$. □

Theorem 3.5. *Let $F_j(x) = \int_{L_2[0,T]} \exp\{i\langle v_j, x'\rangle\}\Phi_j(v_j)\,df_j(v_j)$, $x \in D[0,T]$, where Φ_j is a bounded measurable functional defined on $L_2[0,T]$ for $j=1,2$. Then for each nonzero real number q, the generalized convolution product $(F_1 * F_2)_{q,h}$ exists, belongs to $\hat{\mathcal{S}}$ and is given by*

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle v_1+v_2, y'\rangle - \frac{i}{4q}\|(v_1-v_2)h\|_2^2\Big\} \times \Phi_1(v_1)\Phi_2(v_2)\,df_1(v_1)\,df_2(v_2) \tag{3.7}$$

for $y \in D[0,T]$. Furthermore, as a function of $y \in D[0,T]$, $(F_1 * F_2)_{q,h}(y)$ is an element of $\hat{\mathcal{S}}$. In fact

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2[0,T]} \exp\{i\langle w, y'\rangle\}\,d\tilde{f}^c_{q,h}(w), \tag{3.8}$$

for $y \in D[0,T]$, where $\tilde{f}^c_{q,h} = \tilde{f}_{q,h} \circ \phi^{-1}$ is the measure in $\mathcal{M}$ with $\tilde{f}_{q,h}$ defined by (3.4) replaced f_1 and f_2 with $\tilde{f}_1$ and $\tilde{f}_2$, respectively, and $\tilde{f}_1$ and $\tilde{f}_2$ are the measures defined in the proof below.

Proof. We know from the proof of Theorem 2.4 of [16], that F_j belongs to $\hat{\mathcal{S}}$ and can be expressed as

$$F_j(x) = \int_{L_2[0,T]} \exp\{i\langle v_j, x'\rangle\}\,d\tilde{f}_j(v_j)$$

for $y \in D[0,T]$, where $\tilde{f}_j \in \mathcal{M}$ is given by $\tilde{f}_j(E) = \int_E \Phi_j(v_j)\,df_j(v_j)$ for $E \in \mathcal{B}(L_2[0,T])$. Now we apply Theorem 3.3 to obatin

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle v_1+v_2, y'\rangle - \frac{i}{4q}\|(v_1-v_2)h\|_2^2\Big\} \times d\tilde{f}_1(v_1)\,d\tilde{f}_2(v_2)$$

for $y \in D[0,T]$. Replacing $d\tilde{f}_j(v_j)$ by $\Phi(v_j)\,df_j(v_j)$, we completes the proof. Moreover it is obvious to see that $(F_1 * F_2)_{q,h}(y)$ can be expressed as (3.8) and that $(F_1 * F_2)_{q,h}$ belongs to $\hat{\mathcal{S}}$. □

In Theorems 3.3, 3.4 and 3.5, we considered the generalized convolution product $(F_1 * F_2)_{q,h}$ of the same type of functionals F_1 and F_2. That is, for example, in Theorem 3.3 F_1 and F_2 were given by (2.1), while in Theorem 3.4, F_j was given by $G(x)\Psi_j(x(T))$ for $j = 1, 2$.

But F_1 and F_2 are not necessarily of the same type of functionals. In the following theorem we express explicitly $(F_1 * F_2)_{q,h}$ when F_1 and F_2 are different type of functionals. We state these results without proofs because they can be proved by similar methods as in Theorems 3.3, 3.4 and 3.5.

