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Theory of Light Scattering by a Circular Cylinder over a Planar Substrate: Normal Incidence

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The problem of polarized light scattering by a cylinder on or close to a planar substrate is analytically solved. The light is assumed to be normally incident to the axis of the cylinder. Transverse magnetic (TM) and transverse electric (TE) polarizations are treated separately. The solution for each polarization is composed of a coupled set of linear equations which couples the scattering characteristics of the cylinder and the planar substrate. The coupling comes from the scattering by the planar substrate and by the cylinder. The solution of the coupled set of equations obtained by iterative substitution consists of infinite series, where each term represents the contribution of single and multiple scatterings of all orders.

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I. INTRODUCTION

Light scattering by an object in the presence of a planar substrate is encountered in various situations, and proper theoretical solutions for these problems would be of great value in many possible applications. Accordingly, research dealing with these problems has been reported in many articles [1–13]. The presence of two scattering objects inevitably induces multiple scattering, and its effect is especially important when the objects are close to each other. Theoretical solutions for the problem should elucidate such multiple scattering effects systematically through the corresponding terms extended up to higher orders.

Given the importance of the problem, theoretical analysis of light scattering by a cylinder or cylinders in the presence of a planar substrate has been attempted under various restrictive conditions, such as using a perfectly conducting cylinder [7, 8], a perfectly reflecting plane [6, 10], or inclusion of multiple scattering terms, only up to second order, ignoring higher-order terms [8–13]. Thus, it is desirable to develop a more general theory with no condition on the properties of the scattering objects to include all possible situations, at least for simple geometries.

Recently, analytic solutions to the problem of scalar wave scattering by a cylinder and a planar substrate [14] and by a sphere and a planar substrate [15], which do not impose any restriction on the properties of the scattering objects, have been reported. Light is a transverse vector wave with two independent polarization components, and thus vector theory should be properly employed for a rigorous treatment.

This study analyzes the scattering of polarized light by a cylinder on or close to a planar substrate to obtain a rigorous analytic solution. For mathematical simplicity, we limit the direction of propagation of the incident light to be normal to the cylinder, but not necessarily to the surface of the substrate, where the transverse magnetic (TM) and transverse electric (TE) polarizations are separately treated. The

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solutions reported in this paper apply to any combination of cylinder and planar substrate, no matter whether they are dielectric or conducting, so long as they are composed of linear and isotropic media.

II. THEORY

Figure 1 shows the schematic of the problem in a cross-sectional view: the scattering of light by a circular cylinder and a planar substrate. The substrate fills the right half space. To the left is air and a cylinder of radius a, with its axis placed parallel to and at a distance d ($\geq a$) from the substrate. The electric susceptibilities and magnetic permeabilities of air, the cylinder, and substrate are ϵ_0, and μ_0, ϵ_c and μ_c, and ϵ_p and μ_p, respectively. The axis of the cylinder is taken as the z-axis, and the plane perpendicular to the axis as the xy-plane (see Fig. 1).

For notational simplicity, the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ of light are denoted by a single symbol $\boldsymbol{F}$ in some sections of this paper. The subscripts i, r, and σ of the field vectors in air respectively denote the incident light, light reflected from the substrate, and light scattered by the cylinder; the subscripts t and, τ respectively denote the light transmitted to the substrate and the cylinder.

The incident light field $\boldsymbol{F}_i$ is assumed to be of angular frequency ω and is a superposition of plane waves coming from the left, with the wave vector $\boldsymbol{k}$ normal to the cylinder axis. Thus, this problem is reduced to a two-dimensional one. In air, the field $\boldsymbol{F}_\sigma$ is scattered by the cylinder and the field $\boldsymbol{F}_r$ reflected from the planar substrate, while the transmitted fields $\boldsymbol{F}_\tau$ and $\boldsymbol{F}_t$ are in the cylinder and substrate, respectively. Thus, the whole field $\boldsymbol{F}$ can be represented as follows:

$$\boldsymbol{F} = \begin{cases} \boldsymbol{F}_i + \boldsymbol{F}_\sigma + \boldsymbol{F}_r & \textit{in air} \\ \boldsymbol{F}_\tau & \textit{in the cylinder} \\ \boldsymbol{F}_t & \textit{in the substrate} \end{cases}. \tag{1}$$

