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ON OPIAL-TYPE INEQUALITIES VIA A NEW GENERALIZED INTEGRAL OPERATOR

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Abstract. Opial inequality and its consequences are useful in establishing existence and uniqueness of solutions of initial and boundary value problems for differential and difference equations. In this paper we analyze Opial-type inequalities for convex functions. We have studied different versions of these inequalities for a generalized integral operator. Further difference of Opial-type inequalities are utilized to obtain generalized mean value theorems, which further produce various interesting derivations for fractional and conformable integral operators.

1. Introduction and Preliminary Results

Opial obtained the following integral inequality in 1960 [22].

Theorem 1.1. *Let $g \in C^1[0,h]$ be such that $g(0) = g(h) = 0$ and $g(t) > 0$ for $t \in (0,h)$. Then*

$$\int_0^h |g(t)g'(t)|dt \leq \frac{h}{4}\int_0^h (g'(t))^2 dt.$$

Here $\frac{h}{4}$ is a best possibility constant.

Many researchers have given generalizations and extensions of this well known Opial inequality in different time spans, see [5–7, 9, 20, 23, 24]. Recently Opial-type inequalities are studied for fractional derivatives and integral operators involving Caputo, Canavati and Riemann-Liouville, see [2–4, 10] and the references therein. In this paper our aim is to produce new Opial-type inequalities for convex functions using a generalized integral operator. The differences of these inequalities have been investigated by mean value theorems. To proceed further, we need the following characterizations [25]:

Andrić at el. in [4] gave the following Opial-type inequality for convex functions.

Theorem 1.2. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{1/q})$ is convex and $\phi(0) = 0$. Let $\eta \in U_1(\theta, K)$ where*

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$\left(\int_a^x (K(x,t))^p dt\right)^{1/p} \le M$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\int_a^b |\eta(x)|^{1-q}\phi'(|\eta(x)|)|\theta(x)|^q dx \le \frac{q}{M^q}\phi\left(M\left(\int_a^b |\theta(x)|^q dx\right)^{1/q}\right) \le \frac{q}{M^q(b-a)}\int_a^b \phi((b-a)^{1/q}M|\theta(x)|)dx. \tag{1}$$

If the function $\phi(x^{1/q})$ *is concave, then the inequality holds in reverse direction.*

The following extensions of Opial-type inequalities of Mitrinović and Pečarić are given by Farid et al. in [11]:

THEOREM 1.3. *Let* $\phi,\ g : [0,\infty) \to \mathbb{R}$ *be differentiable convex and increasing functions with* $\phi(g(0)) = 0$. *Also let* $\eta \in U_1(g \circ \theta, K)$ *and* $|K(x,t)| \le M$, *where* M *is a constant. Then the following inequalities hold:*

$$\int_a^b \phi'(g(|\eta(x)|))g'(|\eta(x)|)|g\circ\theta(x)|dx \le \frac{1}{M}\phi\left(g\left(M\int_a^b |g\circ\theta(x)|dx\right)\right) \le \frac{1}{M(b-a)}\int_a^b \phi(g(M(b-a)|g\circ\theta(x)|))dx. \tag{2}$$

THEOREM 1.4. *Let* $\phi : [0,\infty) \to \mathbb{R}$ *be a differentiable convex and increasing function with* $\phi(g(0)) = 0$. *Also let* $\eta \in U_1(\theta, K)$ *and* $|K(x,t)| \le M$, *where* M *is a constant. Then for* $q > 1$ *the following inequalities hold:*

$$\int_a^b \phi'((|\eta(t)|)^q)|\eta(t)|^{q-1}|\theta(t)|^q dt \le \frac{1}{qM}\phi\left(\left(M\int_a^b |\theta(t)|^q dt\right)^q\right) \le \frac{1}{qM(b-a)}\int_a^b \phi((M(b-a)\,|\theta(t)|^q)^q)dt. \tag{3}$$

The following mean value theorems are given in [13]:

