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A VISCOSITY TYPE PROJECTION METHOD FOR SOLVING PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract. A plethora of applications from mathematical programmings, such as minimax, mathematical programming, penalization and fixed point problems can be framed as variational inequality problems. Most of the methods that used to solve such problems involve iterative methods, that is why, in this paper, we introduce a new extragradient-like method to solve pseudomonotone variational inequalities in a real Hilbert space. The proposed method has the advantage of a variable step size rule that is updated for each iteration based on previous iterations. The main advantage of this method is that it operates without the previous knowledge of the Lipschitz constants of an operator. A strong convergence theorem for the proposed method is proved by letting the mild conditions on an operator $\mathcal{G}$. Numerical experiments have been studied in order to validate the numerical performance of the proposed method and to compare it with existing methods.

1. Introduction

In this article, we consider classical variational inequalities [33] and the *variational inequality problem* (VIP) for an operator $\mathcal{G} : \mathbb{E} \to \mathbb{E}$ is defined in the following way:

$$\text{Find } u^* \in \mathbb{K} \text{ such that } \left\langle \mathcal{G}(u^*), v - u^* \right\rangle \geq 0, \ \forall\, v \in \mathbb{K}, \tag{VIP}$$

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where $\mathbb{K}$ is a nonempty, convex and closed subset of a real Hilbert space $\mathbb{E}$, $\langle .,. \rangle$ and $\|.\|$ denote an inner product and the induced norm on $\mathbb{E}$, respectively. Moreover, $\mathbb{R}$, $\mathbb{N}$ are the set of real numbers and natural numbers, respectively. It is useful to note that the problem (VIP) is equivalent to solve the following problem:

$$\text{Find } u^* \in \mathbb{K} \text{ such that } u^* = P_{\mathbb{K}}[u^* - \rho\mathcal{G}(u^*)],$$

where ρ is any positive real number and $P_{\mathbb{K}}$ is a metric projection on $\mathbb{K}$.

The theory of variational inequalities has been used as an important tool to study a wide range of topics, that is, physics, engineering, economics and optimization theory. This problem was presented by Stampacchia [33] in 1964 and also well established that the problem (VIP) is a crucial problem in nonlinear analysis. This is an important mathematical problem that includes several important topics of applied mathematics, such as network equilibrium problems, the necessary optimality conditions, the complementarity problems and the systems of nonlinear equations (for more details [1, 2, 8, 13, 14, 16, 17, 27, 35]).

On the other hand, the projection methods are important iterative methods to solve variational inequalities. Many iterative methods for solving variational inequalities have been proposed and analyzed (see for more details [5, 6, 12, 15, 18, 24, 25, 26, 28, 29, 30, 36, 37, 38, 42]).

The extragradient method was introduced by Korpelevich [18] and Antipin [3]. The method is of the form:

$$\begin{cases} u_0 \in \mathbb{K}, \\ v_n = P_{\mathbb{K}}[u_n - \rho\mathcal{G}(u_n)], \\ u_{n+1} = P_{\mathbb{K}}[u_n - \rho\mathcal{G}(v_n)] \end{cases} \tag{1.1}$$

where $0 < \rho < \frac{1}{L}$ and L is Lipschitz constant of an operator $\mathcal{G}$.

Yang et al. [41] proposed two explicit subgradient extragradient methods to solve monotone variational inequalities. An iterative sequence $\{u_n\}$ was generated in the following way:

Algorithm 1.1. (i) *Let* $u_0 \in \mathbb{K}$, $\mu \in (0,1)$ *and* $\rho_0 > 0$.

(ii) *Compute iterative sequence* $\{u_n\}$ *for* $n \geq 1$ *as follows:*

$$\begin{cases} v_n = P_{\mathbb{K}}[u_n - \rho_n\mathcal{G}(u_n)], \\ u_{n+1} = P_{\mathbb{E}_n}[u_n - \rho_n\mathcal{G}(v_n)], \end{cases} \tag{1.2}$$

where $\mathbb{E}_n = \{z \in \mathbb{E} : \langle u_n - \rho_n\mathcal{G}(u_n) - v_n, z - v_n\rangle \leq 0\}$.

(iii) *Update the step size rule in the following way:*

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n, \frac{\mu\|u_n - v_n\|^2 + \mu\|u_{n+1} - v_n\|^2}{\langle \mathcal{G}(u_n) - \mathcal{G}(v_n), u_{n+1} - v_n\rangle}\right\} \\ \quad \textit{if} \quad \langle \mathcal{G}(u_n) - \mathcal{G}(v_n), u_{n+1} - v_n\rangle > 0, \\ \rho_n \qquad\qquad\qquad\qquad \textit{otherwise.} \end{cases}$$

(iv) *If* $u_n = v_n$*, then stop. Otherwise, set* $n := n + 1$ *and return to Step* (ii).

Algorithm 1.2. (i) *Let* $u_0 \in \mathbb{K}$*,* $\mu \in (0, 1)$*,* $\rho_0 > 0$ *and a sequence* $\phi_n \subset (0, 1)$ *with* $\phi_n \to 0$ *and* $\sum_{n=1}^{\infty} \phi_n = +\infty$*.*

(ii) *Compute iterative sequence* $\{u_n\}$ *for* $n \geq 1$ *as follows:*

$$\begin{cases} v_n = P_{\mathbb{K}}[u_n - \rho_n \mathcal{G}(u_n)], \\ t_n = P_{\mathbb{E}_n}[u_n - \rho_n \mathcal{G}(v_n)], \\ u_{n+1} = \phi_n u_0 + (1 - \phi_n) t_n, \end{cases} \tag{1.3}$$

where $\mathbb{E}_n = \{z \in \mathbb{E} : \langle u_n - \rho_n \mathcal{G}(u_n) - v_n, z - v_n\rangle \leq 0\}$*.*

(iii) *Update the step size rule in the following way:*

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n, \frac{\mu\|u_n - v_n\|^2 + \mu\|t_n - v_n\|^2}{\langle \mathcal{G}(u_n) - \mathcal{G}(v_n), t_n - v_n\rangle}\right\} \\ \quad \textit{if} \quad \langle \mathcal{G}(u_n) - \mathcal{G}(v_n), t_n - v_n\rangle > 0, \\ \rho_n \qquad\qquad\qquad\qquad \textit{otherwise.} \end{cases}$$

(iv) *If* $u_n = v_n$*, then stop. Otherwise, set* $n := n + 1$ *and return to Step* (ii).

Inspired by the methods in [19, 26, 41], in this paper, we introduces a modified subgradient extragradient algorithm for solving pseudomonotone variational inequalities in real Hilbert spaces. In contrast to the results of Yang et al. [41], the primary goal of this paper is to solve pseudomonotone variational inequalities in real Hilbert spaces. It is important to note that the proposed algorithm is more efficient between existing algorithms. In particular, by comparing the results of Yang et al. [41], the proposed algorithm is effective in most situations. Similar to the results of Yang et al. [41], proof of the strong convergence of the proposed algorithm is well established without knowing the Lipschitz constant of the operator $\mathcal{G}$.

The proposed algorithm can be seen as a modification of the methods shown in [18, 19, 41]. Numerical findings have been studied and confirmed so that the new method is more effective than the existing method in [41].

The rest of this article was arranged as follows: Section 2 contains some definitions and basic results used in the paper. Section 3 includes the main algorithm and convergence theorem. Section 4 performs the numerical results that show the algorithmic effectiveness of the proposed method.

2. Preliminaries

We assume that the following requirements have been met.

(B1) The solution set of problem (VIP) is denoted by Ω and it is nonempty.

(B2) An operator $\mathcal{G} : \mathbb{E} \to \mathbb{E}$ is pseudomonotone, that is,

$$\big\langle \mathcal{G}(v_1), v_2 - v_1 \big\rangle \geq 0 \Longrightarrow \big\langle \mathcal{G}(v_2), v_1 - v_2 \big\rangle \leq 0,\ \forall\, v_1, v_2 \in \mathbb{E}.$$

(B3) An operator $\mathcal{G} : \mathbb{E} \to \mathbb{E}$ is *Lipschitz continuous* with constant $L > 0$, that is, there exists a positive constants L such that

$$\|\mathcal{G}(v_1) - \mathcal{G}(v_2)\| \leq L\|v_1 - v_2\|,\ \forall\, v_1, v_2 \in \mathbb{E}.$$

(B4) An operator $\mathcal{G} : \mathbb{E} \to \mathbb{E}$ is *sequentially weakly continuous*, that is, $\{\mathcal{G}(u_n)\}$ converges weakly to $\mathcal{G}(u)$ for each sequence $\{u_n\}$ weakly converges to u.

The *metric projection* $P_{\mathbb{K}}(v_1)$ for $v_1 \in \mathbb{E}$ onto a closed and convex subset $\mathbb{K}$ of $\mathbb{E}$ is defined by $P_{\mathbb{K}}(v_1) = \arg\min\limits_{v_2 \in \mathbb{K}}\{\|v_1 - v_2\|\}$.