These fields in the three regions are the solutions to the following Helmholtz Eqs. [16]:

$$(\nabla^2 + k^2)\,\boldsymbol{F}_{i,\sigma,r} = 0, \tag{2a}$$

$$(\nabla^2 + n_c^2 k^2)\,\boldsymbol{F}_\tau = 0, \tag{2b}$$

$$(\nabla^2 + n_p^2 k^2)\,\boldsymbol{F}_t = 0, \tag{2c}$$

where all the fields are assumed to have the time-harmonic factor $e^{-i\omega t}$, which will be omitted throughout the paper. k is the angular wave number in air and is related to the propagation speed v of the light in air and the angular frequency ω through $k = \omega/v$, where $n_c = \sqrt{\epsilon_c\mu_c/\epsilon_0\mu_0}$ and $n_p = \sqrt{\epsilon_p\mu_p/\epsilon_0\mu_0}$ respectively are the refractive indices of the cylinder and the planar substrate, relative to air. At the boundaries, the tangential components of the electric and magnetic fields are continuous.

To find the analytic solutions of Eqs. (2a)–(2c) that satisfy the boundary conditions, the fields need to be expressed in coordinates that geometrically match the boundaries. Because the boundaries are an infinite plane and an infinite circular cylindrical surface, the proper coordinate systems are rectangular and cylindrical respectively.

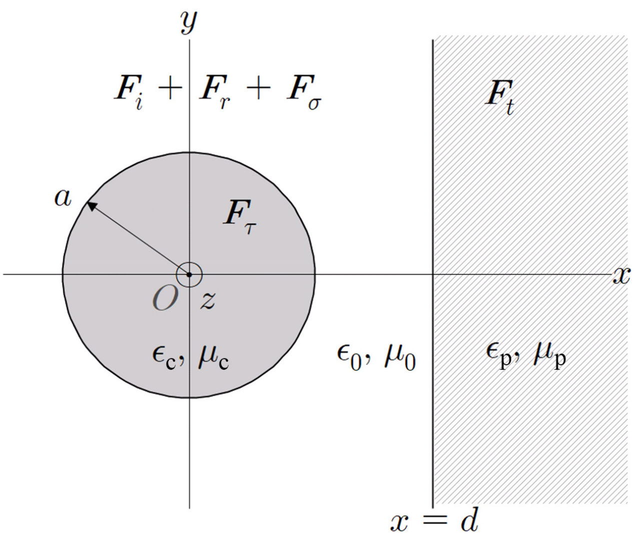

FIG. 1. Light scattering by a circular cylinder over a planar substrate. The cylinder has a radius of a and its axis coincides with the z-axis. The surface of the substrate is located at $x = d$, parallel to the yz-plane. The electric susceptibilities and magnetic permeabilities of air, the cylinder, and the substrate are ϵ_0 and μ_0, ϵ_c and μ_c, and ϵ_p and μ_p, respectively. The field vector $\boldsymbol{F}$ denotes either of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$.

2.1. TM Polarization

When the incident light is TM polarized, its magnetic field $\boldsymbol{H}_i$ can be written in terms of plane waves as follows [16]:

$$\boldsymbol{H}_i = \frac{1}{iZ_0}\int_{-\infty}^{\infty} dk_y\, a(k_y)\, e^{i\left(k_y y + \sqrt{k^2 - k_y^2}\,x\right)}\hat{\mathbf{z}}. \tag{3}$$

Here, $a(k_y)$ is the angular spectrum of the incident light and $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of air.

The magnetic fields $\boldsymbol{H}_r$ and $\boldsymbol{H}_t$ of the light reflected from the surface of the substrate ($x = d$) and transmitted to the substrate can be written:

$$\boldsymbol{H}_r = \frac{1}{iZ_0}\int_{-\infty}^{\infty} dk_y\, r_{\mathrm{TM}}(k_y)\, e^{i\left[k_y y - \sqrt{k^2 - k_y^2}(x-d)\right]}\hat{\mathbf{z}}, \tag{4}$$

$$\boldsymbol{H}_t = \frac{1}{iZ_p}\int_{-\infty}^{\infty} dk_y\, t_{\mathrm{TM}}(k_y)\, e^{i\left[k_y y + \sqrt{n_p^2 k^2 - k_y^2}(x-d)\right]}\hat{\mathbf{z}}, \tag{5}$$

where $r_{\mathrm{TM}}(k_y)$ and $t_{\mathrm{TM}}(k_y)$ are respectively the angular spectra of the reflected and transmitted light. $Z_p = \sqrt{\mu_p/\epsilon_p}$ is the impedance of the substrate material.

