Nonlinear Functional Analysis and Applications
Vol. 26, No. 2 (2021), pp. 373-380
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2021.26.02.09
http://nfaa.kyungnam.ac.kr/journal-nfaa

ON AN INEQUALITY OF S. BERNSTEIN

Barchand Chanam[1], Khangembam Babina Devi[1], Kshetrimayum Krishnadas[2], Maisnam Triveni Devi[1], Reingachan Ngamchui[1] and Thangjam Birkramjit Singh[1]

[1]Department of Mathematics, National Institute of Technology Manipur
Langol, Imphal 795004, Manipur, India
emails: barchand_2004yahoo.co.in, khangembambabina@gmail.com, trivenimaisnam@gmail.com, reinga14@gmail.com, birkramth@gmail.com

[2]Department of Mathematics, Shaheed Bhagat Singh College
University of Delhi, Sheikh Sarai Phase II, New Delhi 110017, India
e-mail: kshetrimayum.krishnadas@sbs.du.ac.in

Abstract. If $p(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then Govil [3] proved that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |p(z)|.$$

In this paper, by involving certain coefficients of $p(z)$, we not only improve the above inequality but also improve a result proved by Dewan and Mir [2].

1. Introduction

For a polynomial $p(z)$ of degree n, let $M(p,r) = \max\limits_{|z|=r} |p(z)|$ and $q(z) = z^n \overline{p(1/\overline{z})}$ be the reciprocal polynomial of $p(z)$. Bernstein [1] proved that

$$M(p',1) \leq nM(p,1). \tag{1.1}$$

Equality in (1.1) is attained for the polynomial $p(z) = \alpha z^n$, $\alpha \neq 0$.

---

[0]Received September 29, 2020. Revised October 23, 2020. Accepted February 9, 2021.
[0]2010 Mathematics Subject Classification: 30C10, 30C15, 30A10.
[0]Keywords: Bernstein, derivative, polynomial, inequality, zeros.
[0]Corresponding author: Barchand Chanam(barchand_2004@yahoo.co.in).

If we restrict ourselves to the class of polynomials $p(z)$ of degree n having no zero in $|z| < 1$, then Erdös conjectured and Lax [8] later proved that

$$M(p', 1) \leq \frac{n}{2} M(p, 1). \tag{1.2}$$

Inequality (1.2) is best possible and the extremal polynomial is $p(z) = \alpha + \beta z^n$ with $|\alpha| = |\beta|$.

The renowned mathematician Boas asked that if $p(z)$ is a polynomial of degree n not vanishing in $|z| < k$, $k > 0$, then how large can $\dfrac{M(p',1)}{M(p,1)}$ be? A partial answer to this problem was given by Malik [7] by proving a generalization of (1.2).

Theorem 1.1. ([7]) *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then*

$$M(p', 1) \leq \frac{n}{1+k} M(p, 1). \tag{1.3}$$

Equality in (1.3) *occurs for $p(z) = (z + k)^n$.*

The question as to what happens to the inequality (1.3) if $k < 1$ remains unanswered. For quite sometime, it was believed that the inequality analogous to (1.3) for $k < 1$ would be

$$M(p', 1) \leq \frac{n}{1+k^n} M(p, 1) \tag{1.4}$$

until Professor E.B. Saff countered the belief with the example $p(z) = \left(z - \frac{1}{2}\right) \times \left(z + \frac{1}{3}\right)$. Though, Govil [4] proved the validity of (1.4), it was achieved with additional condition that both $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$. Further in this quest, Govil [3] could only prove.

Theorem 1.2. ([3]) *If $p(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$M(p', 1) \leq \frac{n}{k^n + k^{n-1}} M(p, 1). \tag{1.5}$$

Inequality (1.5) was further improved by Dewan and Mir [2] by involving certain coefficients of $p(z)$.

Theorem 1.3. *If $p(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$M(p',1) \le \frac{n}{k^n}\left\{\frac{n|a_n|k^2+|a_{n-1}|}{n|a_n|(1+k^2)+2|a_{n-1}|}\right\}M(p,1). \tag{1.6}$$

2. Lemmas

We need the following lemmas to prove our result.