THEOREM 1.5. *[13] Let* $\phi,\ g : [0,\infty) \to \mathbb{R}$ *be differentiable convex and increasing functions with* $\phi(g(0)) = 0$. *Also let* $\eta \in U_1(g\circ\theta, K)$ *and* $|K(x,t)| \le M$. *If* $\phi \in C^2(I)$, *where* $I \subseteq (0,\infty)$ *is compact interval. Then there exists* $\xi_i \in I$ *such that the following equation holds:*

$${}_g\mathbb{F}_i^{\phi}(\eta,\theta;M) = \frac{\phi''(\xi_i)}{2}\,{}_g\mathbb{F}_i^{x^2}(\eta,\theta;M),\ i=1,2. \tag{4}$$

THEOREM 1.6. *[13] Let* ϕ_1, ϕ_2 *and* g *be the functions with assumptions of Theorem 1.5. If* $\phi_1, \phi_2 \in C^2(I)$, *where* $I \subseteq (0,\infty)$ *is a compact interval and* ${}_gF_i^{x^2}(\eta,\theta;M) \neq 0$, *then there exists* $\xi_i \in I$ *such that we have*

$$\frac{{}_g\mathbb{F}_i^{\phi_1}(\eta,\theta;M)}{{}_g\mathbb{F}_i^{\phi_2}(\eta,\theta;M)} = \frac{\phi_1''(\xi_i)}{\phi_2''(\xi_i)},\ i=1,2.$$

Provided denominators are not zero.

Further the following functionals due to nonnegative differences of inequalities (1) are studied by Farid et al. in [1, 12]:

$$F_1^{\phi}(\eta,\theta,M)=\frac{q}{M^q}\phi\left(M\left(\int_a^b|\theta(t)|^q dt\right)^{1/q}\right)-\int_a^b|\eta(t)|^{1-q}\phi'(|\eta(t)|)|\theta(x)|^q dt$$

$$F_2^{\phi}(\eta,\theta,M)=\frac{q}{M^q(b-a)}\int_a^b\phi((b-a)^{1/q}M|\theta(t)|)dt-\int_a^b|\eta(t)|^{1-q}\phi'(|\eta(t)|)|\theta(t)|^q dt.$$

THEOREM 1.7. *Let $\phi:[0,\infty)\to\mathbb{R}$ be a differentiable function such that for $q>1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0)=0$. Let $\eta\in U_1(\theta,K)$ where $\left(\int_a^x(K(x,t))^p dt\right)^{\frac{1}{p}}\leq M$ and $\frac{1}{p}+\frac{1}{q}=1$. If $\phi\in C^2(I)$, where $I\subset\mathbb{R}_+$ is a compact interval. Then there exists $\xi_i\in I$ such that the following result holds*

$$\mathbb{F}_i^{\phi}(\eta,\theta,M)=\frac{\xi_i\phi''(\xi_i)-(q-1)\phi'(\xi_i)}{2q^2(\xi_i^{2q-1})}\mathbb{F}_i^{x^2}(\eta,\theta,M),\ i=1,2. \tag{5}$$

THEOREM 1.8. *Let $\phi_1,\phi_2:[0,\infty)\to\mathbb{R}$ be differentiable functions such that for $q>1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0)=0,\ i=1,2$. Let $\eta\in U_1(\theta,K)$ where $\left(\int_a^x(K(x,t))^p dt\right)^{\frac{1}{p}}\leq M$ and $\frac{1}{p}+\frac{1}{q}=1$. If $\phi\in C^2(I)$, where $I\subset\mathbb{R}_+$ is a compact interval and $\mathbb{F}_i^{x^2}(\eta,\theta,M)\neq 0$, then there exists $\xi_i\in I$ such that we have*

$$\frac{\mathbb{F}_i^{\phi_1}(\eta,\theta,M)}{\mathbb{F}_i^{\phi_2}(\eta,\theta,M)}=\frac{\xi_i\phi_1''(\xi_i)-(q-1)\phi_1'(\xi_i)}{\xi_i\phi_2''(\xi_i)-(q-1)\phi_2'(\xi_i)},\ i=1,2.$$

Provided the denominators are not zero.

Next we present fractional integral operators which we will utilize to prove the fractional calculus results.