The magnetic fields $\boldsymbol{H}_\sigma$ and $\boldsymbol{H}_\tau$, which are scattered by and transmitted into the cylinder respectively, can be written:

$$\boldsymbol{H}_\sigma = \frac{1}{iZ_0}\sum_{m=-\infty}^{\infty} \sigma_{\mathrm{TM},m} H_m^{(1)}(k\rho)e^{im\varphi}\hat{\mathbf{z}}, \quad (6)$$

$$\boldsymbol{H}_\tau = \frac{1}{iZ_c}\sum_{m=-\infty}^{\infty} \tau_{\mathrm{TM},m} J_m(n_c k\rho)e^{im\varphi}\hat{\mathbf{z}}, \quad (7)$$

where $H_m^{(1)}(x)$ and $J_m(x)$ are respectively the m-th order Hankel functions of the first kind and the m^{th} order Bessel function. ρ and ϕ are respectively the radius and azimuthal angle in the cylindrical coordinates. $Z_c = \sqrt{\mu_c/\epsilon_c}$ is the impedance of the cylinder material. $\sigma_{\mathrm{TM},m}$ and $\tau_{\mathrm{TM},m}$ are respectively the scattering and transmission coefficients of the m-th-order cylindrical wave of TM polarization.

The electric fields matched with each of the magnetic fields above can be obtained by using the Ampere-Maxwell's law for harmonic fields, $\boldsymbol{E} = (iZ/k)\nabla \times \boldsymbol{H}$ in their respective regions. Now, the problem is reduced to finding expressions for $r_{\mathrm{TM}}(k_y)$, $t_{\mathrm{TM}}(k_y)$, $\sigma_{\mathrm{TM},m}$ and $\tau_{\mathrm{TM},m}$.

2.1.1. Boundary conditions

The boundary condition dictates that the tangential components of the electric and magnetic fields should be continuous at the interfaces. Thus, at the cylinder's surface ($\rho = a$), the ϕ-component of the electric fields and the z-component of the magnetic fields should be continuous:

$$\left[(\boldsymbol{F}_i + \boldsymbol{F}_\sigma + \boldsymbol{F}_r)_{\phi,z}\right]_{\rho=a} = \left[(\boldsymbol{F}_\tau)_{\phi,z}\right]_{\rho=a}. \quad (8)$$

In addition, at the surface of the planar substrate ($x = d$), the y-component of the electric fields and the z-component of the magnetic fields should be continuous:

$$\left[(\boldsymbol{F}_i + \boldsymbol{F}_\sigma + \boldsymbol{F}_r)_{y,z}\right]_{x=d} = \left[(\boldsymbol{F}_t)_{y,z}\right]_{x=d}. \quad (9)$$

To apply the boundary conditions, Eq. (8) should be written in terms of cylindrical waves, and Eq. (9) in terms of plane waves. Therefore, two types of expansions are required: one that expands a vector plane wave in terms of vector cylindrical waves and the other a vector cylindrical wave in terms of vector plane waves.

2.1.2. Transformation from cylindrical to plane waves and vice versa

A divergent cylindrical wave function $H_m^{(1)}(k\rho)e^{im\varphi}$ can be expanded in terms of plane waves as follows [17, 18]:

$$H_m^{(1)}(k\rho)e^{im\varphi} = \begin{cases} \int_{-\infty}^{\infty} dk_y\ F_{m+}(k_y)\, e^{i\left(k_y y + \sqrt{k^2-k_y^2}x\right)}, & x \ge 0, \\ \int_{-\infty}^{\infty} dk_y\ F_{m-}(k_y)\, e^{i\left(k_y y - \sqrt{k^2-k_y^2}x\right)}, & x < 0, \end{cases} \quad (10)$$

where the expansion functions $F_{m\pm}(k_y)$ are

$$F_{m\pm}(k_y) = \frac{1}{\pi\sqrt{k^2-k_y^2}} \exp\left[\pm im \tan^{-1}\left(\frac{k_y}{\sqrt{k^2-k_y^2}}\right)\right]. \quad (11)$$

Therefore, the light scattered by the cylinder can be expanded in terms of plane waves as follows:

$$\boldsymbol{H}_\sigma = \frac{1}{iZ_0}\sum_{m=-\infty}^{\infty} \sigma_{\mathrm{TM},m} \int_{-\infty}^{\infty} dk_y\ F_{m\pm}(k_y)\, e^{i\left(k_y y \pm \sqrt{k^2-k_y^2}x\right)}\hat{\mathbf{z}}. \quad (12)$$