Lemma 2.1. ([3, Lemma 3]) *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z|<k$, $k\le 1$ *and* $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$, *then*

$$k^n\max_{|z|=1}|p'(z)|\le \max_{|z|=1}|q'(z)|. \tag{2.1}$$

Lemma 2.2. ([5, page 511, inequality (3.2)]) *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* n *and* $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$, *then on* $|z|=1$,

$$|p'(z)|+|q'(z)|\le n\max_{|z|=1}|p(z)|. \tag{2.2}$$

Lemma 2.3. ([6, page 327, Proof of Theorem 1]) *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z|\ge k$, $k\ge 1$ *and* $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$, *then for* $|z|=1$

$$|p'(z)|\le \frac{1}{k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(|\lambda|+k^2)+k(n-1)|\mu-\lambda^2|}|q'(z)|, \tag{2.3}$$

where $\lambda=\frac{k}{n}\frac{a_1}{a_0}$ *and* $\mu=\frac{2k^2}{n(n-1)}\frac{a_2}{a_0}$.

Lemma 2.4. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros on* $|z|=k$, $k\le 1$ *and* $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$, *then for* $|z|=1$

$$|p'(z)|\ge \frac{1}{k}\frac{(1-|\tau|)(1+k^2|\tau|)+k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(|\tau|+k^2)+k(n-1)|\sigma-\tau^2|}|q'(z)|, \tag{2.4}$$

where

$$\tau = \frac{1}{kn}\frac{\overline{a}_{n-1}}{\overline{a}_n} \text{ and } \sigma = \frac{2}{k^2 n(n-1)}\frac{\overline{a}_{n-2}}{\overline{a}_n}.$$

Proof. Since $p(z)$ has all its zeros on $|z| = k$, $q(z) = \sum_{\nu=0}^{n} \overline{a}_{n-\nu} z^{\nu}$ has all its zeros on $|z| = \frac{1}{k}$, where $\frac{1}{k} \geq 1$. Thus, $q(z)$ has no zeros in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$. Applying Lemma 2.3 to the polynomial $q(z)$, the desired conclusion of the lemma follows. □

Lemma 2.5. ([6, page 320, inequality (9)]) *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then*

$$\frac{k}{n}\left|\frac{a_1}{a_0}\right| \leq 1 \tag{2.5}$$

and

$$(n-1)\left|\frac{2k^2}{n(n-1)}\frac{a_2}{a_0} - \frac{k^2}{n^2}\left(\frac{a_1}{a_0}\right)^2\right| \leq 1 - \frac{k^2}{n^2}\left|\frac{a_1}{a_0}\right|^2. \tag{2.6}$$

Lemma 2.6. *If* $p(z) = \sum_{\nu}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \leq 1$, *then*

$$\frac{1}{kn}\left|\frac{a_{n-1}}{a_n}\right| \leq 1 \tag{2.7}$$

and

$$(n-1)\left|\frac{2}{k^2 n(n-1)}\frac{\overline{a}_{n-2}}{\overline{a}_n} - \frac{1}{k^2 n^2}\left(\frac{\overline{a}_{n-1}}{\overline{a}_n}\right)^2\right| \leq 1 - \frac{1}{k^2 n^2}\left|\frac{a_{n-1}}{a_n}\right|^2. \tag{2.8}$$

Proof. Since $p(z)$ has all its zeros in $|z| \leq k$, then $q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}$ has no zeros in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$. Thus, applying Lemma 2.5 to $q(z)$, we obtain inequalities (2.7) and (2.8) respectively from (2.5) and (2.6). □

Lemma 2.7. *Let* $a, b, c, d > 0$ *be real numbers such that* $c \leq d$. *If* $a \leq b$, *then*

$$\frac{a+c}{b+c} \leq \frac{a+d}{b+d}. \tag{2.9}$$

Proof. Suppose that $c < d$. Then (2.9) follows easily as

$$a \le b \quad \text{implies} \quad a(d-c) \le b(d-c),$$

which simplifies to

$$(a+c)(b+d) \le (a+d)(b+c),$$

giving the desired conclusion of the lemma. For $c = d$, the equality in (2.9) holds trivially. □