THEOREM 3.6. *Let F_1, F_2 and F_3 are given as in Theorems 3.3, 3.4 and 3.5, respectively. Then for each nonzero real number q, the following three generalized convolution products exist, belong to $\hat{\mathcal{S}}$ and are given by*

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0,T]} \int_{\mathbb{R}} \exp\Big\{ \frac{i}{\sqrt{2}} \langle v_1 + v_2 + s_2, y' \rangle - \frac{i}{4q} \|(v_1 - v_2 - s_2)h\|_2^2 \Big\}\, d\rho_2(s_2)\, df_1(v_1)\, dg_2(v_2), \tag{3.9}$$

$$(F_1 * F_3)_{q,h}(y) = \int_{L_2^2[0,T]} \exp\Big\{ \frac{i}{\sqrt{2}} \langle v_1 + v_3, y' \rangle - \frac{i}{4q} \|(v_1 - v_3)h\|_2^2 \Big\} \Phi_3(v_3)\, df_1(v_1)\, df_3(v_3) \tag{3.10}$$

and

$$(F_2 * F_3)_{q,h}(y) = \int_{L_2^2[0,T]} \int_{\mathbb{R}} \exp\Big\{ \frac{i}{\sqrt{2}} \langle v_2 + v_3 + s_2, y' \rangle - \frac{i}{4q} \|(v_2 - v_3 + s_2)h\|_2^2 \Big\} \Phi_3(v_3)\, d\rho_2(s_2)\, dg_2(v_2)\, df_3(v_3) \tag{3.11}$$

for $y \in D[0,T]$.

4. Transform and convolution product

In this section, we examine the generalized sequential Fourier-Feynman transform of the generalized convolution product of functionals in $\hat{\mathcal{S}}$. It turns out that the relationship (4.1) below is the same as Theorem 3.3 in [12] for the generalized analytic Fourier-Feynman transform.

THEOREM 4.1. *Let F_j, $j = 1, 2$, be given as in Theorem 3.3. Then for all nonzero real number q, $\Gamma_{q,h}((F_1 * F_2)_{q,h})(y)$ exists and*

$$\Gamma_{q,h}((F_1 * F_2)_{q,h})(y) = \Gamma_{q,h}(F_1)\Big(\frac{y}{\sqrt{2}}\Big) \Gamma_{q,h}(F_2)\Big(\frac{y}{\sqrt{2}}\Big), \tag{4.1}$$

for $y \in D[0,T]$.

Proof. From the expression (3.3) for $(F_1 * F_2)_{q,h}(y)$ and Theorem 2.2, we have

$$\Gamma_{q,h}((F_1 * F_2)_{q,h})(y) = \int_{L_2[0,T]} \exp\Big\{ i\langle w, y'\rangle - \frac{i}{2q}\|wh\|_2^2 \Big\}\, df^c_{q,h}(w)$$

for $y \in D[0,T]$. Using the definition of $f^c_{q,h}$ in Theorem 3.3, we rewrite $\Gamma_{q,h}(F_1 * F_2)_{q,h}(y)$ as

$$\begin{aligned}\Gamma_{q,h}((F_1 * F_2)_{q,h})(y) = \int_{L_2^2[0,T]} \exp\Big\{ &\frac{i}{\sqrt{2}}\langle v_1 + v_2, y'\rangle \\ &- \frac{i}{2q}(\|v_1h\|_2^2 + \|v_2h\|_2^2)\Big\}\, df_1(v_1)\, df_2(v_2),\end{aligned}$$

and, by Theorem 2.2, it is easy to see that the last expression is equal to the right hand side of (4.1) and this completes the proof. □

Next we show that the Parseval's relation for the generalized sequential Fourier-Feynman transform holds for functionals in $\hat{\mathcal{S}}$, which is a generalization of the Parseval's relation for the sequential Fourier-Feynman transform established in Theorem 3.3 of [6].

THEOREM 4.2. *Let F_j, $j = 1, 2$, be given as in Theorem 3.3. Then for all nonzero real number q, the Parseval's relation*

$$\begin{aligned}&\int^{\mathrm{sf}_{-q}} \Gamma_{q,h}(F_1)\Big(\frac{Z_h(y,\cdot)}{\sqrt{2}}\Big)\Gamma_{q,h}(F_2)\Big(\frac{Z_h(y,\cdot)}{\sqrt{2}}\Big)\, dy \\ &= \int^{\mathrm{sf}_{q}} F_1\Big(\frac{Z_h(y,\cdot)}{\sqrt{2}}\Big) F_2\Big(-\frac{Z_h(y,\cdot)}{\sqrt{2}}\Big)\, dy\end{aligned} \tag{4.2}$$

for the generalized sequential Fourier-Feynman transform holds.