Lemma 2.1. ([20]) *Let $\mathbb{K}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathbb{E}$ and $P_{\mathbb{K}} : \mathbb{E} \to \mathbb{K}$ be a metric projection from $\mathbb{E}$ onto $\mathbb{K}$.*

(i) *Let $v_1 \in \mathbb{K}$ and $v_2 \in \mathbb{E}$, we have*

$$\|v_1 - P_{\mathbb{K}}(v_2)\|^2 + \|P_{\mathbb{K}}(v_2) - v_2\|^2 \leq \|v_1 - v_2\|^2.$$

(ii) $v_3 = P_{\mathbb{K}}(v_1)$ *if and only if* $\langle v_1 - v_3, v_2 - v_3\rangle \leq 0,\ \forall\, v_2 \in \mathbb{K}$.

(iii) *For $v_2 \in \mathbb{K}$ and $v_1 \in \mathbb{E}$* $\|v_1 - P_{\mathbb{K}}(v_1)\| \leq \|v_1 - v_2\|$.

Lemma 2.2. ([4]) *For each $v_1, v_2 \in \mathbb{E}$ and $\delta \in \mathbb{R}$, the following relationships hold.*

(i) $\|\delta v_1 + (1-\delta)v_2\|^2 = \delta\|v_1\|^2 + (1-\delta)\|v_2\|^2 - \delta(1-\delta)\|v_1 - v_2\|^2$.

(ii) $\|v_1 + v_2\|^2 \leq \|v_1\|^2 + 2\langle v_2, v_1 + v_2\rangle$.

Lemma 2.3. ([40]) *Let $\{\Psi_n\}$ be a sequence of nonnegative real numbers such that*

$$\Psi_{n+1} \leq (1 - \tau_n)\Psi_n + \tau_n\delta_n,\ \forall\, n \in \mathbb{N},$$

where $\{\tau_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions:

$$\lim_{n\to\infty} \tau_n = 0,\ \sum_{n=1}^{\infty} \tau_n = +\infty \text{ and } \limsup_{n\to\infty} \delta_n \leq 0.$$

Then $\lim_{n\to\infty} \Psi_n = 0$.

Lemma 2.4. ([23]) *Let* $\{\Psi_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *with* $\Psi_{n_i} < \Psi_{n_{i+1}}$ *for all* $i \in \mathbb{N}$. *Then, there exists a nondecreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *as* $k \to \infty$, *and the following conditions are fullfilled by all (sufficiently large) numbers* $k \in \mathbb{N}$*:*

$$\Psi_{m_k} \leq \Psi_{m_{k+1}} \text{ and } \Psi_k \leq \Psi_{m_{k+1}},$$

where $m_k = \max\{j \leq k : \Psi_j \leq \Psi_{j+1}\}$.

Lemma 2.5. ([34]) *Assume that* $\mathcal{G} : \mathbb{K} \to \mathbb{E}$ *is a pseudomonotone and continuous operator. Then,* u^* *is a solution of the problem* (VIP) *if and only if* u^* *is a solution of the following problem:*

$$\text{Find } u \in \mathbb{K} \text{ such that } \langle \mathcal{G}(v), v - u \rangle \geq 0, \ \forall\, v \in \mathbb{K}.$$

3. Viscosity method for pseudomonotone variational inequality

In this section, we provide a method consisting of one convex minimization problem through viscosity and an explicit step size rule which are being used to improve the convergence rate of the iterative sequence. Suppose that $\mathrm{g} : \mathbb{E} \to \mathbb{E}$ is a strict contraction mapping with constant $\xi \in [0, 1)$. The main algorithm is defined as follows:

Algorithm A: (An explicit method for variational inequality problem)

Step 0: Choose $u_0 \in \mathbb{K}$, $\mu \in (0, 1)$, $\rho_0 > 0$ and a sequence $\phi_n \subset (0, 1)$ satisfying the following conditions:

$$\lim_{n \to \infty} \phi_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \phi_n = +\infty.$$

Step 1: Compute

$$v_n = P_{\mathbb{K}}[u_n - \rho_n \mathcal{G}(u_n)].$$

If $u_n = v_n$, then Stop. Otherwise, go to Step 2.

Step 2: Compute

$$t_n = v_n + \rho_n[\mathcal{G}(u_n) - \mathcal{G}(v_n)].$$

Step 3: Compute

$$u_{n+1} = \phi_n \mathrm{g}(u_n) + (1 - \phi_n) t_n,$$

where $\mathrm{g} : \mathbb{E} \to \mathbb{E}$ is a strict contraction mapping with constant $\xi \in [0, 1)$.

Step 4: Evaluate

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n, \frac{\mu\|u_n - v_n\|}{\|\mathcal{G}(u_n) - \mathcal{G}(v_n)\|}\right\}, & \text{if} \quad \mathcal{G}(u_n) \neq \mathcal{G}(v_n), \\ \rho_n, & \text{else.} \end{cases} \tag{3.1}$$

Set $n := n + 1$ and go back to Step 1.

Lemma 3.1. *The step size sequence* $\{\rho_n\}$ *generated in* (3.1) *is monotonically decreasing with a lower bound is* $\min\left\{\frac{\mu}{L}, \rho_0\right\}$ *and converges to a fixed* $\rho > 0$.

Proof. It is easy to see that by definition $\{\rho_n\}$ is monotone and non-increasing sequence. It is given that $\mathcal{G}$ is Lipschitz-continuous with constant $L > 0$. Let $\mathcal{G}(u_n) \neq \mathcal{G}(v_n)$ such that

$$\begin{aligned} \frac{\mu\|u_n - v_n\|}{\|\mathcal{G}(u_n) - \mathcal{G}(v_n)\|} &\geq \frac{\mu\|u_n - v_n\|}{L\|u_n - v_n\|} \\ &\geq \frac{\mu}{L}. \end{aligned} \tag{3.2}$$

The above expression implies that the sequence $\{\rho_n\}$ have a lower bound $\min\left\{\frac{\mu}{L}, \rho_0\right\}$. Moreover, there exits $\rho > 0$ such that $\lim_{n\to\infty} \rho_n = \rho$. □

Lemma 3.2. *Assume that* $\mathcal{G} : \mathbb{E} \to \mathbb{E}$ *satisfies the conditions* (B1)-(B4). *Let* $\{u_n\}$ *be a sequence which is generated by Algorithm A. Moreover, sequence* $\phi_n \subset (0, 1)$ *satisfying the following conditions:*

$$\lim_{n\to\infty} \phi_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \phi_n = +\infty.$$

Then for each $u^* \in \Omega$, *we have*

$$\|t_n - u^*\|^2 \leq \left\|u_n - u^*\right\|^2 - \left(1 - \mu^2 \frac{\rho_n^2}{\rho_{n+1}^2}\right)\left\|u_n - v_n\right\|^2.$$

Proof. Let $u^* \in \Omega$ and by definition of t_n, we have

$$\begin{aligned} \left\|t_n - u^*\right\|^2 &= \left\|v_n + \rho_n[\mathcal{G}(u_n) - \mathcal{G}(v_n)] - u^*\right\|^2 \\ &= \left\|v_n - u^*\right\|^2 + \rho_n^2\left\|\mathcal{G}(u_n) - \mathcal{G}(v_n)\right\|^2 \\ &\quad + 2\rho_n\langle v_n - u^*, \mathcal{G}(u_n) - \mathcal{G}(v_n)\rangle \\ &= \left\|v_n + u_n - u_n - u^*\right\|^2 + \rho_n^2\left\|\mathcal{G}(u_n) - \mathcal{G}(v_n)\right\|^2 \\ &\quad + 2\rho_n\langle v_n - u^*, \mathcal{G}(u_n) - \mathcal{G}(v_n)\rangle \\ &= \left\|v_n - u_n\right\|^2 + \left\|u_n - u^*\right\|^2 + 2\langle v_n - u_n, u_n - u^*\rangle \\ &\quad + \rho_n^2\left\|\mathcal{G}(u_n) - \mathcal{G}(v_n)\right\|^2 + 2\rho_n\langle v_n - u^*, \mathcal{G}(u_n) - \mathcal{G}(v_n)\rangle \\ &= \left\|u_n - u^*\right\|^2 + \left\|v_n - u_n\right\|^2 \\ &\quad + 2\langle v_n - u_n, v_n - u^*\rangle + 2\langle v_n - u_n, u_n - v_n\rangle \\ &\quad + \rho_n^2\left\|\mathcal{G}(u_n) - \mathcal{G}(v_n)\right\|^2 + 2\rho_n\langle v_n - u^*, \mathcal{G}(u_n) - \mathcal{G}(v_n)\rangle. \end{aligned} \tag{3.3}$$

It is given that $v_n = P_{\mathbb{K}}[u_n - \rho_n\mathcal{G}(u_n)]$ and it further implies that

$$\langle u_n - \rho_n\mathcal{G}(u_n) - v_n, y - v_n\rangle \leq 0, \ \forall\, y \in \mathbb{K} \tag{3.4}$$

or equivalently for some $u^* \in \Omega$, we can write

$$\langle u_n - v_n, u^* - v_n \rangle \leq \rho_n \langle \mathcal{G}(u_n), u^* - v_n \rangle. \tag{3.5}$$