DEFINITION 1. [17] Let $f:[a,b]\to\mathbb{R}$ be an integrable function. Let h be an increasing and positive function on $[a,b]$, having a continuous derivatives h' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ of order $\alpha,k>0$ are defined by

$$I_{h,a^+}^{\alpha,k}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x(h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)f(t)dt,\ x>a, \tag{6}$$

and

$$I_{h,b_-}^{\alpha,k}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b(h(t)-h(x))^{\frac{\alpha}{k}-1}h'(t)f(t)dt,\ x<b. \tag{7}$$

A new generalized integral operator is defined in the following definition:

DEFINITION 2. [14] Let $f,h:[a,b]\to\mathbb{R}$ be the functions such that f be positive and $f\in L_1[a,b]$, and h be differentiable and increasing. Also let $\frac{\Phi}{x}$ be an increasing function on $[0,\infty)$. Then for $x\in[a,b]$ the left and right integral operators are defined by

$$\left(F_{a^+}^{\Phi,h}f\right)(x)=\int_a^x\frac{\Phi(h(x)-h(t))}{h(x)-h(t)}h'(t)f(t)dt,\ x>a \tag{8}$$

and

$$\left(F_{b_-}^{\Phi,h}f\right)(x)=\int_x^b\frac{\Phi(h(t)-h(x))}{h(t)-h(x)}h'(t)f(t)dt,\ x<b. \tag{9}$$

The following lemma is given in [12]:

Lemma 1.9. *Let $\phi \in C^2(I)$, where $I \subseteq (0,\infty)$ and $g(x) = x^q$, $q > 1$ with*

$$\gamma \leq \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{q^2\xi^{2q-1}} \leq \Gamma \text{ for all } \xi \in I. \tag{10}$$

Then the functions ϕ_1, ϕ_2 defined as $\phi_1(x) = \frac{\Gamma x^2}{2} - \phi(x)$, $\phi_2(x) = \phi(x) - \frac{\gamma x^2}{2}$, are convex functions with respect to $g(x) = x^q$, that is $\phi_i(x^{\frac{1}{q}})$, $i = 1, 2$, are convex functions.

Lemma 1.10. *[13] Let $\phi \in C^2(I)$, where $I \subseteq (0,\infty)$, $\gamma' \leq \phi''(y) \leq \Gamma'$ for all $y \in I$. Then the functions $\phi_1(x) = \frac{\Gamma' x^2}{2} - \phi(x)$, $\phi_2(x) = \phi(x) - \frac{\gamma' x^2}{2}$, are convex functions.*

The paper is organized as follows:
In Section 2 new Opial-type inequalities for convex functions using generalized integral operators (8) and (9) are obtained. In Section 3, Opial-type inequalities are obtained for fractional operators by defining particular kernels. In Section 4, mean value theorems are given for the differences of Opial-type inequalities containing generalized, fractional and conformable integral operators.

2. Opial-type inequalities for a generalized integral operator

Theorem 2.1. *Let $\phi,\ g : [0,\infty) \to \mathbb{R}$ be the functions with assumptions of Theorem 1.3. Also let Φ be positive and $\frac{\Phi}{x}$, h are increasing functions on $[0,\infty)$. If $\theta \in L_1[a,b]$, $0 \leq a < b$ and $h' \in L_\infty[a,b]$, then the following inequalities hold:*

$$\begin{aligned} &\int_a^b \phi'\left(g\left(\left|F_{a^+}^{\Phi,h}\theta(t)\right|\right)\right) g'\left(\left|F_{a^+}^{\Phi,h}\theta(t)\right|\right) |g\circ\theta(t)|dt \\ &\leq \frac{h(b)-h(a)}{\Phi(h(b)-h(a))||h'||_\infty}\phi\left(g\left(\frac{\Phi(h(b)-h(a))||h'||_\infty}{h(b)-h(a)}\int_a^b |g\circ\theta(t)|dt\right)\right) \\ &\leq \frac{h(b)-h(a)}{(b-a)\Phi(h(b)-h(a))||h'||_\infty} \\ &\qquad \times \int_a^b \phi\left(g\left(\frac{(b-a)\Phi(h(b)-h(a))||h'||_\infty}{h(b)-h(a)}|g\circ\theta(t)|\right)\right)dt. \end{aligned} \tag{11}$$