Proof. Since we know from Theorems 2.2 and 3.3 that both sides of (4.1) belong to the Banach algebra $\hat{\mathcal{S}}$ as a function of y, by Theorem 2.2 of [16], they are generalized sequential Feynman integrable and

$$\begin{aligned}&\int^{\mathrm{sf}_{-q}} \Gamma_{q,h}(F_1)\Big(\frac{Z_h(y,\cdot)}{\sqrt{2}}\Big)\Gamma_{q,h}(F_2)\Big(\frac{Z_h(y,\cdot)}{\sqrt{2}}\Big)\, dy \\ &= \int^{\mathrm{sf}_{-q}} \Gamma_{q,h}((F_1 * F_2)_{q,h})(Z_h(y,\cdot))\, dy.\end{aligned}$$

To evaluate the generalized sequential Feynman integral on the right hand side of the above expression, we apply Theorem 2.2 to the expression (3.3) for $(F_1 * F_2)_{q,h}(y)$ to obtain

$$\Gamma_{q,h}((F_1 * F_2)_{q,h})(y) = \int_{L_2[0,T]} \exp\{ i\langle v, y'\rangle\}\, d(f^c_{q,h})^t_{q,h}(v),$$

where $(f^c_{q,h})^t_{q,h}$ is the measure in $\mathcal{M}$ defined by (2.8) replaced f with $f^c_{q,h}$, and $f^c_{q,h}$ is the measure defined in Theorem 3.3. Now applying the evaluation formula for the generalized sequential Feynman integral in Theorem 2.2 of [16], we obtain

$$\begin{aligned}
&\int^{\mathrm{sf}_{-q}} \Gamma_{q,h}((F_1 * F_2)_{q,h}(Z_h(y,\cdot))\,dy \\
&= \int_{L_2[0,T]} \exp\Big\{\frac{i}{2q}\|vh\|_2^2\Big\}\,d(f_{q,h}^c)_{q,h}^t(v) \\
&= \int_{L_2^2[0,T]} \exp\Big\{-\frac{i}{4q}\|(v_1-v_2)h\|_2^2\Big\}\,df_1(v_1)\,df_2(v_2),
\end{aligned}$$

where the second equality is obtained by the definition of the measure $(f_{q,h}^c)_{q,h}^t$. On the other hand, since $\hat{\mathcal{S}}$ is a Banach space,

$$F_1\Big(\frac{y}{\sqrt{2}}\Big)F_2\Big(-\frac{y}{\sqrt{2}}\Big) = \int_{L_2^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle v_1-v_2, y'\rangle\Big\}\,df_1(v_1)\,df_2(v_2)$$

belongs to $\hat{\mathcal{S}}$, and we apply the evaluation formula in Theorem 2.2 of [16] once more to obtain

$$\begin{aligned}
&\int^{\mathrm{sf}_{q}} F_1\Big(\frac{Z_h(y,\cdot)}{\sqrt{2}}\Big)F_2\Big(-\frac{Z_h(y,\cdot)}{\sqrt{2}}\Big)\,dy \\
&= \int_{L_2^2[0,T]} \exp\Big\{-\frac{i}{4q}\|(v_1-v_2)h\|_2^2\Big\}\,df_1(v_1)\,df_2(v_2),
\end{aligned}$$

which completes the proof. □

Remark 4.3. From a careful look at the relationship (4.2), we can modify the proof of Theorem 4.2 to obtain the following alternative form of the Parseval's relation for the generalized sequential Fourier-Feynman transform:

$$\begin{aligned}
&\int^{\mathrm{sf}_{-q}} \Gamma_{q/2,h}(F_1)(Z_h(y,\cdot))\Gamma_{q/2,h}(F_2)(Z_h(y,\cdot))\,dy \\
&= \int^{\mathrm{sf}_{q}} F_1(Z_h(y,\cdot))F_2(-Z_h(y,\cdot))\,dy,
\end{aligned} \tag{4.3}$$

for F_j, $j=1,2$, as in Theorem 3.3 and for every nonzero real number q.