Combining expressions (3.3) and (3.5), we have

$$\begin{aligned}
&\|t_n - u^*\|^2 \\
&\leq \|u_n - u^*\|^2 + \|v_n - u_n\|^2 + 2\rho_n \langle \mathcal{G}(u_n), u^* - v_n \rangle - 2\langle u_n - v_n, u_n - v_n \rangle \\
&\quad + \rho_n^2 \|\mathcal{G}(u_n) - \mathcal{G}(v_n)\|^2 - 2\rho_n \langle \mathcal{G}(u_n) - \mathcal{G}(v_n), u^* - v_n \rangle \\
&= \|u_n - u^*\|^2 - \|u_n - v_n\|^2 + \rho_n^2 \|\mathcal{G}(u_n) - \mathcal{G}(v_n)\|^2 - 2\rho_n \langle \mathcal{G}(v_n), v_n - u^* \rangle.
\end{aligned} \tag{3.6}$$

It is given that u^* is the solution of the problem (VIP), implies that

$$\langle \mathcal{G}(u^*), y - u^* \rangle \geq 0, \ \forall \, y \in \mathbb{K}.$$

Due to the pseudomontonicity of $\mathcal{G}$ on $\mathbb{K}$, we obtain

$$\langle \mathcal{G}(y), y - u^* \rangle \geq 0, \ \forall \, y \in \mathbb{K}.$$

Substituting $y = v_n \in \mathbb{K}$, we have

$$\langle \mathcal{G}(v_n), v_n - u^* \rangle \geq 0. \tag{3.7}$$

Combining expressions (3.6) and (3.7), we obtain

$$\begin{aligned}
\|t_n - u^*\|^2 &\leq \|u_n - u^*\|^2 - \|u_n - v_n\|^2 + \mu^2 \frac{\rho_n^2}{\rho_{n+1}^2} \|u_n - v_n\|^2 \\
&= \|u_n - u^*\|^2 - \left(1 - \mu^2 \frac{\rho_n^2}{\rho_{n+1}^2}\right) \|u_n - v_n\|^2.
\end{aligned} \tag{3.8}$$

□

Lemma 3.3. *Suppose that conditions* (B1)-(B4) *are hold. Let* $\{u_n\}$ *be a sequence which is generated by Algorithm A. Moreover, sequence* $\phi_n \subset (0,1)$ *satisfying the following conditions:*

$$\lim_{n \to \infty} \phi_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \phi_n = +\infty.$$

If there is a weakly convergent subsequence $\{u_{n_k}\}$ *to* $\hat{u} \in \mathbb{E}$ *and*

$$\lim_{n \to \infty} \|u_n - v_n\| = 0,$$

then $\hat{u} \in \Omega$.

Proof. It is given that $v_{n_k} = P_{\mathbb{K}}[u_{n_k} - \rho_{n_k}\mathcal{G}(u_{n_k})]$ which is equivalent to

$$\langle u_{n_k} - \rho_{n_k}\mathcal{G}(u_{n_k}) - v_{n_k}, y - v_{n_k}\rangle \leq 0, \ \forall\, y \in \mathbb{K}. \tag{3.9}$$

The above inequality implies that

$$\langle u_{n_k} - v_{n_k}, y - v_{n_k}\rangle \leq \rho_{n_k}\langle \mathcal{G}(u_{n_k}), y - v_{n_k}\rangle, \ \forall\, y \in \mathbb{K}. \tag{3.10}$$

Thus, we obtain

$$\frac{1}{\rho_{n_k}}\langle u_{n_k} - v_{n_k}, y - v_{n_k}\rangle + \langle \mathcal{G}(u_{n_k}), v_{n_k} - u_{n_k}\rangle \leq \langle \mathcal{G}(u_{n_k}), y - u_{n_k}\rangle, \ \forall\, y \in \mathbb{K}. \tag{3.11}$$

Due to boundedness of the sequence $\{u_{n_k}\}$ implies that $\{\mathcal{G}(u_{n_k})\}$ is also bounded. Now, using $\lim_{k\to\infty}\|u_{n_k} - v_{n_k}\| = 0$ and $\lim_{k\to\infty}\rho_{n_k} = \rho > 0$, and $k \to \infty$ in (3.11), we obtain

$$\liminf_{k\to\infty}\langle \mathcal{G}(u_{n_k}), y - u_{n_k}\rangle \geq 0, \ \forall\, y \in \mathbb{K}. \tag{3.12}$$

Moreover, we have

$$\begin{aligned}\langle \mathcal{G}(v_{n_k}), y - v_{n_k}\rangle &= \langle \mathcal{G}(v_{n_k}) - \mathcal{G}(u_{n_k}), y - u_{n_k}\rangle \\ &\quad + \langle \mathcal{G}(u_{n_k}), y - u_{n_k}\rangle + \langle \mathcal{G}(v_{n_k}), u_{n_k} - v_{n_k}\rangle.\end{aligned} \tag{3.13}$$

Since $\lim_{k\to\infty}\|u_{n_k} - v_{n_k}\| = 0$ and $\mathcal{G}$ is L-Lipschitz continuous on $\mathbb{E}$ implies that

$$\lim_{k\to\infty}\|\mathcal{G}(u_{n_k}) - \mathcal{G}(v_{n_k})\| = 0, \tag{3.14}$$

which together with (3.13) and (3.14), we obtain

$$\liminf_{k\to\infty}\langle \mathcal{G}(v_{n_k}), y - v_{n_k}\rangle \geq 0, \ \forall\, y \in \mathbb{K}. \tag{3.15}$$

Next, we need to prove that $\hat{u}$ belongs to solution set Ω. Let consider a sequence of positive numbers $\{\epsilon_k\}$ that is decreasing and converge to zero.

For each k, we denote m_k by the smallest positive integer such that

$$\langle \mathcal{G}(u_{n_i}), y - u_{n_i}\rangle + \epsilon_k \geq 0, \ \forall\, i \geq m_k. \tag{3.16}$$

Due to $\{\epsilon_k\}$ is decreasing and $\{m_k\}$ is increasing.

Case 1: If there is a subsequence $\{u_{n_{m_{k_j}}}\}$ of $\{u_{n_{m_k}}\}$ such that $\mathcal{G}(u_{n_{m_{k_j}}}) = 0$ for all j. Let $j \to \infty$, we obtain

$$\langle \mathcal{G}(\hat{u}), y - \hat{u}\rangle = \lim_{j\to\infty}\langle \mathcal{G}(u_{n_{m_{k_j}}}), y - \hat{u}\rangle = 0. \tag{3.17}$$

Hence $\hat{u} \in \mathbb{K}$, therefore we obtain $\hat{u} \in \Omega$.

Case 2: If there exits $N_0 \in \mathbb{N}$ such that for all $n_{m_k} \geq N_0$, $\mathcal{G}(u_{n_{m_k}}) \neq 0$. Consider that

$$\Im_{n_{m_k}} = \frac{\mathcal{G}(u_{n_{m_k}})}{\|\mathcal{G}(u_{n_{m_k}})\|^2}, \ \forall\, n_{m_k} \geq N_0. \tag{3.18}$$

Due to the above definition, we obtain

$$\langle \mathcal{G}(u_{n_{m_k}}), \Im_{n_{m_k}} \rangle = 1, \ \forall\, n_{m_k} \geq N_0. \tag{3.19}$$

Moreover, expressions (3.16) and (3.19), for all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{G}(u_{n_{m_k}}), \ y + \epsilon_k \Im_{n_{m_k}} - u_{n_{m_k}} \rangle \geq 0. \tag{3.20}$$

Due to the pseudomonotonicity of $\mathcal{G}$ for $n_{m_k} \geq N_0$,

$$\langle \mathcal{G}(y + \epsilon_k \Im_{n_{m_k}}), \ y + \epsilon_k \Im_{n_{m_k}} - u_{n_{m_k}} \rangle \geq 0. \tag{3.21}$$

For all $n_{m_k} \geq N_0$, we have

$$\begin{aligned} \langle \mathcal{G}(y), y - u_{n_{m_k}} \rangle \geq & \langle \mathcal{G}(y) - \mathcal{G}(y + \epsilon_k \Im_{n_{m_k}}), \ y + \epsilon_k \Im_{n_{m_k}} - u_{n_{m_k}} \rangle \\ & - \epsilon_k \langle \mathcal{G}(y), \Im_{n_{m_k}} \rangle. \end{aligned} \tag{3.22}$$

Due to $\{u_{n_k}\}$ weakly converges to $\hat{u} \in \mathbb{K}$ and through $\mathcal{G}$ is sequentially weakly continuous on the set $\mathbb{K}$, we get $\{\mathcal{G}(u_{n_k})\}$ weakly converges to $\mathcal{G}(\hat{u})$.

Suppose that $\mathcal{G}(\hat{u}) \neq 0$, we have

$$\|\mathcal{G}(\hat{u})\| \leq \liminf_{k \to \infty} \|\mathcal{G}(u_{n_k})\|. \tag{3.23}$$

Since $\{u_{n_{m_k}}\} \subset \{u_{n_k}\}$ and $\lim_{k \to \infty} \epsilon_k = 0$, we have

$$\begin{aligned} 0 & \leq \lim_{k \to \infty} \|\epsilon_k \Im_{n_{m_k}}\| \\ & = \lim_{k \to \infty} \frac{\epsilon_k}{\|\mathcal{G}(u_{n_{m_k}})\|} \\ & \leq \frac{0}{\|\mathcal{G}(\hat{u})\|} = 0. \end{aligned} \tag{3.24}$$

Next, consider $k \to \infty$ in (3.22), we obtain

$$\langle \mathcal{G}(y), y - \hat{u} \rangle \geq 0, \ \forall\, y \in \mathbb{K}. \tag{3.25}$$

By the use of Minty Lemma 2.5, we infer $\hat{u} \in \Omega$. □

Theorem 3.4. *Assume that an operator $\mathcal{G} : \mathbb{K} \to \mathbb{E}$ satisfies the conditions* (B1)-(B4) *and u^* belongs to the solution set Ω. Moreover, sequence $\{\phi_n\} \subset$* (0, 1) *satisfying the following conditions:*

$$\lim_{n \to \infty} \phi_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \phi_n = +\infty.$$

Then the sequences $\{u_n\}$, $\{v_n\}$ and $\{t_n\}$ generated by Algorithm A converge strongly to $u^* = P_\Omega \circ \mathrm{g}(u^*)$.