Proof. For $x \in [a,b]$, we define the following kernel:

$$K(x,t) = \begin{cases} \dfrac{\Phi(h(x)-h(t))}{h(x)-h(t)}h'(t), & a \leq t \leq x, \\ 0, & x < t \leq b. \end{cases}$$

Also define the function η by

$$\eta(x) = F_{a^+}^{\Phi,h}\theta(x) = \int_a^x \frac{\Phi(h(x)-h(t))}{h(x)-h(t)}h'(t)\theta(t)dt. \tag{12}$$

Then we have

$$|K(x,t)| \leq \frac{\Phi(h(x)-h(t))}{h(x)-h(t)}||h'||_\infty. \tag{13}$$

As $\frac{\Phi}{x}$ and h are increasing on $[a,b]$, therefore we have

$$|K(x,t)| \leq \frac{\Phi(h(b)-h(a))}{h(b)-h(a)}||h'||_\infty = M. \tag{14}$$

Hence applying Theorem 1.3 we get inequalities in (11). □

For right sided generalized operator following result holds.

Theorem 2.2. *Let $\phi,\ g : [0,\infty) \to \mathbb{R}$ be the functions with assumptions of Theorem 1.3. Also let Φ be positive and $\frac{\Phi}{x}$, h are increasing functions on $[0,\infty)$. If $\theta \in L_1[a,b]$, $0 \leq a < b$ and $h' \in L_\infty[a,b]$, then the following inequalities hold:*

$$\begin{aligned}
&\int_a^b \phi'\left(g\left(\left|F_{b_-}^{\Phi,h}\theta(t)\right|\right)\right) g'\left(\left|F_{b_-}^{\Phi,h}\theta(t)\right|\right)|g\circ\theta(t)|dt \\
&\leq \frac{h(b)-h(a)}{\Phi(h(b)-h(a))||h'||_\infty}\phi\left(g\left(\frac{\Phi(h(b)-h(a))||h'||_\infty}{h(b)-h(a)}\int_a^b |g\circ\theta(t)|dt\right)\right) \\
&\leq \frac{h(b)-h(a)}{(b-a)\Phi(h(b)-h(a))||h'||_\infty} \\
&\quad\times\int_a^b \phi\left(g\left(\frac{(b-a)\Phi(h(b)-h(a))||h'||_\infty}{h(b)-h(a)}|g\circ\theta(t)|\right)\right)dt.
\end{aligned} \tag{15}$$

Proof. The proof is similar to the proof of Theorem 2.1 with kernel

$$K(x,t) = \begin{cases} 0, & a \leq t \leq x, \\ \dfrac{\Phi(h(t)-h(x))}{h(t)-h(x)}h'(t), & x < t \leq b. \end{cases}$$

□

Next we present the results for power function.

Theorem 2.3. *Under the assumptions of Theorem 2.1. If $q > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\int_a^b \phi'\left(\left|F_{a^+}^{\Phi,h}\theta(t)\right|^q\right)\left|F_{a^+}^{\Phi,h}\theta(t)\right|^{q-1}|\theta(t)|^q dt \\
&\leq \frac{h(b)-h(a)}{\Phi(h(b)-h(a))||h'||_\infty q}\phi\left(\left(\frac{\Phi(h(b)-h(a))||h'||_\infty}{h(b)-h(a)}\int_a^b|\theta(t)|^q dt\right)^q\right) \\
&\leq \frac{h(b)-h(a)}{q(b-a)\Phi(h(b)-h(a))||h'||_\infty}\int_a^b \phi\left(\left(\frac{(b-a)\Phi(h(b)-h(a))||h'||_\infty}{h(b)-h(a)}|\theta(t)|^q\right)^q\right)dt.
\end{aligned} \tag{16}$$

Proof. Let $g(t) = t^q$ in Theorem 2.1. Then g is convex and increasing for $q > 1$. Therefore using this power function g in inequalities (11) we get inequalities in (16). □

For right sided generalized operator following result holds.