In the opposite direction, the plane waves $e^{i\left(k_y y \pm \sqrt{k^2-k_y^2}x\right)}$ can be expanded in terms of cylindrical waves $J_m(k\rho)e^{im\varphi}$ as follows [19]:

$$e^{i\left(k_y y \pm \sqrt{k^2-k_y^2}x\right)} = \sum_{m=-\infty}^{\infty} e^{i\Theta_{m\pm}(k_y)} J_m(k\rho)e^{im\varphi}, \quad (13)$$

where $\Theta_m \pm (k_y)$ is given by

$$\Theta_{m\pm}(k_y) = m\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{k_y}{\pm\sqrt{k^2-k_y^2}}\right)\right]. \quad (14)$$

Therefore, the magnetic fields of the incident and reflected light can be expanded in terms of vector cylindrical waves as follows:

$$\boldsymbol{H}_i = \frac{1}{iZ_0}\sum_{m=-\infty}^{\infty} \alpha_m J_m(k\rho)e^{im\varphi}\,\hat{\mathbf{z}}, \quad (15)$$

$$\boldsymbol{H}_r = \frac{1}{iZ_0}\sum_{m=-\infty}^{\infty} \beta_m J_m(k\rho)e^{im\varphi}\,\hat{\mathbf{z}}, \quad (16)$$

where the expansion coefficients α_m and β_m are

$$\alpha_m = \int_{-\infty}^{\infty} dk_y\, a(k_y)e^{i\Theta_{m+}(k_y)}, \quad (17)$$

$$\beta_m = \int_{-\infty}^{\infty} dk_y\, r_{\mathrm{TM}}(k_y)e^{i[\Theta_{m-}(k_y) + \vartheta(k_y)]}. \quad (18)$$

Here, $\vartheta(k_y)$ is the phase change of the plane wave $e^{i\left(k_y y + \sqrt{k^2-k_y^2}x\right)}$ when it goes a distance d along the x-axis,

$$\vartheta(k_y) = \sqrt{k^2-k_y^2}\,d. \quad (19)$$

2.1.3. Application of a boundary condition at the cylindrical surface

For notational simplicity, the range of the summation index and the range of integration is omitted hereinafter. From the boundary condition at the cylinder surface, Eq. (8), and the orthogonality of the harmonic functions, the following relations are obtained:

$$(\alpha_m + \beta_m) J_m(ka) + \sigma_{TM,m} H_m^{(1)}(ka) = \frac{Z_0}{Z_c} \tau_{TM,m} J_m(n_c ka), \quad (20a)$$

$$(\alpha_m + \beta_m) J'_m(ka) + \sigma_{TM,m} H_m^{(1)\prime}(ka) = n_c \frac{Z_0}{Z_c} \tau_{TM,m} J'_m(n_c ka), \quad (20b)$$

where the prime indicates a derivative with respect to the argument in parentheses. Eqs. (20a) and (20b) can be solved to find $\sigma_{TM,m}$ and $\tau_{TM,m}$. Then substituting Eq. (18) for β_m, the following result is obtained:

$$\sigma_{TM,m} = \sigma_{0TM,m}\left\{\alpha_m + \int dk_y\, r_{TM}(k_y) e^{i[\Theta_{m-}(k_y) + \vartheta(k_y)]}\right\}, \quad (21a)$$

$$\tau_{TM,m} = \tau_{0TM,m}\left\{\alpha_m + \int dk_y\, r_{TM}(k_y) e^{i[\Theta_{m-}(k_y) + \vartheta(k_y)]}\right\}, \quad (21b)$$

where $\sigma_{0TM,m}$ and $\tau_{0TM,m}$ represent the following quantities:

$$\sigma_{0TM,m} = -\frac{J'_m(n_c ka) J_m(ka) - (Z_0/Z_c) J_m(n_c ka) J'_m(ka)}{J'_m(n_c ka) H_m^{(1)}(ka) - (Z_0/Z_c) J_m(n_c ka) H_m^{(1)\prime}(ka)}, \quad (22a)$$

$$\tau_{0TM,m} = \frac{J'_m(ka) H_m^{(1)}(ka) - J_m(ka) H_m^{(1)\prime}(ka)}{J'_m(n_c ka) H_m^{(1)}(ka) - (Z_0/Z_c) J_m(n_c ka) H_m^{(1)\prime}(ka)}. \quad (22b)$$

These are the coefficients of scattering and transmission of the TM-polarized vector cylindrical wave of m^{th} order when the cylinder is by itself [20].