Proof. By using Lemma 3.2, we have

$$\|t_n - u^*\|^2 \le \big\|u_n - u^*\big\|^2 - \left(1 - \mu^2 \frac{\rho_n^2}{\rho_{n+1}^2}\right)\big\|u_n - v_n\big\|^2. \tag{3.26}$$

Given that $\rho_n \to \rho$, so there exists a fixed number $\epsilon \in (0, 1-\mu^2)$ such that

$$\lim_{n\to\infty}\left(1 - \mu^2 \frac{\rho_n^2}{\rho_{n+1}^2}\right) = 1 - \mu^2 > \epsilon > 0.$$

Thus, there is a finite number $N_1 \in \mathbb{N}$ such that

$$\left(1 - \mu^2 \frac{\rho_n^2}{\rho_{n+1}^2}\right) > \epsilon > 0,\ \forall\, n \ge N_1. \tag{3.27}$$

Thus, we obtain

$$\|t_n - u^*\|^2 \le \|u_n - u^*\|^2,\ \forall\, n \ge N_1. \tag{3.28}$$

It is given that $u^* \in \Omega$. From sequence $\{u_{n+1}\}$ and the reason that g is a contraction with constant $\xi \in [0,1)$ and $n \ge N_1$, we have

$$\begin{aligned}
\big\|u_{n+1} - u^*\big\| &= \big\|\phi_n \mathrm{g}(u_n) + (1-\phi_n)t_n - u^*\big\| \\
&= \big\|\phi_n[\mathrm{g}(u_n) - u^*] + (1-\phi_n)[t_n - u^*]\big\| \\
&= \big\|\phi_n[\mathrm{g}(u_n) + \mathrm{g}(u^*) - \mathrm{g}(u^*) - u^*] \\
&\quad + (1-\phi_n)[t_n - u^*]\big\| \\
&\le \phi_n\big\|\mathrm{g}(u_n) - \mathrm{g}(u^*)\big\| + \phi_n\big\|\mathrm{g}(u^*) - u^*\big\| \\
&\quad + (1-\phi_n)\big\|t_n - u^*\big\| \\
&\le \phi_n\xi\big\|u_n - u^*\big\| + \phi_n\big\|\mathrm{g}(u^*) - u^*\big\| \\
&\quad + (1-\phi_n)\big\|t_n - u^*\big\|.
\end{aligned} \tag{3.29}$$

Combining expressions (3.28) with (3.29) and $\{\phi_n\} \subset (0,1)$, we deduce that

$$\begin{aligned}
\big\|u_{n+1} - u^*\big\| &\le \phi_n\xi\big\|u_n - u^*\big\| + \phi_n\big\|\mathrm{g}(u^*) - u^*\big\| \\
&\quad + (1-\phi_n)\big\|u_n - u^*\big\| \\
&= [1 - \phi_n + \xi\phi_n]\big\|u_n - u^*\big\| \\
&\quad + \phi_n(1-\xi)\frac{\big\|\mathrm{g}(u^*) - u^*\big\|}{(1-\xi)} \\
&\le \max\left\{\big\|u_n - u^*\big\|, \frac{\big\|\mathrm{g}(u^*) - u^*\big\|}{(1-\xi)}\right\} \\
&\le \max\left\{\big\|u_{N_1} - u^*\big\|, \frac{\big\|\mathrm{g}(u^*) - u^*\big\|}{(1-\xi)}\right\}.
\end{aligned} \tag{3.30}$$

Therefore, we deduce that $\{u_n\}$ is a bounded sequence. Due to the continuity and monotonicity of the operator $\mathcal{G}$ implies that the solution set Ω is a closed and convex set (for more details see [22, 21]). Since the mapping is a contraction and so does $P_\Omega \circ \mathrm{g}$.

Now, we are in position to use the Banach contraction theorem for the existence of a fixed point of $u^* \in \Omega$ such that

$$u^* = P_\Omega(\mathrm{g}(u^*)).$$

By using Lemma 2.1 (ii), we have

$$\langle \mathrm{g}(u^*) - u^*, y - u^* \rangle \leq 0, \ \forall\, y \in \Omega. \tag{3.31}$$

It is given that $u_{n+1} = \phi_n \mathrm{g}(u_n) + (1-\phi_n)t_n$, and using Lemma 2.2 (i) and Lemma 3.2, we have

$$\begin{aligned}
\big\|u_{n+1} - u^*\big\|^2 &= \big\|\phi_n \mathrm{g}(u_n) + (1-\phi_n)t_n - u^*\big\|^2 \\
&= \big\|\phi_n[\mathrm{g}(u_n) - u^*] + (1-\phi_n)[t_n - u^*]\big\|^2 \\
&= \phi_n\|\mathrm{g}(u_n) - u^*\|^2 + (1-\phi_n)\|t_n - u^*\|^2 \\
&\quad - \phi_n(1-\phi_n)\|\mathrm{g}(u_n) - t_n\|^2 \\
&\leq \phi_n\|\mathrm{g}(u_n) - u^*\|^2 \\
&\quad + (1-\phi_n)\Big[\big\|u_n - u^*\big\|^2 - \Big(1 - \mu^2\frac{\rho_n^2}{\rho_{n+1}^2}\Big)\big\|u_n - v_n\big\|^2\Big] \\
&\quad - \phi_n(1-\phi_n)\|\mathrm{g}(u_n) - t_n\|^2 \\
&\leq \phi_n\|\mathrm{g}(u_n) - u^*\|^2 + \|u_n - u^*\|^2 \\
&\quad - (1-\phi_n)\Big(1 - \mu^2\frac{\rho_n^2}{\rho_{n+1}^2}\Big)\big\|u_n - v_n\big\|^2.
\end{aligned} \tag{3.32}$$

The remainder of the proof shall be divided into the following two parts:

Case 1: Assume that there is a fixed number $N_2 \in \mathbb{N}$ $(N_2 \geq N_1)$ such that

$$\|u_{n+1} - u^*\| \leq \|u_n - u^*\|, \ \forall\, n \geq N_2. \tag{3.33}$$

Then, $\lim_{n\to\infty}\|u_n - u^*\|$ exists and let $\lim_{n\to\infty}\|u_n - u^*\| = l$. From expression (3.32), we have

$$\begin{aligned}
&(1-\phi_n)\Big(1 - \mu^2\frac{\rho_n^2}{\rho_{n+1}^2}\Big)\big\|u_n - v_n\big\|^2 \\
&\leq \phi_n\|\mathrm{g}(u_n) - u^*\|^2 + \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2.
\end{aligned} \tag{3.34}$$

Due to the existence of $\lim_{n\to\infty}\|u_n - u^*\| = l$, and $\phi_n \to 0$, we infer that

$$\lim_{n\to\infty}\|u_n - v_n\| = 0. \tag{3.35}$$

It follows that

$$\begin{aligned}\|t_n - v_n\| &= \|v_n + \rho_n[\mathcal{G}(u_n) - \mathcal{G}(v_n)] - v_n\| \\ &\le \rho_0 L\|u_n - v_n\|.\end{aligned}$$

The above expression implies that

$$\lim_{n\to\infty} \|t_n - v_n\| = 0. \tag{3.36}$$

It follows that

$$\lim_{n\to\infty} \|u_n - t_n\| \le \lim_{n\to\infty} \|u_n - v_n\| + \lim_{n\to\infty} \|v_n - t_n\| = 0. \tag{3.37}$$

We can also obtain

$$\begin{aligned}\big\|u_{n+1} - u_n\big\| &= \big\|\phi_n \mathrm{g}(u_n) + (1-\phi_n)t_n - u_n\big\| \\ &= \big\|\phi_n[\mathrm{g}(u_n) - u_n] + (1-\phi_n)[t_n - u_n]\big\| \\ &\le \phi_n\big\|\mathrm{g}(u_n) - u_n\big\| + (1-\phi_n)\big\|t_n - u_n\big\| \\ &\longrightarrow 0, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.38}$$

The above expression implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.39}$$

The sequence $\{u_n\}$ is bounded and implies that the sequences $\{v_n\}$ and $\{t_n\}$ are also bounded. Thus, we can take a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly converges to some $\hat{u} \in \mathbb{E}$. Moreover, due to $\|u_n - v_n\| \to 0$, we have $\hat{u} \in \Omega$. It follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle \mathrm{g}(u^*) - u^*, u_n - u^*\rangle &= \limsup_{k\to\infty}\langle \mathrm{g}(u^*) - u^*, u_{n_k} - u^*\rangle \\ &= \langle \mathrm{g}(u^*) - u^*, \hat{u} - u^*\rangle \\ &\le 0. \end{aligned} \tag{3.40}$$

