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NORMAL STRUCTURE, FIXED POINTS AND MODULUS OF n-DIMENSIONAL U-CONVEXITY IN BANACH SPACES X AND X^*

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Abstract. Let X and X^* be a Banach space and its dual, respectively, and let $B(X)$ and $S(X)$ be the unit ball and unit sphere of X, respectively. In this paper, we study the relation between Modulus of n-dimensional U-convexity in X^* and normal structure in X. Some new results about fixed points of nonexpansive mapping are obtained, and some existing results are improved. Among other results, we proved: if X is a Banach space with $U^n_{X^*}(n+1) > 1 - \frac{1}{n+1}$ where $n \in \mathbb{N}$, then X has weak normal structure.

1. Introduction

Let X be a normed linear space. Let $B(X) = \{x \in X : \|x\| \leq 1\}$ and $S(X) = \{x \in X : \|x\| = 1\}$ be the unit ball and the unit sphere of X, respectively. Let X^* be the dual space of X, and $\nabla_x \in S(X^*)$ denotes the set of norm one supporting functionals of $x \in S(X)$.

Brodskiĭ and Mil'man [3] introduced the following geometric concepts in 1948:

Definition 1.1. A bounded and convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more

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than one point contains a point $x_0 \in H$, such that

$$\sup\{\|x_0 - y\| : y \in H\} < d(H),$$

where $d(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H.

A Banach space X is said to have normal structure if every bounded and convex subset of X has normal structure. A Banach space X is said to have weak normal structure if every weakly compact convex set K in X has normal structure. A Banach space X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any bounded closed convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} \leq c \cdot d(K).$$

For a reflexive Banach space, the normal structure and weak normal structure coincide.

Let C be a non-empty subset of a Banach space X. A mapping $T : C \to C$ is said to be nonexpansive whenever $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A Banach space has the fixed point property if for every nonempty bounded closed and convex subset C of X and for each nonexpansive mapping $T : C \to C$, there is a point $x \in C$ such that $x = Tx$ ([11], [16], [22]).

Kirk [16] proved that if a Banach space X has weak normal structure then it has the weak fixed point property, that is, every nonexpansive mapping from a weakly compact and convex subset of X into itself has a fixed point.

In [4], Clarkson introduced the concept of modulus of convexity:

$$\delta_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in S(X), \ \ \|x - y\| \geq \varepsilon\},$$

where $0 \leq \varepsilon \leq 2$. In [9], Gao introduced the concept of modulus of U-convexity which is a generalization of $\delta_X(\varepsilon)$:

$$U_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in S(X), \langle x - y, f_x\rangle \geq \varepsilon \text{ for some } f_x \in \nabla_x\},$$

where $0 \leq \varepsilon \leq 2$.

It is clear that $\delta_X(\varepsilon) \leq U_X(\varepsilon), 0 < \varepsilon < 2$. In general, $\delta_X(\varepsilon) \neq \delta_{X^*}(\varepsilon)$ and $U_X(\varepsilon) \neq U_{X^*}(\varepsilon)$, for $0 < \varepsilon < 2$. Both $\delta_X(\varepsilon)$ and $U_X(\varepsilon)$ are continuous and increasing function in [0, 2) ([6], [12], [18]).

Saejung [20] proved that:

Theorem 1.2. *let X be a Banach space with $U_X(1+t) > \frac{t}{2}$, for any $0 \leq t < 1$. Then both X and its dual X^* have uniform normal structures.*

Definition 1.3. ([5], [7]) Let X and Y be Banach spaces. We say that Y is finitely representable in X if for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$ there is an isomorphism $T : N \to T(N)$ such that for any $y \in N$,

$$(1-\varepsilon)\|y\| \leq \|Ty\| \leq (1+\varepsilon)\|y\|.$$

The Banach space X is called super-reflexive if any space Y which is finitely representable in X is reflexive.

Definition 1.4. ([13]) A Banach space X is called uniformly non-square if there exists $\delta > 0$ such that if $x, y \in S(X)$, then either $\frac{\|x+y\|}{2} \leq 1-\delta$ or $\frac{\|x-y\|}{2} \leq 1-\delta$.

Seajung [19] also proved that:

Theorem 1.5. *A Banach space X is unifomly nonsquere if and only if there exists $\varepsilon > 0$, such that $U_X(2-\varepsilon) > 0$.*

Remark 1.6. It is well known that:

(a) if X is uniformly non-square then X is supper-reflexive and therefore X is reflexive.
(b) X is super-reflexive if and only if X^* is supper-reflexive.