Lemma 2.8. *If* $p(z) = \sum_{\nu}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \le 1$, *then*

$$\begin{aligned} &\frac{n}{k^n + k^{n-1}} \left(\frac{(1-|\tau|)(k^2+|\tau|) + k(n-1)|\sigma - \tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|) + k(n-1)|\sigma-\tau^2|} \right) \\ &\le \frac{n}{k^{n-1}} \left(\frac{k+|\tau|}{1+k^2+2k|\tau|} \right), \end{aligned} \tag{2.10}$$

where

$$\tau = \frac{1}{kn} \frac{\overline{a}_{n-1}}{\overline{a}_n} \quad \text{and} \quad \sigma = \frac{2}{k^2 n(n-1)} \frac{\overline{a}_{n-2}}{\overline{a}_n}.$$

Proof. Let $a = (1-|\tau|)(k^2+|\tau|)$, $b = (1-|\tau|)(1-k+k^2+k|\tau|)$, $c = k(n-1)|\sigma-\tau^2|$ and $d = k(1-|\tau|^2)$. Then, by inequality (2.8) of Lemma 2.6, we have $c \le d$. It is also easy to verify that $a \le b$ as $k \le 1$ and $|\tau| \le 1$ by inequality (2.7) of Lemma 2.6. Thus, by Lemma 2.7, we have

$$\begin{aligned} &\frac{n}{k^n + k^{n-1}} \left(\frac{(1-|\tau|)(k^2+|\tau|) + k(n-1)|\sigma - \tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|) + k(n-1)|\sigma-\tau^2|} \right) \\ &\le \frac{n}{k^n + k^{n-1}} \left(\frac{(1-|\tau|)(k^2+|\tau|) + k(1-|\tau|^2)}{(1-|\tau|)(1-k+k^2+k|\tau|) + k(1-|\tau|^2)} \right) \\ &= \frac{n}{k^{n-1}(k+1)} \left(\frac{k^2+|\tau|+k+k|\tau|}{1-k+k^2+k+k|\tau|} \right) \\ &= \frac{n}{k^{n-1}} \left(\frac{k+|\tau|}{1+k^2+2k|\tau|} \right). \end{aligned}$$

□

3. Main results

In this section, we prove an improvement of (1.5) due to Govil [3]. Moreover, our result also improves (1.6) proved by Dewan and Mir [2].

Theorem 3.1. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \leq 1$, *then*

$$M(p',1) \leq \frac{n}{k^n + k^{n-1}} \left(\frac{(1-|\tau|)(k^2+|\tau|) + k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|) + k(n-1)|\sigma-\tau^2|} \right) \times M(p,1), \tag{3.1}$$

where

$$\tau = \frac{1}{kn}\frac{\overline{a}_{n-1}}{\overline{a}_n} \quad \text{and} \quad \sigma = \frac{2}{k^2 n(n-1)}\frac{\overline{a}_{n-2}}{\overline{a}_n}. \tag{3.2}$$

Proof. Let z_0 be a point on $|z| = 1$ such that $|q'(z_0)| = M(q',1)$. Then, by Lemma 2.2, we have

$$|p'(z_0)| + M(q',1) \leq nM(p,1), \tag{3.3}$$

which on combining with inequality (2.4) of Lemma 2.4, we get

$$\frac{1}{k}\left(\frac{(1-|\tau|)(1+k^2|\tau|) + k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(|\tau|+k^2) + k(n-1)|\sigma-\tau^2|} \right) |q'(z_0)| + M(q',1) \leq nM(p,1), \tag{3.4}$$

which simplifies to

$$\left(\frac{(1-|\tau|)(1-k+k^2+k|\tau|) + k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(|\tau|+k^2) + k(n-1)|\sigma-\tau^2|} \right) M(q',1) \leq \frac{kn}{k+1} M(p,1). \tag{3.5}$$