Since all the functionals we worked with in Theorems 3.4 through 3.6 belong to the Banach algebra $\hat{\mathcal{S}}$, by Theorems 4.1 and 4.2, we have the following results.

Theorem 4.4. *Let F_1 and F_2 be given as in Theorems* 3.4, 3.5 *or* 3.6. *Then for all nonzero real number q, $\Gamma_{q,h}((F_1 * F_2)_{q,h})(y)$ exists for $y \in D[0,T]$ and relationship* (4.1) *holds.*

Theorem 4.5. *Let F_1 and F_2 be given as in Theorems* 3.4, 3.5 *or* 3.6. *Then for all nonzero real number q, the Parseval's relations* (4.2) *and* (4.3) *for the generalized sequential Fourier-Feynman transform holds.*

References

[1] R.H. Cameron and D.A. Storvick, *A simple definition of the Feynman integral, with applications*, Mem. Amer. Math. Soc. No. **288**, Amer. Math. Soc., 1983.

[2] R.H. Cameron and D.A. Storvick, *Sequential Fourier-Feynman transforms*, Annales Acad. Scient. Fenn. **10** (1985), 107–111.

[3] R.H. Cameron and D.A. Storvick, *New existence theorems and evaluation formulas for sequential Feynman integrals*, Proc. London Math. Soc. **52** (1986), 557–581.

[4] R.H. Cameron and D.A. Storvick, *New existence theorems and evaluation formulas for analytic Feynman integrals*, Deformations Math. Struct., Complex Analy. Phys. Appl., Kluwer Acad. Publ., Dordrecht (1989), 297–308.

[5] K.S. Chang, D.H. Cho, B.S. Kim, T.S. Song and I. Yoo, *Relationships involving generalized Fourier-Feynman transform, convolution and first variation*, Integral Transform. Spec. Funct. **16** (2005), 391–405.

[6] K.S. Chang, D.H. Cho, B.S. Kim, T.S. Song and I. Yoo, *Sequential Fourier-Feynman transform, convolution and first variation*, Trans. Amer. Math. Soc. **360** (2008), 1819–1838.

[7] K.S. Chang, B.S. Kim and I. Yoo, *Fourier-Feynman transform, convolution and first variation of functionals on abstract Wiener space*, Integral Transform. Spec. Funct. **10** (2000), 179–200.

[8] S.J. Chang and J.G. Choi, *Rotation of Gaussian paths on Wiener space with applications*, Banach J. Math. Anal. **12** (2018), 651–672.

[9] D.M. Chung, C. Park and D. Skoug, *Generalized Feynman integrals via conditional Feynman integrals*, Michigan Math. J. **40** (1993), 377–391.

[10] T. Huffman, C. Park and D. Skoug, *Analytic Fourier-Feynman transforms and convolution*, Trans. Amer. Math. Soc. **347** (1995), 661–673.

[11] T. Huffman, C. Park and D. Skoug, *Convolutions and Fourier-Feynman transforms of functionals involving multiple integrals*, Michigan Math. J. **43** (1996), 247–261.

[12] T. Huffman, C. Park and D. Skoug, *Generalized transforms and convolutions*, Int. J. Math. Math. Sci. **20** (1997), 19–32.

[13] C. Park and D. Skoug, *A Kac-Feynman integral equation for conditional Wiener integrals*, J. Integral Equations Appl. **3** (1991), 411–427.

[14] C. Park, D. Skoug and D. Storvick, *Relationships among the first variation, the convolution product, and the Fourier-Feynman transform*, Rocky Mountain J. Math. **28** (1998), 1447–1468.

[15] D. Skoug and D. Storvick, *A survey of results involving transforms and convolutions in function space*, Rocky Mountain J. Math. **34** (2004), 1147–1176.

[16] I. Yoo and B.S. Kim, *Generalized sequential Feynman integral and Fourier-Feynman transform*, Rocky Mountain J. Math. accepted. (2021) .

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