The following result refer to a Banach space with weak* sequentially compact unit ball of the dual. Notice that this property is satisfied by reflexive or separable Banach spaces, and by those that admit an equivalent smooth norm (see [8], Ch. XIII).

Lemma 1.7. ([21]) *If X is a Banach space with $B(X^*)$ which is weak* sequentially compact and fails to have weak normal structure, then for any $\varepsilon > 0$ there are a sequence $\{x_n\} \subseteq S(X)$ and a sequence $\{f_n\} \subseteq S(X^*)$ such that*

(a) $|\|x_i - x_j\| - 1| < \varepsilon$, *whenever* $i \neq j$;
(b) $\langle x_i, f_i\rangle = 1$, *whenever* $1 \leq i \leq \infty$;
(c) $|\langle x_j, f_i\rangle| < \varepsilon$, *whenever* $i \neq j$; *and*
(d) $\|f_i - f_j\| > 2-\varepsilon$, *whenever* $i \neq j$.

2. Main results

For two sets of vectors $\{x_1, x_2, \ldots, x_{n+1}\} \subseteq X$ and $\{f_1, f_2, \ldots, f_n\} \subseteq X^*$ where $n \in \mathbb{N}$, the following matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \langle x_1, f_1\rangle & \langle x_2, f_1\rangle & \cdots & \langle x_{n+1}, f_1\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1, f_n\rangle & \langle x_2, f_n\rangle & \cdots & \langle x_{n+1}, f_n\rangle \end{bmatrix}$$

is denoted by $M(x_1, x_2, \ldots, x_{n+1}; f_1, f_2, \ldots, f_n)$.

In 1951, Silverman [23] introduced the concept of volume of the convex hull of $x_1, x_2, \ldots, x_{n+1}$ in X by

$$V(x_1, x_2, \ldots, x_{n+1}) := \sup\{\det M(x_1, x_2, \ldots, x_{n+1}; f_1, f_2, \ldots, f_n) : f_1, f_2, \ldots, f_n \in S(X^*)\}.$$

In 1988, Kirk introduced the modulus of n-dimensional uniform convexity as follows [17]:

Definition 2.1. Let X be a Banach space. Then

$$\delta_X^n(\varepsilon) := \inf\left\{1 - \frac{1}{n+1}\|x_1 + x_2 + \cdots + x_{n+1}\| : \begin{array}{c} x_1, x_2, \ldots x_{n+1} \in S(X), \\ V(x_1, x_2, \ldots, x_{n+1}) \geq \varepsilon \end{array}\right\},$$

where $0 \leq \varepsilon \leq 2$ is called the modulus of n-dimensional uniform convexity of X.

For two sets of vectors $\{x_1, x_2, \ldots, x_{n+1}\} \subseteq X$ and $\{f_2 \in \nabla_{x_2}, f_3 \in \nabla_{x_3}, \ldots, f_{n+1} \in \nabla_{x_{n+1}}\} \subseteq X^*$, where $n \in \mathbb{N}$, the following matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \langle x_1, f_2\rangle & \langle x_2, f_2\rangle & \cdots & \langle x_{n+1}, f_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1, f_{n+1}\rangle & \langle x_2, f_{n+1}\rangle & \cdots & \langle x_{n+1}, f_{n+1}\rangle \end{bmatrix}$$

is denoted by $m(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1})$.

In 2015, Saejung and Gao [21] introduced another concept of volume by the convex hull of $x_1, x_2, \ldots, x_{n+1}$ in X by

$$v(x_1, x_2, \ldots, x_{n+1}) := \sup\{\det m(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1}) : f_2 \in \nabla_{x_2}, f_3 \in \nabla_{x_3}, \ldots, f_{n+1} \in \nabla_{x_{n+1}}\}.$$

It is clear from the definition that:

Proposition 2.2. $v(x_1, x_2, \ldots, x_{n+1}) \leq V(x_1, x_2, \ldots, x_{n+1})$.

Definition 2.3. ([21]) Let $\nu_X^n = \sup\{v(x_1, x_2, \ldots, x_{n+1}) : x_1, x_2, \ldots x_{n+1} \in S(X)\}$ be the upper bound of all n-dimmensional volume in X.