Inequality (3.5), in conjunction with Lemma 2.1 yields

$$k^n \left(\frac{(1-|\tau|)(1-k+k^2+k|\tau|) + k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(|\tau|+k^2) + k(n-1)|\sigma-\tau^2|} \right) M(p',1) \leq \frac{kn}{k+1} M(p,1), \tag{3.6}$$

which is equivalent to

$$M(p',1) \leq \frac{n}{k^n + k^{n-1}} \times \left(\frac{(1-|\tau|)(|\tau|+k^2) + k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|) + k(n-1)|\sigma-\tau^2|} \right) M(p,1),$$

which proves Theorem 3.1 completely. □

Remark 3.2. To prove that the bound given by (3.1) improves over the bound given by (1.5) of Theorem 1.2 proved by Govil [3], we show that

$$\frac{n}{k^n+k^{n-1}}\left(\frac{(1-|\tau|)(k^2+|\tau|)+k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|)+k(n-1)|\sigma-\tau^2|}\right)\leq\frac{n}{k^n+k^{n-1}},$$

for which it is sufficient to show that

$$\frac{(1-|\tau|)(k^2+|\tau|)+k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|)+k(n-1)|\sigma-\tau^2|}\leq 1$$

or

$$(k^2+|\tau|)\leq(1-k+k^2+k|\tau|),$$

which clearly holds as $k\leq 1$ and $|\tau|\leq 1$ by (2.7) of Lemma 2.6.

Remark 3.3. Inequality (3.1) is also an improvement of inequality (1.6) of Theorem 1.3 due to Dewan and Mir [2]. It is enough to show that

$$\frac{n}{k^n+k^{n-1}}\left(\frac{(1-|\tau|)(k^2+|\tau|)+k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|)+k(n-1)|\sigma-\tau^2|}\right)$$
$$\leq\frac{n}{k^n}\left\{\frac{n|a_n|k^2+|a_{n-1}|}{n|a_n|(1+k^2)+2|a_{n-1}|}\right\}. \tag{3.7}$$

Dividing both numerator and denominator of $\frac{n}{k^n}\left\{\frac{n|a_n|k^2+|a_{n-1}|}{n|a_n|(1+k^2)+2|a_{n-1}|}\right\}$ by $kn|a_n|$, we get, from (3.2)

$$\frac{n}{k^n}\left(\frac{k+\frac{1}{kn}\frac{|a_{n-1}|}{|a_n|}}{\frac{1+k^2}{k}+\frac{2}{kn}\frac{|a_{n-1}|}{|a_n|}}\right)=\frac{n}{k^{n-1}}\left(\frac{k+|\tau|}{1+k^2+2k|\tau|}\right)$$

which, in conjunction with inequality (2.10) of Lemma 2.8 proves (3.7).

As discussed earlier, Theorem 3.1 improves both Theorem 1.2 and Theorem 1.3. We illustrate this by means of the following example.

Example 3.4. Let $p(z)=z^3-\frac{3}{50}z^2-\frac{3}{500}z+\frac{1}{1000}$, $k=\frac{1}{10}$. By Theorem 1.2 and Theorem 1.3, we have

$$M(p',1)\leq 272.727\ M(p,1)$$

and

$$M(p',1)\leq 85.7143\ M(p,1)$$

respectively, while by Theorem 3.1, we obtain

$$M(p',1)\leq 74.3802\ M(p,1),$$

which shows improvement in the bound of about 72.73 % and 13.22 % over bounds given by Theorem 1.2 and Theorem 1.3, respectively.

Acknowledgement: The authors are extremely grateful to the referee for his valuable comments and suggestions about the paper.

References

[1] S. Bernstein, *Lecons sur les propriétés extrémales et la meilleure approximation desfonctions analytiques d'une variable réelle*, Gauthier Villars, Paris, 1926.

[2] K.K. Dewan and A. Mir, *Note on a theorem of S. Bernstein*, Southeast Asian Bull. Math., **31** (2007), 691–695.

[3] N.K. Govil, *On a Theorem of S. Bernstein*, J. Math. Phy. Sci., **14**(2) (1980), 183–187.

[4] N.K. Govil, *On a Theorem of S. Bernstein*, Proc. Nat. Acad. Sci., **50** (1980), 50–52.

[5] N.K. Govil and Q.I. Rahman, *Functions of exponential type not vanishing in half plane and related polynomials*, Trans. Amer. Math. Soc., **137** (1969), 501–517.

[6] N.K. Govil, Q. I. Rahman and G. Schmeisser, *On the derivative of a polynomial*, Illinois J. Math., **23**(2) (1979), 319–329.

[7] M.A. Malik, *On the derivative of a polynomial*, J. London Math. Soc., **1** (1969), 57–60.

[8] P.D. Lax, *Proof of a Conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50** (1944), 509–513.