Theorem 2.4. *Under the assumptions of Theorem 2.1. If $q > 1$, then the following inequalities hold:*

(17)
$$\int_a^b \phi'\left(\left|F_{b_-}^{\Phi,h}\theta(t)\right|^q\right)\left|F_{b_-}^{\Phi,h}\theta(t)\right|^{q-1}|\theta(t)|^q dt$$
$$\leq \frac{h(b)-h(a)}{\Phi(h(b)-h(a))||h'||_\infty q}\phi\left(\left(\frac{\Phi(h(b)-h(a))||h'||_\infty}{h(b)-h(a)}\int_a^b|\theta(t)|^q dt\right)^q\right)$$
$$\leq \frac{h(b)-h(a)}{(b-a)\Phi(h(b)-h(a))||h'||_\infty q}\int_a^b\phi\left(\left(\frac{(b-a)\Phi(h(b)-h(a))||h'||_\infty}{h(b)-h(a)}|\theta(t)|^q\right)^q\right)dt.$$

Proof. The proof is similar to the proof of Theorem 2.3. □

3. Opial-type inequalities for fractional integral operators

In this section we present different fractional versions of results proved in Section 2.

Theorem 3.1. *Let $\phi,\ g:[0,\infty)\to\mathbb{R}$ be functions with assumption of Theorem 1.3. Also let $\theta\in L_1[a,b]$, $0\leq a<b$, $\alpha\geq k$, $k>0$ and h be an increasing and positive function on $(a,b]$, having a continuous derivative on (a,b). Then the following inequalities hold:*

$$(18)\quad \int_a^b \phi'\left(g\left(\left|I_{h,a^+}^{\alpha,k}\theta(t)\right|\right)\right)g'\left(\left|I_{h,a^+}^{\alpha,k}\theta(t)\right|\right)|g\circ\theta(t)|dt$$
$$\leq \frac{k\Gamma_k(\alpha)}{(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}\phi\left(g\left(\frac{(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}{k\Gamma_k(\alpha)}\int_a^b|g\circ\theta(t)|dt\right)\right)$$
$$\leq \frac{k\Gamma_k(\alpha)}{(b-a)(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}$$
$$\times\int_a^b\phi\left(g\left(\frac{(b-a)(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}{k\Gamma_k(\alpha)}|g\circ\theta(t)|\right)\right)dt.$$

Proof. If we put $\Phi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.1, then required result is obtained. □

Remark 3.2. *In the above Theorem 3.1.*
(ii) If $k=1$ and $h(x)=x$, then we get [11, Theorem 3.1].

For right sided generalized Riemann-Liouville fractional integral we have the following result.

Theorem 3.3. *Let $\phi,\ g:[0,\infty)\to\mathbb{R}$ be functions with assumptions of Theorem 1.3. Also let $\theta\in L_1[a,b]$, $0\leq a<b$, $\alpha\geq k$, $k>0$ and h be an increasing and positive function on $(a,b]$, having a continuous derivative on (a,b). Then the following* *inequalities hold:*

$$\begin{aligned}
(19)\qquad & \int_a^b \phi'\left(g\left(\left|I_{h,b_-}^{\alpha,k}\theta(t)\right|\right)\right) g'\left(\left|I_{h,b_-}^{\alpha,k}\theta(t)\right|\right)|g\circ\theta(t)|dt \\
&\leq \frac{k\Gamma_k(\alpha)}{(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}\phi\left(g\left(\frac{(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}{k\Gamma_k(\alpha)}\int_a^b|g\circ\theta(t)|dt\right)\right) \\
&\leq \frac{k\Gamma_k(\alpha)}{(b-a)(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty} \\
&\qquad \times\int_a^b\phi\left(g\left(\frac{(b-a)(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}{k\Gamma_k(\alpha)}|g\circ\theta(t)|\right)\right)dt.
\end{aligned}$$

Proof. If we put $\Phi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 2.2, then required result is obtained. □

REMARK 3.4. *In the above Theorem 3.3.*
(ii) If $k=1$ and $h(x)=x$, then we get [11, Theorem 3.2].

