



## SOME RADIUS RESULTS OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE SRIVASTAVA-ATTIYA OPERATOR

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**Abstract.** The main object of the present paper is to investigate some radius results of the functions  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  ( $|z| < 1$ ) with  $|a_n| \leq n$  for all  $n \in \mathbb{N}$ . Some applications for certain operator defined through convolution are also considered.

### 1. INTRODUCTION AND DEFINITIONS

Let  $\mathcal{A}$  denote the class of functions  $f(z)$  of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk  $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ . Also let  $\mathcal{S}$  denote the class of functions in  $\mathcal{A}$  which are univalent in the unit disk  $\mathbb{U}$ .

Here and in the following, let  $\mathbb{C}$ ,  $\mathbb{N}$  and  $\mathbb{Z}_0^-$  be the sets of complex numbers, positive integers, and non-positive integers, respectively. Further we let  $\mathcal{P}(\alpha)$

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denote the subclass of  $\mathcal{S}$  consisting of those functions satisfying

$$\operatorname{Re}\{f'(z)\} > \alpha \quad (0 \leq \alpha < 1). \tag{1.2}$$

Here the class  $\mathcal{P}(\alpha)$  was introduced by Hallenbeck [6]. In particular, the class  $\mathcal{P}(0) = \mathcal{P}$  was investigated by MacGregor [11]. Here we recall Noshiro-Warschawski theorem (see, e.g., [3, Theorem 2.16]): *If  $f$  is analytic in a convex domain  $D$  and  $\operatorname{Re}\{f'(z)\} > 0$  there, then  $f$  is univalent in  $D$ .*

In view of this theorem, we find that the class of functions in  $\mathcal{A}$  which satisfy (1.2) become a subclass of  $\mathcal{S}$ . In the regard, the condition “ $\mathcal{P}(\alpha)$  is a subclass of  $\mathcal{S}$ ” is redundant.

Let  $\alpha_j$  ( $j = 1, \dots, p$ ) and  $\beta_j$  ( $j = 1, \dots, q$ ) be complex numbers with  $\beta_j \notin \mathbb{Z}_0^- := \{0, -1, -2, \dots\}$ . Then the generalized hypergeometric function  ${}_pF_q(z)$  is defined by

$$\begin{aligned} {}_pF_q(z) &\equiv {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \quad (p \leq q + 1), \end{aligned} \tag{1.3}$$

where  $(\lambda)_n$  is the Pochhammer symbol defined, in terms of the Gamma function, by

$$\begin{aligned} (\lambda)_\nu &:= \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \\ &= \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \end{aligned}$$

it being understood conventionally that  $(0)_0 = 1$ . We note that the series  ${}_pF_q(z)$  in (1.3) converges absolutely for  $|z| < \infty$  if  $p < q + 1$ , and for  $z \in \mathbb{U}$  if  $p = q + 1$ . The condition  $p \leq q + 1$  stated with the definition (1.3) will be assumed to hold true throughout this paper.

For functions  $f_j(z) \in \mathcal{A}$ , given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2),$$

we define the *Hadamard product (or convolution)* of  $f_1(z)$  and  $f_2(z)$  by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z) \quad (z \in \mathbb{U}).$$

With a view to defining the Srivastava-Attiya transform, we recall here a general Hurwitz-Lerch zeta function, which is defined in [15] by the following

series:

$$\Phi(z, \lambda, \delta) := \frac{1}{\delta^\lambda} + \sum_{k=1}^{\infty} \frac{z^k}{(k + \delta)^\lambda}$$

$(\delta \in \mathbb{C} \setminus \mathbb{Z}_0^-; \lambda \in \mathbb{C} \text{ when } z \in \mathcal{U}; \operatorname{Re}(\lambda) > 1 \text{ when } |z| = 1).$

For further interesting properties and characteristics of the Hurwitz-Lerch Zeta function and other related special functions, see for example [4], [10] and [16].