We have $\lim_{n\to\infty} \big\|u_{n+1} - u_n\big\| = 0$. It follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle \mathrm{g}(u^*) - u^*, u_{n+1} - u^*\rangle &\le \limsup_{n\to\infty}\langle \mathrm{g}(u^*) - u^*, u_{n+1} - u_n\rangle \\ &\quad + \limsup_{n\to\infty}\langle \mathrm{g}(u^*) - u^*, u_n - u^*\rangle \\ &\le 0. \end{aligned} \tag{3.41}$$

From Lemma 2.2(ii) and Lemma 3.2 for all $n \geq N_2$, we obtain

$$\begin{aligned}
\|u_{n+1} - u^*\|^2 &= \|\phi_n \mathrm{g}(u_n) + (1-\phi_n)t_n - u^*\|^2 \\
&= \|\phi_n[\mathrm{g}(u_n) - u^*] + (1-\phi_n)[t_n - u^*]\|^2 \\
&\leq (1-\phi_n)^2\|t_n - u^*\|^2 \\
&\quad + 2\phi_n\langle \mathrm{g}(u_n) - u^*, (1-\phi_n)[t_n - u^*] + \phi_n[\mathrm{g}(u_n) - u^*]\rangle \\
&= (1-\phi_n)^2\|t_n - u^*\|^2 \\
&\quad + 2\phi_n\langle \mathrm{g}(u_n) - \mathrm{g}(u^*) + \mathrm{g}(u^*) - u^*, u_{n+1} - u^*\rangle \\
&= (1-\phi_n)^2\|t_n - u^*\|^2 + 2\phi_n\langle \mathrm{g}(u_n) - \mathrm{g}(u^*), u_{n+1} - u^*\rangle \\
&\quad + 2\phi_n\langle \mathrm{g}(u^*) - u^*, u_{n+1} - u^*\rangle \\
&\leq (1-\phi_n)^2\|t_n - u^*\|^2 + 2\phi_n\xi\|u_n - u^*\|\|u_{n+1} - u^*\| \\
&\quad + 2\phi_n\langle \mathrm{g}(u^*) - u^*, u_{n+1} - u^*\rangle \\
&\leq (1+\phi_n^2 - 2\phi_n)\|u_n - u^*\|^2 + 2\phi_n\xi\|u_n - u^*\|^2 \\
&\quad + 2\phi_n\langle \mathrm{g}(u^*) - u^*, u_{n+1} - u^*\rangle \\
&= (1-2\phi_n)\|u_n - u^*\|^2 + \phi_n^2\|u_n - u^*\|^2 + 2\phi_n\xi\|u_n - u^*\|^2 \\
&\quad + 2\phi_n\langle \mathrm{g}(u^*) - u^*, u_{n+1} - u^*\rangle \\
&= \left[1 - 2\phi_n(1-\xi)\right]\|u_n - u^*\|^2 \\
&\quad + 2\phi_n(1-\xi)\left[\frac{\phi_n\|u_n - u^*\|^2}{2(1-\xi)} + \frac{\langle \mathrm{g}(u^*) - u^*, u_{n+1} - u^*\rangle}{1-\xi}\right].
\end{aligned} \tag{3.42}$$

It follows from expressions (3.41) and (3.42), we obtain

$$\limsup_{n\to\infty}\left[\frac{\phi_n\|u_n - u^*\|^2}{2(1-\xi)} + \frac{\langle \mathrm{g}(u^*) - u^*, u_{n+1} - u^*\rangle}{1-\xi}\right] \leq 0. \tag{3.43}$$

Let choose $n \geq N_3 \in \mathbb{N}$ ($N_3 \geq N_2$) large enough such that $2\phi_n(1-\xi) < 1$. Now, by using expressions (3.42) and (3.43) and applying Lemma 2.3, we conclude that $\|u_n - u^*\| \to 0$, as $n \to \infty$.

Case 2: Assume there is a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - u^*\| \leq \|u_{n_{i+1}} - u^*\|, \ \forall i \in \mathbb{N}.$$

Then, by Lemma 2.4, there is a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to \infty$, such that

$$\|u_{m_k} - u^*\| \leq \|u_{m_{k+1}} - u^*\|$$

and

$$\|u_k - u^*\| \leq \|u_{m_{k+1}} - u^*\|, \text{ for all } k \in \mathbb{N}. \tag{3.44}$$

As similar to Case 1, from (3.32), we have

$$\begin{aligned}&(1-\phi_{m_k})\left(1-\mu^2\frac{\rho_{m_k}^2}{\rho_{m_k+1}^2}\right)\big\|u_{m_k}-v_{m_k}\big\|^2\\&\leq \phi_{m_k}\|\mathrm{g}(u_{m_k})-u^*\|^2+\|u_{m_k}-u^*\|^2-\|u_{m_k+1}-u^*\|^2.\end{aligned} \tag{3.45}$$

Due to $\phi_{m_k} \to 0$, we deduce the following:

$$\lim_{k\to\infty}\|u_{m_k}-v_{m_k}\| = 0. \tag{3.46}$$

Similar to above case we can prove that

$$\begin{aligned}\lim_{k\to\infty}\|u_{m_k}-t_{m_k}\| &= \lim_{k\to\infty}\|v_{m_k}-t_{m_k}\|\\&= 0.\end{aligned} \tag{3.47}$$

Also, we obtain

$$\begin{aligned}\big\|u_{m_k+1}-u_{m_k}\big\| &= \big\|\phi_{m_k}\mathrm{g}(u_{m_k})+(1-\phi_{m_k})t_{m_k}-u_{m_k}\big\|\\&= \big\|\phi_{m_k}[\mathrm{g}(u_{m_k})-u_{m_k}]+(1-\phi_{m_k})[t_{m_k}-u_{m_k}]\big\|\\&\leq \phi_{m_k}\big\|\mathrm{g}(u_{m_k})-u_{m_k}\big\|+(1-\phi_{m_k})\big\|t_{m_k}-u_{m_k}\big\|\\&\longrightarrow 0.\end{aligned} \tag{3.48}$$

We have to use the same justification as in the Case 1, such that

$$\limsup_{k\to\infty}\langle \mathrm{g}(u^*)-u^*, u_{m_k+1}-u^*\rangle \leq 0. \tag{3.49}$$

By the use of expressions (3.42) and (3.44), we have

$$\begin{aligned}\big\|u_{m_k+1}-u^*\big\|^2 &\leq \big[1-2\phi_{m_k}(1-\xi)\big]\big\|u_{m_k}-u^*\big\|^2+2\phi_{m_k}(1-\xi)\\&\quad\times\left[\frac{\phi_{m_k}\big\|u_{m_k}-u^*\big\|^2}{2(1-\xi)}+\frac{\langle \mathrm{g}(u^*)-u^*, u_{m_k+1}-u^*\rangle}{1-\xi}\right]\\&\leq \big[1-2\phi_{m_k}(1-\xi)\big]\big\|u_{m_k+1}-u^*\big\|^2+2\phi_{m_k}(1-\xi)\\&\quad\times\left[\frac{\phi_{m_k}\big\|u_{m_k}-u^*\big\|^2}{2(1-\xi)}+\frac{\langle \mathrm{g}(u^*)-u^*, u_{m_k+1}-u^*\rangle}{1-\xi}\right].\end{aligned} \tag{3.50}$$

It follows that

$$\begin{aligned}\big\|u_{m_k+1}-u^*\big\|^2 &\leq \frac{\phi_{m_k}\big\|u_{m_k}-u^*\big\|^2}{2(1-\xi)}\\&\quad+\frac{\langle \mathrm{g}(u^*)-u^*, u_{m_k+1}-u^*\rangle}{1-\xi}.\end{aligned} \tag{3.51}$$

Since $\phi_{m_k} \to 0$, as $k \to \infty$ and $\left\|u_{m_k} - u^*\right\|$ is a bounded sequence, expressions (3.49) and (3.51) implies that

$$\|u_{m_k+1} - u^*\|^2 \to 0, \text{ as } k \to \infty. \tag{3.52}$$

The above expression with (3.44) implies that

$$\begin{aligned} \lim_{k\to\infty} \|u_k - u^*\|^2 &\leq \lim_{k\to\infty} \|u_{m_k+1} - u^*\|^2 \\ &\leq 0. \end{aligned} \tag{3.53}$$

Consequently, $u_n \to u^*$ as $n \to \infty$. This completes the proof. □

4. Numerical Illustrations

The numerical results given in this section show the efficacy of Algorithm A for six test problems, two of which are monotone and the other four are pseudomonotone variational inequalities.