THEOREM 3.5. *Under the assumptions of Theorem 3.1. If $q>1$, then the following inequalities hold:*

$$\begin{aligned}
(20)\qquad & \int_a^b \phi'\left(\left|I_{h,a^+}^{\alpha,k}\theta(t)\right|^q\right)\left|I_{h,a^+}^{\alpha,k}\theta(t)\right|^{q-1}|\theta(t)|^q dt \\
&\leq \frac{k\Gamma_k(\alpha)}{(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty q}\phi\left(\left(\frac{(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}{k\Gamma_k(\alpha)}\int_a^b|\theta(t)|^q dt\right)^q\right) \\
&\leq \frac{k\Gamma_k(\alpha)}{(b-a)(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty q} \\
&\qquad \times\int_a^b\phi\left(\left(\frac{(b-a)(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}{k\Gamma_k(\alpha)}|\theta(t)|^q\right)^q\right)dt.
\end{aligned}$$

Proof. Let $g(t)=t^q$ in Theorem 3.1. Then g is convex and increasing for $q>1$. Therefore using this power function g in inequalities (18) we get inequalities in (20). □

REMARK 3.6. *In the above Theorem 3.5.*
(ii) If $k=1$ and $h(x)=x$, then we get [11, Theorem 3.3].

THEOREM 3.7. *Under the assumptions of Theorem 3.1. If $q>1$, then the following inequalities hold:*

$$\begin{aligned}
(21)\qquad & \int_a^b \phi'\left(\left|I_{h,b_-}^{\alpha,k}\theta(t)\right|^q\right)\left|I_{h,b_-}^{\alpha,k}\theta(t)\right|^{q-1}|\theta(t)|^q dt \\
&\leq \frac{k\Gamma_k(\alpha)}{(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty q}\phi\left(\left(\frac{(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}{k\Gamma_k(\alpha)}\int_a^b|\theta(t)|^q dt\right)^q\right) \\
&\leq \frac{k\Gamma_k(\alpha)}{(b-a)(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty q} \\
&\qquad \times\int_a^b\phi\left(\left(\frac{(b-a)(h(b)-h(a))^{\frac{\alpha}{k}-1}||h'||_\infty}{k\Gamma_k(\alpha)}|\theta(t)|^q\right)^q\right)dt.
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.5, with kernel

$$K(x,t)=\begin{cases} 0, & a\leq t\leq x,\\ \frac{1}{k\Gamma_k(\alpha)}(h(t)-h(x))^{\frac{\alpha}{k}-1}h'(t), & x<t\leq b.\end{cases}$$

□

REMARK 3.8. *In the above Theorem 3.7.*
(ii) If $k=1$ and $h(x)=x$, then we get [11, Theorem 3.4].

4. generalized mean value theorems

In this section we present several mean value theorems for generalized, fractional and conformable integral operators.

THEOREM 4.1. *Let $\phi,\ g:[0,\infty)\to\mathbb{R}$ be functions with assumptions of Theorem 1.5. If $\phi\in C^2(I)$, where $I\subseteq\mathbb{R}_+$ is compact interval. Also let Φ be positive and $\frac{\Phi}{x}$, h are increasing functions on $[0,\infty)$. If $\theta\in L_1[a,b]$, $0\leq a<b$ and $h'\in L_\infty[a,b]$, then there exists an $\xi_i\in I$, such that the following result holds:*

$$ {}_g\mathbb{F}_i^{\phi}(F_{a^+}^{\Phi,h}\theta,\theta;M)=\frac{\phi''(\xi_i)}{2}\ {}_g\mathbb{F}_i^{x^2}(F_{a^+}^{\Phi,h}\theta,\theta;M),\ i=1,2, \tag{22}$$

where $M=\dfrac{\Phi(h(b)-h(a))}{h(b)-h(a)}||h'||_\infty.$

Proof. It can easily be proved by using function η defined in (12) and M calculated in (14) in Theorem 1.5. □