2.1.4. Application of a boundary condition at the planar surface

From the boundary condition at the substrate surface, Eq. (9), and the orthogonality of the plane waves, the following equations are obtained:

$$e^{i\vartheta(k_y)}\left[a(k_y) + \sum \sigma_{TM,m} F_{m+}(k_y)\right] + r_{TM}(k_y) = \frac{Z_0}{Z_p} t_{TM}(k_y), \quad (23a)$$

$$e^{i\vartheta(k_y)}\left[a(k_y) + \sum \sigma_{TM,m} F_{m+}(k_y)\right] - r_{TM}(k_y) = \frac{Z_0}{Z_p} \frac{\sqrt{n_p^2 k^2 - k_y^2}}{\sqrt{k^2 - k_y^2}} t_{TM}(k_y). \quad (23b)$$

Equations (23a) and (23b) can be solved to find $r_{TM}(k_y)$ and $t_{TM}(k_y)$ and the result is

$$r_{TM}(k_y) = r_{0TM}(k_y)\left[a(k_y) + \sum \sigma_{TM,m} F_{m+}(k_y)\right] e^{i\vartheta(k_y)}, \quad (24a)$$

$$t_{TM}(k_y) = t_{0TM}(k_y)\left[a(k_y) + \sum \sigma_{TM,m} F_{m+}(k_y)\right] e^{i\vartheta(k_y)}, \quad (24b)$$

where $r_{0TM}(k_y)$ and $t_{0TM}(k_y)$ are the abbreviated symbols for the following quantities:

$$r_{0TM}(k_y) = \frac{n_p(Z_0/Z_p) - \sqrt{n_p^2 k^2 - k_y^2}\Big/\sqrt{k^2 - k_y^2}}{n_p(Z_0/Z_p) + \sqrt{n_p^2 k^2 - k_y^2}\Big/\sqrt{k^2 - k_y^2}}, \quad (25a)$$

$$t_{0TM}(k_y) = \frac{2n_p}{n_p(Z_0/Z_p) + \sqrt{n_p^2 k^2 - k_y^2}\Big/\sqrt{k^2 - k_y^2}}. \quad (25b)$$

These are the reflection and transmission coefficients of the incident plane wave of TM polarization at a planar interface [21].

2.1.5. Interpretation of higher-order terms

Equations (21a), (21b), (24a), and (24b) are the basic expressions that comprise the solution. $\sigma_{TM,m}$ and $r_{TM}(k_y)$ are mutually coupled through Eqs. (21a) and (24a), respectively. They can be separated by substituting Eq. (24a) for $r_{TM}(k_y)$ in Eq. (21a), and Eq. (21a) for $\sigma_{TM,m}$ in Eq. (24a).

Equations (21a) and (21b) are changed to the following form when Eq. (24a) is substituted for $r_{TM}(k_y)$:

$$\sigma_{TM,m} = \sigma_{0TM,m} \times \left\{\alpha_m + \int dk_y\, r_{0TM}(k_y)\left[a(k_y) + \sum \sigma_{TM,m'} F_{m'+}(k_y)\right] e^{i[\Theta_{m-}(k_y) + 2\vartheta(k_y)]}\right\}, \quad (26a)$$

$$\tau_{TM,m} = \tau_{0TM,m} \times \left\{\alpha_m + \int dk_y\, r_{0TM}(k_y)\left[a(k_y) + \sum \sigma_{TM,m'} F_{m'+}(k_y)\right] e^{i[\Theta_{m-}(k_y) + 2\vartheta(k_y)]}\right\}. \quad (26b)$$

In Eq. (26a), the first term corresponds to the light that is directly scattered by the cylinder, for an incident vector cylindrical wave of m^{th} order. The second term corresponds to the light that is reflected at the substrate first and then scattered by the cylinder, thus doubly scattered. The third term corresponds to the light with some previous scattering history, cylinder-substrate-cylinder again. Thus, it implicitly contains multiple scattering terms of orders higher than three, at least. Equation (26b) has the same mathematical structure as Eq. (26a) except that the first factor $\sigma_{0TM,m}$ is replaced by $\tau_{0TM,m}$, meaning that the final process is transmission instead of scattering.

Equation (26a) is a self-recurrent form for $\sigma_{TM,m}$ as it appears on both sides. Therefore, by substituting the entire right side of Eq. (26a) for $\sigma_{TM,m}$ on the right-hand side repeatedly, multiple scattering terms of increasingly higher-order can be obtained.