Example 4.1. First consider the HpHard problem which is taken from [9]. This example was considered by many authors for experimental tests (see, [7, 10, 32]), while $\mathcal{G} : \mathbb{R}^m \to \mathbb{R}^m$ is an operator taken as $\mathcal{G}(u) = Mu + q$ with $q \in \mathbb{R}^m$ and

$$M = NN^T + B + D,$$

where N is an $m \times m$ matrix, B is an $m \times m$ skew-symmetric matrix and D is an $m \times m$ positive definite diagonal matrix. The set $\mathbb{K}$ is taken in the following way:

$$\mathbb{K} = \{u \in \mathbb{R}^m : Qu \leq b\},$$

where Q is an $100 \times m$ matrix and b is a nonnegative vector in $\mathbb{R}^m$. It is clear that $\mathcal{G}$ is monotone and Lipschitz continuous through $L = \|M\|$. During this experiment, the initial point is $u_0 = (1, 1, \cdots, 1)$ and $D_n = \|u_n - v_n\| \leq Tolerance = 10^{-3}$. Furthermore, control conditions $\rho_0 = \frac{0.5}{\|M\|}$ and $\mu = 0.7$ for Algorithm 1 (EgM-1) in [41]; $\rho_0 = \frac{0.5}{\|M\|}$, $\mu = 0.7$ and $\phi_n = \frac{1}{40n+100}$ for Algorithm 2 (EgM-2) in [41]; $\rho_0 = \frac{0.5}{\|M\|}$, $\mu = 0.7$, $\phi_n = \frac{1}{2n+4}$ and $\mathrm{g}(u) = \frac{u}{2}$ for Algorithm A (EgM-3).

The numerical and graphical results of three methods are shown in Figures 1-5 and Table 1.

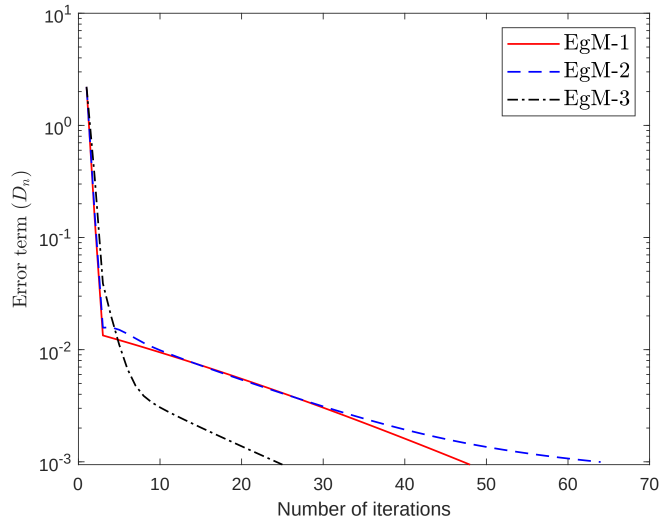

FIGURE 1. Numerical behavior of Algorithm A relative to Algorithm 1 in [41] and Algorithm 2 in [41] for Example 4.1 when $m = 5$.

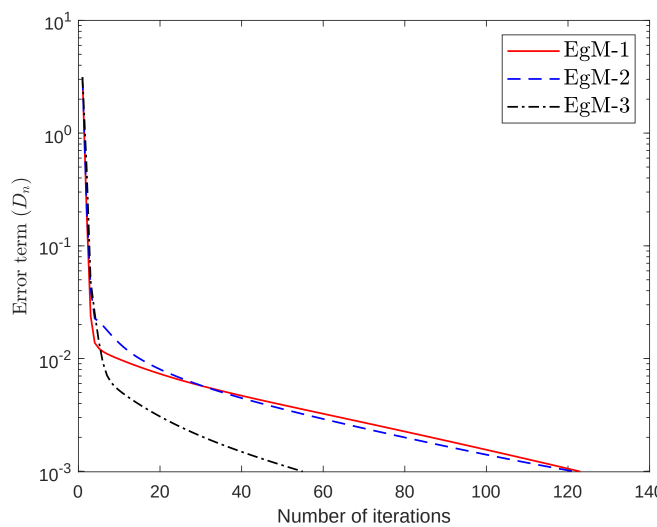

FIGURE 2. Numerical behavior of Algorithm A relative to Algorithm 1 in [41] and Algorithm 2 in [41] for Example 4.1 when $m = 10$.

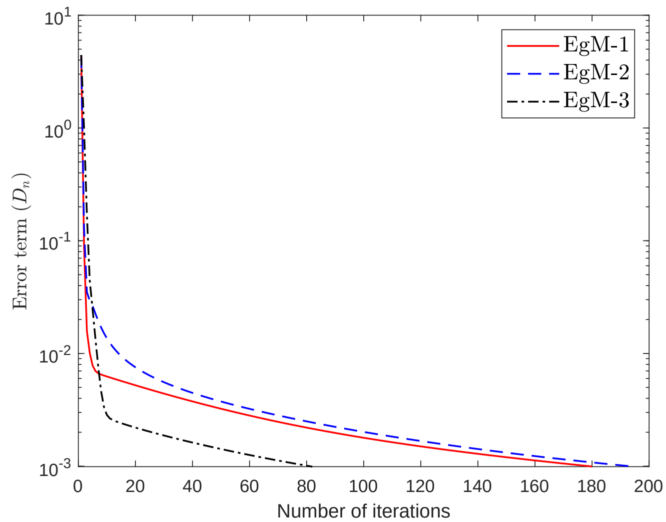

FIGURE 3. Numerical behavior of Algorithm A relative to Algorithm 1 in [41] and Algorithm 2 in [41] for Example 4.1 when $m = 20$.

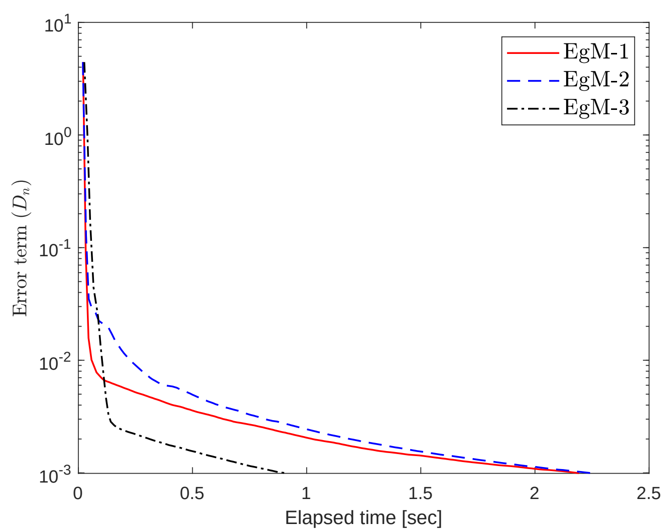

FIGURE 4. Numerical behavior of Algorithm A relative to Algorithm 1 in [41] and Algorithm 2 in [41] for Example 4.1 when $m = 20$.

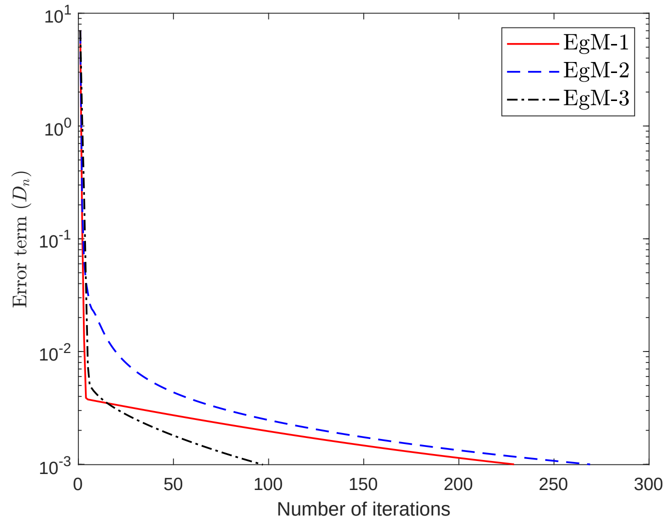

FIGURE 5. Numerical behavior of Algorithm A relative to Algorithm 1 in [41] and Algorithm 2 in [41] for Example 4.1 when $m = 50$.

TABLE 1. Numerical comparison values for Figures 1-5.

	EgM-1 [41]		EgM-2 [41]		EgM-3	
m	iter.	time	iter.	time	iter.	time
5	48	0.5567	64	0.7195	25	0.3387
10	123	1.5047	122	1.4211	55	0.6207
20	180	2.1935	194	2.2479	82	0.9162
50	229	3.3714	269	3.5252	97	1.3614

Example 4.2. Suppose that $\mathbb{E} = L^2([0,1])$ is a Hilbert space through an inner product

$$\langle u, v\rangle = \int_0^1 u(t)v(t)dt, \ \forall\, u, v \in \mathbb{E},$$

where the induced norm

$$\|u\| = \sqrt{\int_0^1 |u(t)|^2 dt}.$$

Let $\mathbb{K} := \{u \in L^2([0,1]) : \|u\| \leq 1\}$ be the unit ball and $\mathcal{G} : \mathbb{K} \to \mathbb{E}$ be defined by

$$\mathcal{G}(u)(t) = \int_0^1 \big(u(t) - H(t,s)f(u(s))\big)ds + g(t),$$

where

$$H(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, \quad f(u) = \cos u, \quad g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

Then, it is easily to see that $\mathcal{G}$ is Lipschitz-continuous with Lipschitz constant $L = 2$ and monotone [39]. Figures 6-8 and Table 2 show the numerical results by choosing different values of u_0. The numerical results of three methods are shown in Figures 6-8 and Table 2.

In this experiment, we take the different initial points u_0 and

$$D_n = \|u_n - v_n\| \leq Tolerance = 10^{-3}.$$

Moreover, the control parameters $\rho_0 = \frac{0.6}{L}$ and $\mu = 0.45$ for Algorithm 1 (EgM-1) in [41]; $\rho_0 = \frac{0.6}{L}$, $\mu = 0.45$ and $\phi_n = \frac{1}{100(n+2)}$ for Algorithm 2 (EgM-2) in [41]; $\rho_0 = \frac{0.6}{L}$, $\mu = 0.45$, $\phi_n = \frac{1}{n+2}$ and $\mathrm{g}(u) = \frac{u}{3}$ for Algorithm A (EgM-3).