Proposition 2.4. ([21]) *For a Banach space X with* $\dim(X) > n$, $\nu_X^n \geq 2$.

Definition 2.5. ([21]) Let X be a Banach space. Then

$$U_X^n(\varepsilon) := \inf\left\{1-\frac{1}{n+1}\|x_1+x_2+\cdots+x_{n+1}\| : \begin{array}{c} x_1,x_2,\ldots x_{n+1}\in S(X), \\ v(x_1,x_2,\ldots,x_{n+1})\geq\varepsilon \end{array}\right\},$$

where $0 \leq \varepsilon \leq \nu_X^n$ is called the modulus of n-dimensional U-convexity of X.

It is clear that for a Banach space X with $\dim(X) > n$, if $0 \leq \varepsilon \leq 2$, then $\delta_X^n(\varepsilon) \leq U_X^n(\varepsilon)$.

Lemma 2.6. ([21]) *$U_X^n(\varepsilon)$ is a continuous function in $[0, \nu_X^n)$.*

Theorem 2.7. ([21]) *If X is a Banach space with $U_X^n(1) > 0$ where $n \in \mathbb{N}$, then X is super-reflexive.*

Theorem 2.8. ([10]) *If X is a Banach space with $U_X^2(\frac{5}{4}) > \frac{2}{3}$, then X is super-reflexive.*

Lemma 2.9. ([2]) *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in B(X)$ and $h \in S(X^*)$ with $1 - \langle z, h\rangle < \frac{\varepsilon^2}{4}$, then there exist $y \in S(X)$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Remark 2.10. It is easy to know that the condition of Theorem 2.9 can be extended to $1 - \langle z, h\rangle \leq \frac{\varepsilon^2}{4}$ for given $z \in B(X)$ and $h \in S(X^*)$.

The following result was proved by James:

Theorem 2.11. ([13]) *Let X be a Banach space. Then X is not reflexive if and only if for any $0 < \eta < 1$ there are two sequences $\{x_n\} \subseteq S(X)$ and sequence $\{f_n\} \subseteq S(X^*)$ such that*

(a) $\langle x_m, f_n\rangle = \eta$ *whenever* $n \leq m$*; and*
(b) $\langle x_m, f_n\rangle = 0$ *whenever* $n > m$.

Theorem 2.12. *If X is a Banach space with $\max\{U_X^n(1), U_{X^*}^n(1)\} > 0$ where $n \in \mathbb{N}$, then X is supre-reflexive.*

Proof. This is a direct result of Theorem 2.7 and Remark 1.6. □

Theorem 2.13. *If X is a Banach space with $\max\{U_X^n(\frac{5}{4}), U_{X^*}^n(\frac{5}{4})\} > \frac{2}{3}$, then X is super-reflexive.*

Proof. This is a direct result of Theorem 2.8 and Remark 1.6. □

Theorem 2.14. *If X is a Banach space with $U_{X^*}^n(n+1) > 1 - \frac{1}{n+1} = \frac{n}{n+1}$ where $n \in \mathbb{N}$, then X has weak normal structure.*

Proof. It is easy to prove by the mathematical induction that:

$$\det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{(n+1)\times(n+1)} = n+1.$$

Suppose X does not have weak normal structure, for $0 < \varepsilon < 1$, let $\{x_i\}$ and $\{f_i\}$ be the sequances satisty four conditions in Lemma 1.7.

Consider

$$\{g_j\} = \{-f_{j+1}\} \subseteq S(X^*), j = 1, 2, ..., n+1$$

and

$$\{y_i\} = \{x_i - x_{i+1}\} \subseteq (1+\varepsilon)U(X) \subseteq (1+\varepsilon)U((X^*)^*),\ i = 1, 2, \dots, n+1.$$

Then, we have:

$$1 - \varepsilon < \langle g_j, y_i \rangle = \langle -f_{j+1}, x_i - x_{i+1} \rangle = 1 + \varepsilon_{i,i} < 1 + \varepsilon,\ \text{if } i = j,$$

$$-1 - \varepsilon < \langle g_j, y_i \rangle = \langle -f_{j+1}, x_i - x_{i+1} \rangle = -1 + \varepsilon_{j.i} < -1 + \varepsilon, \text{ if } i = j+1;$$

and

$$-\varepsilon < \langle g_j, y_i \rangle = \langle -f_{j+1}, x_i - x_{i+1} \rangle = \varepsilon_{j,i} < \varepsilon,\ \text{if } i \neq j,\ \text{and } i \neq j+1,$$

where $1 \leq i, j \leq n+1$.