Likewise, substituting Eq. (21a) for $\sigma_{TM,m}$ alters Eqs. (24a) and (24b) into the following:

$$r_{\mathrm{TM}}(k_y) = r_{0\mathrm{TM}}(k_y) \times \{a(k_y) + \sum \sigma_{0\mathrm{TM},m}[\alpha_m + \int dk'_y r_{\mathrm{TM}}(k'_y) e^{i[\Theta_{m-}(k'_y) + \vartheta(k'_y)]}] F_{m+}(k_y)\} e^{i\vartheta(k_y)}, \quad (27a)$$

$$t_{\mathrm{TM}}(k_y) = t_{0\mathrm{TM}}(k_y) \times \{a(k_y) + \sum \sigma_{0\mathrm{TM},m}[\alpha_m + \int dk'_y r_{\mathrm{TM}}(k'_y) e^{i[\Theta_{m-}(k'_y) + \vartheta(k'_y)]}] F_{m+}(k_y)\} e^{i\vartheta(k_y)}. \quad (27b)$$

The first term of Eq. (27a) corresponds to the direct reflection at the substrate of a plane wave with a y-component k_y. The second term corresponds to the light that is first scattered by the cylinder and then reflected by the planar substrate; thus, it is doubly scattered. The third term corresponds to the substrate-cylinder-substrate scattering, implicitly containing the terms for further scattering. Equation (27b) has the same mathematical structure as Eq. (27a), except that the factor $r_{0\mathrm{TM}}(k_y)$ is replaced by $t_{0\mathrm{TM}}(k_y)$; thus, the corresponding interpretation changes from reflection to transmission. Replacing $r(k'_y)$ in Eqs. (27a) and (27b) with the right side of Eq. (27a) can separate the pure triple-scattering term from multiple-scattering terms of higher orders.

2.1.6. Special cases

(a) Incidence at the Brewster angle

If the incident light is a plane wave of TM polarization with an incident angle equal to the Brewster angle $\phi_B = \tan^{-1} n_p$, then the values of the coefficients of reflection and transmission are $r_{0\mathrm{TM}}(k_{yB}) = 0$ and $t_{0\mathrm{TM}}(k_{yB}) = 1$, where $k_{yB} = k \sin \phi_B$. In this case $\alpha(k_y) = \delta(k_y - k_{yB})$ and thus Eqs. (24a) and (24b) have the following form:

$$r_{\mathrm{TM}}(k_y) = r_{0\mathrm{TM}}(k_y) \sum \sigma_{\mathrm{TM},m} F_{m+}(k_y) e^{i\vartheta(k_y)}, \quad (28a)$$

$$t_{\mathrm{TM}}(k_y) = [\delta(k_y - k_{yB}) + t_{0\mathrm{TM}}(k_y) \sum \sigma_{\mathrm{TM},m} F_{m+}(k_y)] e^{i\vartheta(k_y)}. \quad (28b)$$

Equation (28a) shows that even in the case of Brewster incidence, there is still reflected light due to the light scattered by the cylinder. In Eq. (28b), the first term is the light transmitted into the substrate directly, and the second term is the light scattered by the cylinder and then transmitted into the substrate.

Similarly, Eqs. (26a) and (26b) have the following form:

$$\sigma_{\mathrm{TM},m} = \sigma_{0\mathrm{TM},m}\{\alpha_m(\phi_B) + \int dk_y\, r_{0\mathrm{TM}}(k_y) \sum \sigma_{\mathrm{TM},m'} F_{m'+}(k_y)\, e^{i[\Theta_m(k_y) + 2\vartheta(k_y)]}\}, \quad (29a)$$

$$\tau_{\mathrm{TM},m} = \tau_{0\mathrm{TM},m}\{\alpha_m(\phi_B) + \int dk_y\, r_{0\mathrm{TM}}(k_y) \sum \sigma_{\mathrm{TM},m'} F_{m'+}(k_y)\, e^{i[\Theta_m(k_y) + 2\vartheta(k_y)]}\}. \quad (29b)$$

Comparing Eqs. (29a) and (29b) to Eqs. (26a) and (26b), one term (the double-scattering term initiated by the reflection of the incident light at the planar interface) is missing, as expected.