THEOREM 4.2. *Let ϕ_1,ϕ_2 and g be the functions with assumptions of Theorem 1.5. If $\phi_1,\phi_2\in C^2(I)$, where $I\subseteq\mathbb{R}_+$ is compact interval. Also let h be an increasing function and $h'\in L_\infty[a,b]$. If ${}_g\mathbb{F}_i^{x^2}(F_{a^+}^{\Phi,h}\theta,\theta;M)\neq 0,\ i=1,2$, then there exists an $\xi_i\in I$, such that we have*

$$\frac{{}_g\mathbb{F}_i^{\phi_1}(F_{a^+}^{\Phi,h}\theta,\theta;M)}{{}_g\mathbb{F}_i^{\phi_2}(F_{a^+}^{\Phi,h}\theta,\theta;M)}=\frac{\phi_1''(\xi_i)}{\phi_2''(\xi_i)},\ i=1,2, \tag{23}$$

where $M=\dfrac{\Phi(h(b)-h(a))}{h(b)-h(a)}||h'||_\infty$. *Provided the denominators are not zero.*

Proof. It follows easily for the η defined by (12) and Theorem 1.6. □

THEOREM 4.3. *Let $\phi,\ g:[0,\infty)\to\mathbb{R}$ be functions with assumptions of Theorem 1.5. If $\phi\in C^2(I)$, where $I\subseteq\mathbb{R}_+$ is compact interval. Also let h be an increasing function and $h'\in L_\infty[a,b]$ has fractional integral of order α and $k>0$. If $\alpha>k$, then there exists an $\xi_i\in I$, such that the following result holds:*

$$ {}_g\mathbb{F}_i^{\phi}(I_{h,a+}^{\alpha,k}\theta,\theta;M)=\frac{\phi''(\xi_i)}{2}\ {}_g\mathbb{F}_i^{x^2}(I_{h,a+}^{\alpha,k}\theta,\theta;M),\ i=1,2, \tag{24}$$

where $M=\dfrac{(h(b)-h(a))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}||h'||_\infty.$

Proof. It can easily be proved by taking $\Phi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 4.1. □

REMARK 4.4. *In the above Theorem 4.3.*
(ii) If $k = 1$ and $h(x) = x$, then we get [13, Theorem 3.1].

THEOREM 4.5. *Let ϕ_1, ϕ_2 and g be the functions with assumptions of Theorem 1.5. If $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is compact interval. Also let h be an increasing function and $h' \in L_\infty[a,b]$. If $\alpha > k$, where $k > 0$ and ${}_g\mathbb{F}_i^{x^2}(I_{h,a+}^{\alpha,k}\theta, \theta; M) \neq 0,\ i = 1,2,$ then there exists an $\xi_i \in I$, such that we have*

$$\frac{{}_g\mathbb{F}_i^{\phi_1}(I_{h,a+}^{\alpha,k}\theta, \theta; M)}{{}_g\mathbb{F}_i^{\phi_2}(I_{h,a+}^{\alpha,k}\theta, \theta; M)} = \frac{\phi_1''(\xi_i)}{\phi_2''(\xi_i)},\ i = 1,2, \tag{25}$$

where $M = \dfrac{(h(b)-h(a))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}||h'||_\infty$. Provided the denominators are not zero.

Proof. It can easily be proved by taking $\Phi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 4.2. □

REMARK 4.6. *In the above Theorem 4.5.*
(ii) If $k = 1$ and $h(x) = x$, then we get [13, Theorem 3.2].

THEOREM 4.7. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a function with assumptions of Theorem 1.7. Further, let ϕ' is increasing function and $\phi \in C^2(I)$, where $I \subset \mathbb{R}_+$ is a compact interval. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > \frac{k}{q}$, where $k > 0$. If $\theta \in L_1[a,b]$ and $h' \in L_\infty[a,b]$, then there exists an $\xi_i \in I$ such that the following result holds*

$$\mathbb{F}_i^{\phi}(I_{h,a^+}^{\alpha,k}\theta, \theta, M) = \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{2q^2(\xi^{2q-1})}\mathbb{F}_i^{x^2}(I_{h,a^+}^{\alpha,k}\theta, \theta, M), \tag{26}$$

where $M = \dfrac{||h'||_\infty^{\frac{1}{q}}(h(b)-h(a))^{\frac{\alpha q - k}{kq}}}{k\Gamma_k(\alpha)\left[p\left(\frac{\alpha q-k}{kq}\right)\right]^{\frac{1}{p}}}$.