We therefore have:

$$\det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \langle g_1, y_2 \rangle & \langle g_2, y_2, \rangle & \langle g_3, y_2, \rangle & \cdots & \langle g_n, y_2 \rangle & \langle g_{n+1}, y_2 \rangle \\ \langle g_1, y_3 \rangle & \langle g_2, y_3, \rangle & \langle g_3, y_3, \rangle & \cdots & \langle g_n, y_3 \rangle & \langle g_{n+1}, y_3 \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ \langle g_1, y_n \rangle & \langle g_2, y_n \rangle & \langle g_3, y_n \rangle & \cdots & \langle g_n, y_n \rangle & \langle g_{n+1}, y_n \rangle \\ \langle g_1, y_{n+1} \rangle & \langle g_2, y_{n+1} \rangle & \langle g_3, y_{n+1} \rangle & \cdots & \langle g_n, y_{n+1} \rangle & \langle g_{n+1}, y_{n+1} \rangle \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1+\varepsilon_{1,2} & 1+\varepsilon_{2,2} & \varepsilon_{3,2} & \cdots & \varepsilon_{n,2} & \varepsilon_{n+1,2} \\ \varepsilon_{1,3} & -1+\varepsilon_{2,3} & 1+\varepsilon_{3,3} & \cdots & \varepsilon_{n,3} & \varepsilon_{n+1,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon_{1,n} & \varepsilon_{2,n} & \varepsilon_{3,n} & \cdots & 1+\varepsilon_{n,n} & \varepsilon_{n+1,n} \\ \varepsilon_{1,n+1} & \varepsilon_{2,n+1} & \varepsilon_{3,n+1} & \cdots & -1+\varepsilon_{n,n+1} & 1+\varepsilon_{n+1,n+1} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{(n+1)\times(n+1)} + d\varepsilon$$

$$= n + 1 + d\varepsilon,$$

where d is a constant.

From Lemma 2.9 (since ε can be arbitrarily small, if necessary we can normalize y_i to use Lemma 2.9), for $0 < \varepsilon < 1$, there are $\{h_j\} \subseteq S(X^*), j = 1, 2, \dots, n+1$ and $\{z_i\} \subseteq S((X^*)^*), i = 1, 2, \dots, n+1$ such that $z_n \in \nabla_{h_n}, \|h_j - g_j\| < \varepsilon$, for j=1, 2, ... n+1, and $\|z_i - y_i\| < \varepsilon$ for $i = 1, 2, \dots, n+1$. Hence, we have

$$-2\varepsilon \leq \langle h_j, z_i \rangle - \langle g_j, y_i \rangle \leq 2\varepsilon$$

for $i = 1, 2, \dots, n+1$, and $j = 1, 2, \dots, n+1$. Therefore,

$$\det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \langle h_1, z_2 \rangle & \langle h_2, z_2, \rangle & \langle h_3, z_2, \rangle & \cdots & \langle h_n, z_2 \rangle & \langle h_{n+1}, y_2 \rangle \\ \langle h_1, z_3 \rangle & \langle h_2, z_3, \rangle & \langle h_3, z_3, \rangle & \cdots & \langle h_n, z_3 \rangle & \langle h_{n+1}, y_3 \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ \langle h_1, z_n \rangle & \langle h_2, z_n \rangle & \langle h_3, z_n \rangle & \cdots & \langle h_n, z_n \rangle & \langle h_{n+1}, z_n \rangle \\ \langle h_1, z_{n+1} \rangle & \langle h_2, z_{n+1} \rangle & \langle h_3, z_{n+1} \rangle & \cdots & \langle h_n, z_{n+1} \rangle & \langle h_{n+1}, z_{n+1} \rangle \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \langle g_1, y_2 \rangle & \langle g_2, y_2, \rangle & \langle g_3, y_2, \rangle & \cdots & \langle g_n, y_2 \rangle & \langle g_{n+1}, y_2 \rangle \\ \langle g_1, y_3 \rangle & \langle g_2, y_3, \rangle & \langle g_3, y_3, \rangle & \cdots & \langle g_n, y_3 \rangle & \langle g_{n+1}, y_3 \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ \langle g_1, y_n \rangle & \langle g_2, y_n \rangle & \langle g_3, y_n \rangle & \cdots & \langle g_n, y_n \rangle & \langle g_{n+1}, y_n \rangle \\ \langle g_1, y_{n+1} \rangle & \langle g_2, y_{n+1} \rangle & \langle g_3, y_{n+1} \rangle & \cdots & \langle g_n, y_{n+1} \rangle & \langle g_{n+1}, y_{n+1} \rangle \end{bmatrix}$$