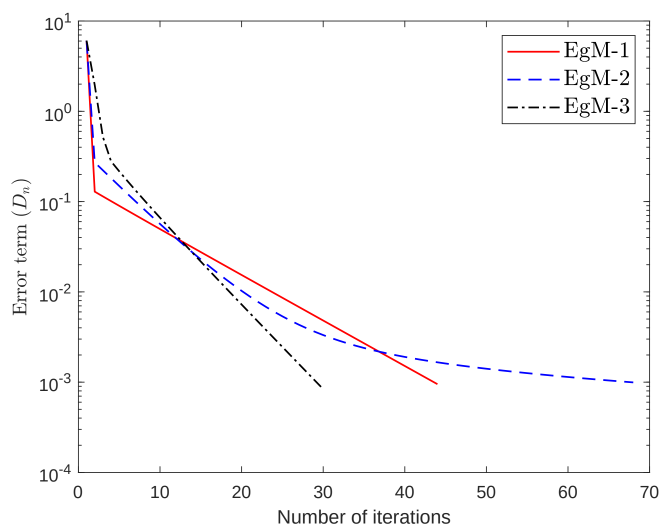

Figure 6. Numerical behavior of Algorithm A relative to Algorithm 1 in [41] and Algorithm 2 in [41] for Example 4.2 when $u_0 = 1$.

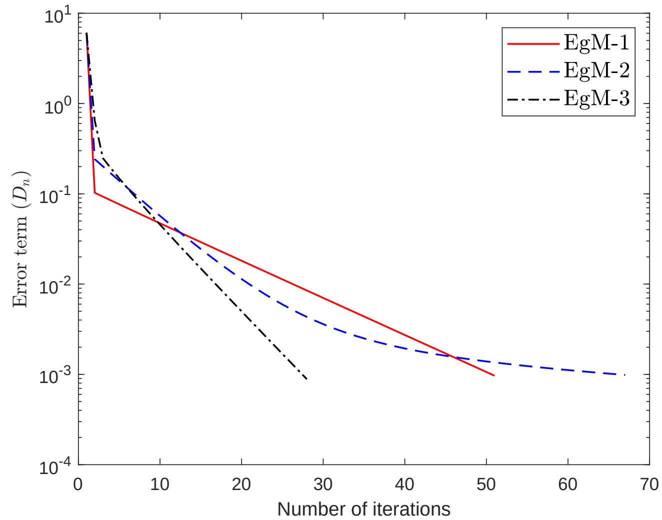

Figure 7. Numerical behavior of Algorithm A relative to Algorithm 1 in [41] and Algorithm 2 in [41] for Example 4.2 when $u_0 = e^t$.

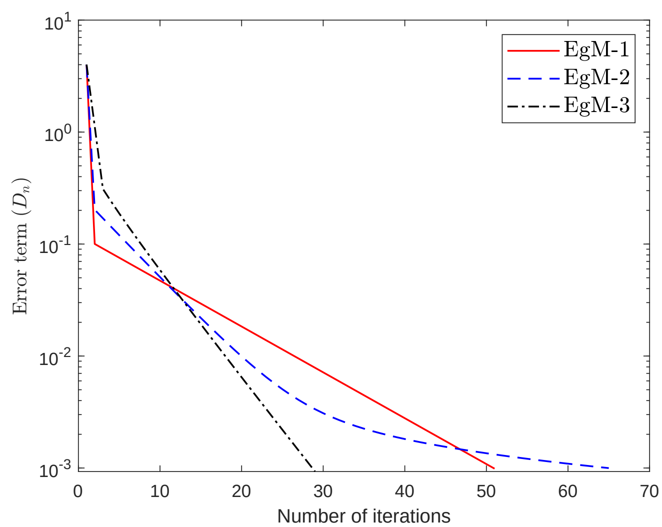

Figure 8. Numerical behavior of Algorithm A relative to Algorithm 1 in [41] and Algorithm 2 in [41] for Example 4.2 when $u_0 = \sin(t)$.

TABLE 2. Numerical comparison values for Figures 6-8.

	EgM-1 [41]		EgM-2 [41]		EgM-3	
u_0	iter.	time	iter.	time	iter.	time
1	44	0.041021	68	0.042249	30	0.019874
e^t	51	0.036009	67	0.039082	28	0.020269
$\sin(t)$	51	0.036100	65	0.038172	29	0.020530

Example 4.3. Consider the nonlinear complementarity problem of Kojima-Shindo where the feasible set $\mathbb{K}$ is

$$\mathbb{K} = \{u \in \mathbb{R}^4 : 1 \leq u_i \leq 5,\ i = 1, 2, 3, 4\}$$

and the mapping $\mathcal{G} : \mathbb{R}^4 \to \mathbb{R}^4$ is evaluated by

$$\mathcal{G}(u) = \begin{pmatrix} u_1 + u_2 + u_3 + u_4 - 4u_2u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_3 \end{pmatrix}.$$

Then, it is easy to see that $\mathcal{G}$ is not monotone on the set $\mathbb{K}$. By using the Monte-Carlo approach [11], it can be shown that $\mathcal{G}$ is pseudo-monotone on $\mathbb{K}$. This problem has a unique solution $u^* = (5, 5, 5, 5)^T$. Actually, in general, it is a very difficult task to check the pseudomonotonicity of any mapping $\mathcal{G}$ in practice. We here employ the Monte Carlo approach according to the definition of pseudomonotonicity: Generate a large number of pairs of points u and v uniformly in $\mathbb{K}$ satisfying $\mathcal{G}(u)^T(v-u) \geq 0$ and then check if $\mathcal{G}(v)^T(v-u) \geq 0$.

In this experiment, we take different initial points and $D_n = \|u_n - v_n\|$. Moreover, control parameters $\rho_0 = 0.33$, $\mu = 0.25$, $\phi_n = \frac{1}{2(n+2)}$ and $\mathrm{g}(u) = \frac{u}{2}$ for Algorithm A. Numerical results regarding the third example are shown in Table 3.

TABLE 3. Numerical behavior of Algorithm A for Example 4.3.

TOL	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
u_0	Iter.	Iter.	Iter.	Iter.	time	time	time	time
$[-2, 2, 8, 10]^T$	11	49	477	4873	0.068822	0.217773	3.051465	41.9378342
$[-1, 1, 5, 6]^T$	10	47	475	4873	0.073831	0.216922	2.108431	42.1511784
$[-5, 2, -1, 2]^T$	8	42	477	4873	0.055413	0.215572	3.234742	43.0306253
$[1, 2, 3, 4]^T$	6	1104	979	4873	0.030871	8.053123	6.136634	42.2317051

Example 4.4. For this example, consider the quadratic fractional programming problem in the following form [11]:

$$\begin{cases} \min f(u) = \dfrac{u^T Q u + a^T u + a_0}{b^T u + b_0}, \\ \text{subject to } u \in \mathbb{K} = \{u \in \mathbb{R}^4 : b^T u + b_0 > 0\}, \end{cases}$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ a_0 = -2, \ \text{ and } \ b_0 = 4.$$

Then, it is easy to verify that Q is symmetric and positive definite on $\mathbb{R}^4$ and consequently f is pseudoconvex on $\mathbb{K}$. Hence, ∇f is pseudomonotone. Using the quotient rule, we obtain

$$\nabla f(u) = \frac{(b^T u + b_0)(2Qu + a) - b(u^T Q + a^T u + a_0)}{(b^T u + b_0)^2}. \tag{4.1}$$

In this point of view, we can set $\mathcal{G} = \nabla f$ in Theorem 3.4. We minimize f over $\mathbb{K} = \{u \in \mathbb{R}^4 : 1 \le u_i \le 10, \ i = 1, 2, 3, 4\}$. This problem has a unique solution $u^* = (1, 1, 1, 1)^T \in \mathbb{K}$.

In this experiment, we take different initial points and $D_n = \|u_n - v_n\|$. Moreover, control parameters $\rho_0 = 0.33$, $\mu = 0.25$, $\phi_n = \frac{1}{3(n+2)}$ and $\mathrm{g}(u) = \frac{u}{2}$ for Algorithm A. Numerical results regarding the fourth example are shown in Table 4.

TABLE 4. Numerical behavior of Algorithm A for Example 4.4.

TOL u_0	10^{-2} Iter.	10^{-3} Iter.	10^{-4} Iter.	10^{-5} Iter.	10^{-2} time	10^{-3} time	10^{-4} time	10^{-5} time
$[10, 10, 10, 10]^T$	40	40	89	867	0.279142	0.209284	0.39155201	7.4805312
$[10, 20, 30, 40]^T$	39	41	89	867	0.261706	0.177554	0.38415240	7.8989278
$[20, -20, 20, -20]^T$	37	33	89	867	0.127673	0.148191	0.37465402	7.1684634

Example 4.5. The fifth example was taken from [31] where $\mathcal{G} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathcal{G}(u) = \begin{pmatrix} 0.5u_1u_2 - 2u_2 - 10^7 \\ -4u_1 - 0.1u_2^2 - 10^7 \end{pmatrix},$$

where $\mathbb{K} = \{u \in \mathbb{R}^2 : (u_1 - 2)^2 + (u_2 - 2)^2 \le 1\}$. It can easily see that $\mathcal{G}$ is Lipschitz continuous with $L = 5$ and $\mathcal{G}$ is not monotone on $\mathbb{K}$ but pseudomonotone. Here, the above problem has unique solution $u^* = (2.707, 2.707)^T$.