Proof. Let us define for $x \in [a,b]$, the kernel $K(x,t)$ as

$$K(x,t) = \begin{cases} \frac{1}{k\Gamma_k(\alpha)}(h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t), & a \le t \le x, \\ 0, & x < t \le b. \end{cases}$$

Also if η is defined by

$$\eta(x) = I_{h,a^+}^{\alpha,k}\theta(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (h(x)-h(t))^{\frac{\alpha}{k}-1}h'(t)\theta(t)dt. \tag{27}$$

$$P(x) = \left(\int_a^x (K(x,t))^p dt\right)^{\frac{1}{p}} = \frac{1}{k\Gamma_k(\alpha)}\left(\int_a^x ((h(x)-h(t))^{\alpha-1}h'(t))^p dt\right)^{\frac{1}{p}}. \tag{28}$$

As $h' \in L_\infty[a,b]$, then we have

$$\begin{aligned} P(x) &< -\frac{||h'||_\infty^{\frac{1}{q}}}{k\Gamma_k(\alpha)}\left(\int_a^x (h(x)-h(t))^{p(\frac{\alpha-k}{k})}(-h'(t))dt\right)^{\frac{1}{p}} \\ &= \frac{||h'||_\infty^{\frac{1}{q}}(h(x)-h(a))^{\frac{\alpha q-k}{kq}}}{k\Gamma_k(\alpha)\left[p\left(\frac{\alpha q-k}{kq}\right)\right]^{\frac{1}{p}}}. \end{aligned} \tag{29}$$

It is easy to see that for $\alpha > \frac{k}{q}$ the function P is increasing on $[a,b]$, thus

$$P(x) \leq \frac{||h'||_{\infty}^{\frac{1}{q}}(h(b)-h(a))^{\frac{\alpha q-k}{kq}}}{k\Gamma_k(\alpha)\left[p\left(\frac{\alpha q-k}{kq}\right)\right]^{\frac{1}{p}}} = M. \tag{30}$$

Hence $\left(\int_a^x (K(x,t))^p dt\right)^{\frac{1}{p}} \leq M$, which with the function η defined in (27) and Theorem 1.7 gives us (26). □

REMARK 4.8. *In the above Theorem 4.7.*

(i) If $i,k=1$ and $h(x)=x$, then we get [1, Theorem 4.1].
(ii) If $i=2, k=1$ and $h(x)=x$, then we get [12, Theorem 6].

THEOREM 4.9. *Let ϕ_1, ϕ_2 and g be the functions with assumptions of Theorem 1.8. If $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is compact interval. Let $\frac{1}{p}+\frac{1}{q}=1$, $\alpha > \frac{k}{q}$, where $k>0$ and $\mathbb{F}_i^{x^2}(I_{h,a^+}^{\alpha,k}\theta,\theta,M) \neq 0, i=1,2$. If $\theta \in L_1[a,b]$ and $h' \in L_\infty[a,b]$, then there exists an $\xi \in I$ such that the following result holds*

$$\frac{\mathbb{F}_i^{\phi_1}(I_{h,a^+}^{\alpha,k}\theta,\theta,M)}{\mathbb{F}_i^{\phi_2}(I_{h,a^+}^{\alpha,k}\theta,\theta,M)} = \frac{\xi\phi_1''(\xi)-(q-1)\phi_1'(\xi)}{\xi\phi_2''(\xi)-(q-1)\phi_2'(\xi)},$$

where $M = \dfrac{||h'||_{\infty}^{\frac{1}{q}}(h(b)-h(a))^{\frac{\alpha q-k}{kq}}}{k\Gamma_k(\alpha)\left[p\left(\frac{\alpha q-k}{kq}\right)\right]^{\frac{1}{p}}}$. Provided the denominators are not zero.

Proof. It follows easily for the η defined by (27) and Theorem 1.8. □

REMARK 4.10. *In the above Theorem 4.9.*

(i) If $i,k=1$ and $h(x)=x$, then we get [1, Theorem 4.2].
(ii) If $i=2, k=1$ and $h(x)=x$, then we get [12, Theorem 7].

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