$$+ e\varepsilon$$

$$= n + 1 + f\varepsilon,$$

where e and f are constant. So, we have

$$\begin{aligned} v(h_1, h_2, \dots, h_{n+1}) := \sup\{&\det m(h_1, h_2, \dots, h_{n+1}; z_2, z_3, \dots, z_{n+1}) : \\ &z_2 \in \nabla_{h_2}, z_3 \in \nabla_{h_3}, \dots, z_{n+1} \in \nabla_{h_{n+1}}\} \\ \geq\ & n + 1 + f\varepsilon. \end{aligned}$$

On the other hand, since

$$\frac{\|h_1+h_2+\cdots+h_{n+1}\|}{n+1} \geq \langle \frac{h_1+h_2+\cdots+h_{n+1}}{n+1}, z_1\rangle - \varepsilon \\ \geq \frac{1}{n+1},$$

we have

$$1-\frac{\|h_1+h_2+\cdots+h_{n+1}\|}{n+1} < 1-\frac{1}{n+1} = \frac{n}{n+1}.$$

From the definition of $U_X^n(\varepsilon)$, we have $U_{X^*}^n(n+1) < 1-\frac{1}{n+1}$. □

3. Uniform normal structure

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces on an index set I, and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I}\|x_i\| < \infty$.

Definition 3.1. ([1], [15], [24]) Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultra-product of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote an element of the ultra-product. It follows from the property of ultra-product [10], and the definition of quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|. \tag{3.1}$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$ for all $i \in \mathbb{N}$ for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the corresponding ultra-product, called an ultra-power of X.

Lemma 3.2. ([1], [15], [24]) *Suppose that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and X is a Banach space. Then $(X^*)_{\mathcal{U}} \cong (X_{\mathcal{U}})^*$ if and only if X is super-reflexive; and in this case, the mapping J defined by*

$$\langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}})\rangle = \lim_{\mathcal{U}}\langle x_i, f_i\rangle, \quad \textit{for all } (x_i)_{\mathcal{U}} \in X_{\mathcal{U}}$$

is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

Therefore, we have:

Theorem 3.3. *Let X be a super-reflexive Banach space. Then for any non-trivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and for all $n \in \mathbb{N}$ and $\varepsilon > 0$, we have*

$$U^n_{X_{\mathcal{U}}}(\varepsilon) = U^n_X(\varepsilon).$$

Lemma 3.4. ([14]) *If X is a super-reflexive Banach space, then X has uniform normal structure if and only if $X_{\mathcal{U}}$ has normal structure.*

From Theorem 2.12, Theorem 2.14, and Lemma 3.4, we have:

Theorem 3.5. *If X is a Banach space with $\max\{U^n_X(1), U^n_{X^*}(1)\} > 0$ and $U^n_{X^*}(n+1) > 1 - \frac{1}{n+1}$, where $n \in \mathbb{N}$, then X has uniform normal structure.*

Since $\frac{5}{4} < 2$, when $n = 2$, from Theorem 2.13, Theorem 2.14, and Theorem 3.4, we have:

Theorem 3.6. *If X is a Banach space with $\max\{U^2_X(\frac{5}{4}), U^2_{X^*}(\frac{5}{4})\} > 1 - \frac{1}{3} = \frac{2}{3}$, then X has uniform normal structure.*

Theorem 3.6 improves Theorem 3.5 of [10].

Corollary 3.7. *If X is a Banach space with $\max\{\delta^n_X(1), \delta^n_{X^*}(1)\} > 0$ and $\delta^n_{X^*}(n+1) > 1 - \frac{1}{n+1}$ where $n \in \mathbb{N}$, then X has uniform normal structure.*

When $n = 2$, we have:

Corollary 3.8. *If X is a Banach space with $\max\{\delta^2_X(\frac{5}{4}), \delta^2_{X^*}(\frac{5}{4})\} > 1 - \frac{1}{3} = \frac{2}{3}$, then X has uniform normal structure.*

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