In this experiment, we take different initial points and $D_n = \|u_n - v_n\|$. Moreover, control parameters $\rho_0 = 0.53$, $\mu = 0.33$, $\phi_n = \frac{1}{3(n+2)}$ and $g(u) = \frac{u}{3}$ for Algorithm A. Numerical results for fifth example are shown in Table 5.

Table 5. Numerical behavior of Algorithm A for Example 4.5.

TOL u_0	10^{-2} Iter.	10^{-3} Iter.	10^{-4} Iter.	10^{-5} Iter.	10^{-2} time	10^{-3} time	10^{-4} time	10^{-5} time
$[0,0]^T$	7	27	263	2560	0.6069327	1.8075131	13.1206618	101.2023872
$[10,10]^T$	7	26	265	2581	0.2718143	1.0576142	11.7641276	103.0158238
$[-5,-5]^T$	6	26	258	2587	0.3247282	1.0951934	11.0242628	103.9937285

Example 4.6. The last example is taken from [31] where $\mathcal{G} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathcal{G}(u) = \begin{pmatrix} (u_1^2 + (u_2-1)^2)(1+u_2) \\ -u_1^3 - u_1(u_2-1)^2 \end{pmatrix},$$

where $\mathbb{K} = \{u \in \mathbb{R}^2 : -10 \le u_i \le 10,\ i = 1,2\}$. It can easily see that $\mathcal{G}$ is Lipschitz continuous with $L = 5$ and $\mathcal{G}$ is not monotone on $\mathbb{K}$ but pseudomonotone.

In this experiment, we take different initial points and $D_n = \|u_n - v_n\|$. Moreover, control parameters $\rho_0 = 0.43$, $\mu = 0.72$, $\phi_n = \frac{1}{4(n+2)}$ and $g(u) = \frac{u}{4}$ for Algorithm A. Numerical results regarding the sixth example are shown in Table 6.

Table 6. Numerical behavior of Algorithm A for Example 4.6.

TOL u_0	10^{-2} Iter.	10^{-3} Iter.	10^{-4} Iter.	10^{-5} Iter.	10^{-2} time	10^{-3} time	10^{-4} time	10^{-5} time
$[0,0]^T$	14	201	2131	28871	0.1451215	1.954013	25.386392	201.565752
$[10,10]^T$	23	179	2001	25043	0.1318482	1.647422	23.264956	190.633297
$[-5,-5]^T$	40	399	3866	44756	0.5731771	3.971161	40.293646	387.086833

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References

[1] M.O. Aibinu and J.K. Kim, *Convergence analysis of viscosity implicit rules of nonexpansive mappings in Banach spaces,* Nonlinear Funct. Anal. Appl., **24**(4) (2019), 691–713.

[2] M.O. Aibinu and J.K. Kim, *On the rate of convergence of viscosity implicit iterative algorithms,* Nonlinear Funct. Anal. Appl., **25**(1) (2020), 135–152.

[3] A.S. Antipin, *On a method for convex programs using a symmetrical modification of the Lagrange function*, Ekonomika i Matematicheskie Metody, **12**(6) (1976), 1164-1173.

[4] H.H. Bauschke and P.L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, Springer, New York, 2011.

[5] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl., **148**(2) (2010), 318–335.

[6] Y. Censor, A. Gibali and S. Reich, *Extensions of korpelevich extragradient method for the variational inequality problem in euclidean space*, Optim., **61**(9) (2012), 1119–1132.

[7] Q.L. Dong, Y.J. Cho, L.L. Zhong and T.M. Rassias, *Inertial projection and contraction algorithms for variational inequalities*, J. Global Optim., **70**(3) (2017), 687–704.

[8] C.M. Elliott, *Variational and Quasivariational Inequalities Applications to Free–Boundary Problems.(Claudio Baiocchi And Antonio Capelo)*, SIAM Rev., **29**(2) (1987), 314–315.

[9] P.T. Harker and J.-S. Pang, *for the linear complementarity problem*, Lectures in Applied Mathematics, **26** (1990).

[10] D.V. Hieu, P.K. Anh and L.D. Muu, *Modified hybrid projection methods for finding common solutions to variational inequality problems*, Comput. Optim. Appl., **66**(1) (2016), 75–96.

[11] X. Hu and J. Wang, *Solving pseudomonotone variational inequalities and pseudoconvex optimization problems using the projection neural network*, IEEE Trans. Neural Networks, **17**(6) (2006), 1487–1499.

[12] A.N. Iusem and B.F. Svaiter, *A variant of Korpelevich's method for variational inequalities with a new search strategy*, Optim., **42**(4) (1997), 309–321.

[13] G. Kassay, J. Kolumbán and Z. Páles. On Nash stationary points, Publ. Math., **54**(3-4) (1999), 267–279.

[14] G. Kassay, J. Kolumbán and Z. Páles, *Factorization of minty and Stampacchia variational inequality systems*, Eur. J. Oper. Res., **143**(2) (2002), 377–389.

[15] J.K. Kim, A.H. Dar and Salahuddin, *Existence theorems for the generalized relaxed pseudomonotone variational inequalities* Nonlinear Funct. Anal. Appl., **25**(1) (2020), 25–34.

[16] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York and London, 1980.

[17] I. Konnov, *Equilibrium models and variational inequalities*, Elsevier, New York, 2007.

[18] G. Korpelevich, *The extragradient method for finding saddle points and other problems*, Matecon, **12** (1976), 747–756.

[19] R. Kraikaew and S. Saejung, *Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim. Theory Appl., **163**(2) (2013), 399–412.

[20] E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley Classics Library, New York, 1989.

[21] Z. Liu, S. Migórski and S. Zeng, Partial differential variational inequalities involving nonlocal boundary conditions in Banach spaces, J. Differ. Equ., **263**(7) (2017), 3989–4006.

[22] Z. Liu, S. Zeng and D. Motreanu, *Evolutionary problems driven by variational inequalities*, J. Differ. Equ., **260**(9) (216), 6787–6799.

[23] P.-E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal., **16**(7-8) (2008), 899–912.

[24] Y.V. Malitsky and V.V. Semenov, *An extragradient algorithm for monotone variational inequalities*, Cybern. Syst. Anal., **50**(2) (2014), 271–277.

[25] A.A. Mogbademu, *New iteration process for a general class of contractive mappings*, Acta Comment. Univ. Tartu. Math., **20**(2) (2016), 117–122.

[26] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl., **241**(1) (2000), 46–55.

[27] A. Nagurney, *A variational inequality approach*, Springer, Dordrecht, Boston, 1999.

[28] M.A. Noor, *General variational inequalities*, Appl. Math. Lett., **1**(2) (1988), 119–122.

[29] M.A. Noor, *An iterative algorithm for variational inequalities*, J. Math. Anal. Appl., **158**(2) (1991), 448–455.

[30] M.A. Noor, *Some iterative methods for nonconvex variational inequalities*, Comput. Math. Model., **21**(1) (2010), 97–108.

[31] Y. Shehu, Q.-L. Dong and D. Jiang, *Single projection method for pseudo-monotone variational inequality in Hilbert spaces*, Optim., **68**(1) (2018), 385–409.

[32] M.V. Solodov and B.F. Svaiter, *A new projection method for variational inequality problems*, SIAM J. Control Optim., **37**(3) (1999), 765–776.

[33] G. Stampacchia, *Formes bilinéaires coercitives sur les ensembles convexes*, Comptes Rendus Hebdomadaires Des Seances De L Academie Des Sciences, **258**(18) (1964), 4413–4416.

[34] W. Takahashi, *Nonlinear functional analysis*, Yokohama Publishers, Yokohama, 2000.

[35] W. Takahashi, *Introduction to nonlinear and convex analysis*, Yokohama Publishers, Yokohama, 2009.

[36] D.V. Thong and D.V. Hieu, *Modified subgradient extragradient method for variational inequality problems*, Numer. Algorithms, **79**(2) (2017), 597–610.

[37] D.V. Thong and D.V. Hieu, *Weak and strong convergence theorems for variational inequality problems*, Numer. Algorithms, **78**(4) (2017), 1045–1060.

[38] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., **38**(2) (2000), 431–446.

[39] D.V. Hieu, P.K. Anh and L.D. Muu, *Modified hybrid projection methods for finding common solutions to variational inequality problems*, Comput. Optim. Appl., **66**(1) (2017), 75–96.

[40] H.K. Xu, Another control condition in an iterative method for nonexpansive mappings, Bull. Aust. Math. Soc., **65**(1) (2002), 109–113.

[41] J. Yang, H. Liu and Z. Liu, *Modified subgradient extragradient algorithms for solving monotone variational inequalities*, Optim., **67**(12) (2018), 2247–2258.

[42] L. Zhang, C. Fang and S. Chen, *An inertial subgradient-type method for solving single-valued variational inequalities and fixed point problems*, Numer. Algorithms, **79**(3) (2018), 941–956.