(b) Total internal reflection

If the incident light is a TM polarized plane wave with angle of incidence equal to or greater than the critical angle of total internal reflection $\phi_c = \sin^{-1} n_p$, then the value of the reflection coefficient becomes a unimodular complex number $r_{0\mathrm{TM}}(k_{yc}) = e^{i\psi}(k_{yc})$, where $k_{yc} \geq k \sin \phi_c$ and $\psi(k_{yc})$ is the phase shift induced in the process of total internal reflection. In this case $a(k_y) = \delta(k_y - k_{yc})$ and thus Eqs. (25a) and (25b) have the following form:

$$r_{\mathrm{TM}}(k_y) = [e^{i\psi(k_{yc})} \delta(k_y - k_{yc}) + r_{0\mathrm{TM}}(k_y) \sum \sigma_{\mathrm{TM},m} F_{m+}(k_y)]\, e^{i\vartheta(k_y)}, \quad (30a)$$

$$t_{\mathrm{TM}}(k_y) = [t_{0\mathrm{TM}}(k_{yc}) \delta(k_y - k_{yc}) + t_{0\mathrm{TM}}(k_y) \sum \sigma_{\mathrm{TM},m} F_{m+}(k_y)] e^{i\vartheta(k_y)}. \quad (30b)$$

In the first term of Eq. (30a), the light is totally reflected at the planar interface with an appropriate phase shift while in Eq. (30b) it is an evanescent wave and the other terms are as before. In this case Eqs. (26a) and (26b) that describe the scattering by and transmission to the cylinder retain the same form, except that the second terms are the light that is first totally reflected internally and then scattered by or transmitted into the cylinder.

It is generally accepted that air has the lowest refractive index of all materials. However, some metamaterials can have refractive indices smaller than air's in certain limited frequency bands, and the condition of total internal reflection can be met [22]. Another case is that in which air is replaced by a material with a refractive index higher than the substrate's where the above result can be used with some change in the values of refractive indices.

2.2. TE Polarization

For TE polarization, the electric field of the incident light can be written in terms of plane waves:

$$\boldsymbol{E}_i = \int dk_y\, a(k_y)\, e^{i\left(k_y y + \sqrt{k^2-k_y^2}x\right)}\hat{\mathbf{z}}. \quad (31)$$

Similarly, the electric fields reflected by and transmitted to the planar substrate are

$$\boldsymbol{E}_r = \int dk_y\, r_{\mathrm{TE}}(k_y) e^{i\left[k_y y - \sqrt{k^2-k_y^2}(x-d)\right]}\hat{\mathbf{z}}, \quad (32)$$

$$\boldsymbol{E}_t = \int dk_y\, t_{\mathrm{TE}}(k_y)\, e^{i\left[k_y y + \sqrt{n_p^2k^2-k_y^2}(x-d)\right]}\hat{\mathbf{z}}. \quad (33)$$

In addition, the electric fields scattered by and transmitted into the cylinder are

$$\boldsymbol{E}_\sigma = \sum \sigma_{\mathrm{TE},m} H_m^{(1)}(k\rho) e^{im\varphi}\hat{\mathbf{z}}, \quad (34)$$

$$\boldsymbol{E}_\tau = \sum \tau_{\mathrm{TE},m} J_m(n_c k\rho) e^{im\varphi}\hat{\mathbf{z}}. \quad (35)$$

The magnetic fields corresponding to each of the electric fields above can be obtained by using Faraday's law for harmonic fields, $\boldsymbol{H} = (-i/Zk)\nabla \times \boldsymbol{E}$ in their respective regions.

The scattering and transmission coefficients of the cylinder $\sigma_{\mathrm{TE},m}$ and $\tau_{\mathrm{TE},m}$, and the angular spectra of the light reflected by and transmitted into the planar substrate, $r_{\mathrm{TE}}(k_y)$ and $t_{\mathrm{TE}}(k_y)$ for TE polarization can be obtained using steps similar to those for TM polarization:

$$\sigma_{\mathrm{TE},m} = \sigma_{0\mathrm{TE},m}\left\{\alpha_m + \int dk_y\, r_{0\mathrm{TE}}(k_y)\left[a(k_y) + \sum \sigma_{\mathrm{TE},m'}F_{m'+}(k_y)\right]e^{i[\Theta_{m-}(k_y)+2\vartheta(k_y)]}\right\}, \quad (36a)$$

$$\tau_{\mathrm{TE},m} = \tau_{0\mathrm{TE},m}\left\{\alpha_m + \int dk_y\, r_{0\mathrm{TE}}(k_y)\left[a(k_y) + \sum \sigma_{\mathrm{TE},m'}F_{m'+}(k_y)\right]e^{i[\Theta_{m-}(k_y)+2\vartheta(k_y)]}\right\}, \quad (36b)$$

$$r_{\mathrm{TE}}(k_y) = r_{0\mathrm{TE}}(k_y)\left\{a(k_y) + \sum \sigma_{0\mathrm{TM},m}\left[\alpha_m + \int dk'_y r_{\mathrm{TE}}(k'_y)e^{i[\Theta_{m-}(k'_y)+\delta(k'_y)]}\right]F_{m+}(k_y)\right\}e^{i\vartheta(k_y)}, \quad (37a)$$

$$t_{\mathrm{TE}}(k_y) = t_{0\mathrm{TE}}(k_y)\left\{a(k_y) + \sum \sigma_{0\mathrm{TM},m}\left[\alpha_m + \int dk'_y r_{\mathrm{TE}}(k'_y)e^{i[\Theta_{m-}(k'_y)+\delta(k'_y)]}\right]F_{m+}(k_y)\right\}e^{i\vartheta(k_y)}, \quad (37b)$$

where $\sigma_{0\mathrm{TE},m}$ and $\tau_{0\mathrm{TE},m}$ are the scattering and the transmission coefficients of the cylinder alone,

$$\sigma_{0\mathrm{TE},m} = -\frac{(Z_0/Z_c)\,J'_m(n_cka)J_m(ka) - J_m(n_cka)\,J'_m(ka)}{(Z_0/Z_c)J'_m(n_cka)H_m^{(1)}(ka) - J_m(n_cka)H_m^{(1)\prime}(ka)}, \quad (38a)$$

$$\tau_{0\mathrm{TE},m} = \frac{J'_m(ka)H_m^{(1)}(ka) - J_m(ka)H_m^{(1)\prime}(ka)}{(Z_0/Z_c)J'_m(n_cka)H_m^{(1)}(ka) - J_m(n_cka)H_m^{(1)\prime}(ka)}. \quad (38b)$$

Similarly, $r_{0\mathrm{TE}}(k_y)$ and $t_{0\mathrm{TE}}(k_y)$ are the coefficients of reflection and transmission of the planar substrate alone,

$$r_{0TE}(k_y) = \frac{n_p - (Z_0/Z_p)\sqrt{n_p^2k^2-k_y^2}\Big/\sqrt{k^2-k_y^2}}{n_p + (Z_0/Z_p)\sqrt{n_p^2k^2-k_y^2}\Big/\sqrt{k^2-k_y^2}}, \quad (39a)$$

$$t_{0\mathrm{TE}}(k_y) = \frac{2}{1+(Z_0/Z_p)\sqrt{n_p^2k^2-k_y^2}\Big/\sqrt{k^2-k_y^2}}. \quad (39b)$$

III. CONCLUSION

A rigorous analytic solution was obtained for the problem of light scattering in air by a circular cylinder situated on or close to a planar substrate. The case of normal incidence to the axis of the cylinder is considered, where both TM and TE polarizations are treated separately. The solution assumes different compositions in three different media: In air, it is a superposition of the incident plane wave, the cylindrical waves scattered by the cylinder, and the plane waves reflected from the planar substrate. Inside the cylinder it is a superposition of cylindrical waves transmitted to the cylinder, and inside the planar substrate, it is a superposition of the plane waves transmitted to the substrate.

The solutions are given by infinite series of cylindrical waves and/or plane waves, consisting of numerous terms that represent single, double, triple, and the rest of the higher-order multiple scattering. The coefficient of each term explicitly shows the history of its scattering process, including the associated scattering and transmission coefficients, the phase shift experienced, and the conversion between cylindrical and plane waves.

Compared to the scalar theory in a similar situation with a unique solution [14], here in vector theory we obtained two independent solutions corresponding to incidence with different polarizations, TE and TM. In addition, in vector theory, we can consider the Brewster angle of incidence for TM polarization. While there is no direct reflection from the planar interface, the scattering by the cylinder gives rise to higher-order reflections from the planar interface. Such polarization-related phenomena cannot be found with the scalar theory.

A possible application of this work is identification and characterization of cylindrical objects like trains on a railway in remote sensing or detection of fibrous contaminants on silicon wafers in a device fabrication process.

The approach developed here can be extended to oblique incidence, in which case the TM and TE polarization components will be coupled, due to being scattered by the cylinder. However, the nature of the scattering process will remain the same and thus the solution is expected to be similar but more complicated. The results of our correlated study of the vector theory of light scattered by a cylinder embedded within a planar substrate will be reported separately.

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