In the literature on special functions one often finds applications of operational techniques involving various integral transforms (see [7], [13] and [17]). Recently, Srivastava and Attiya [14] have introduced the linear operator  $\mathcal{L}_{\lambda, \delta} : \mathcal{A} \rightarrow \mathcal{A}$ , defined in terms of the Hadamard product by

$$\mathcal{L}_{\lambda, \delta} f(z) = \mathcal{G}_{\lambda, \delta}(z) * f(z) \quad (\delta \in \mathbb{C} \setminus \mathbb{Z}_0^-; \lambda \in \mathbb{C}; z \in \mathbb{U}), \tag{1.4}$$

where

$$\mathcal{G}_{\lambda, \delta}(z) = (1 + \delta)^\lambda \left[ \Phi(z, \lambda, \delta) - \delta^{-\lambda} \right] \quad (z \in \mathbb{U}). \tag{1.5}$$

The operator  $\mathcal{L}_{\lambda, \delta}$  is now popularly known in the literature as the Srivastava-Attiya operator. Various class-mapping properties of the operator  $\mathcal{L}_{\lambda, \delta}$  (and its variants) are discussed in the recent works of Srivastava and Attiya [14], Liu [9], Murugusundaramoorthy [12], Yuan and Liu [18] and others.

It is easy to observe from (1.1) and (1.4) that

$$\mathcal{L}_{\lambda, \delta} f(z) = z + \sum_{k=2}^{\infty} \left( \frac{1 + \delta}{k + \delta} \right)^\lambda a_k z^k. \tag{1.6}$$

We note that:

- (i)  $\mathcal{L}_{0, \delta} f(z) = f(z)$ ;
- (ii)  $\mathcal{L}_{1, 0} f(z) = \mathcal{L} f(z) = \int_0^z \frac{f(t)}{t} dt \quad (f \in \mathcal{A})$  (see Alexander [1]);
- (iii)  $\mathcal{L}_{m, 1} f(z) = \mathcal{I}^m f(z) \quad (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\})$  (see Flett [5]);
- (iv)  $\mathcal{L}_{\gamma, 1} f(z) = \mathcal{Q}^\gamma f(z) \quad (\gamma > 0)$  (see Jung et al. [7]);
- (v)  $\mathcal{L}_{m, 0} f(z) = \mathcal{L}^m f(z) \quad (m \in \mathbb{N}_0)$  (see Sălăgean [13]).

In this article, we determine some radii for normalized analytic functions  $f(z)$  defined by (1.1) with  $|a_n| \leq n \ (n = 2, 3, 4, \dots)$  in order to satisfy such inequalities as  $\operatorname{Re}\{f'(z)\} > \alpha \ (0 \leq \alpha < 1)$ . Moreover, some interesting consequences for certain operator defined through convolution are also pointed out.

## 2. MAIN RESULTS

We begin by proving the following theorem.

**Theorem 2.1.** *Let the function  $f(z)$  be defined by (1.1) and suppose that  $|a_n| \leq n$  for  $n = 2, 3, 4, \dots$ . Then we have*

$$f(z) \in \mathcal{P}(\alpha) \quad \text{for } |z| < r_0, \quad (2.1)$$

where  $0 \leq \alpha < 1$  and  $r_0$  is the unique solution of the equation

$$(2 - \alpha)r^3 - 3(2 - \alpha)r^2 + (7 - 3\alpha)r - 1 + \alpha = 0.$$

Furthermore, the radius  $r_0$  is sharp.

*Proof.* From the hypothesis, we obtain

$$\operatorname{Re} \{f'(z)\} \geq 1 - \sum_{n=2}^{\infty} n^2 r^{n-1} \quad (|z| = r).$$

Since

$$\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n \quad \text{and} \quad \frac{1+z}{(1-z)^3} = \sum_{n=1}^{\infty} n^2 z^{n-1}, \quad (2.2)$$

we readily have

$$\begin{aligned} \operatorname{Re}\{f'(z) - \alpha\} &\geq 2 - \frac{1+r}{(1-r)^3} - \alpha \\ &= \frac{-(2-\alpha)r^3 + 3(2-\alpha)r^2 - (7-3\alpha)r + 1 - \alpha}{(1-r)^3}. \end{aligned} \quad (2.3)$$

If we put  $g(r) = (2 - \alpha)r^3 - 3(2 - \alpha)r^2 + (7 - 3\alpha)r - 1 + \alpha$ , it is easily seen that there exist the unique solution  $r_0$  of the equation  $g(r) = 0$ . Since  $g(0) = -1 + \alpha$  and  $g(1) = 2$ ,

$$r_0 \in (0, 1).$$

By applying (2.2) and (2.3), we see that

$$\operatorname{Re}\{f'(z) - \alpha\} > 0 \quad \text{for } |z| < r_0,$$

which implies that  $f(z) \in \mathcal{P}(\alpha)$  for  $|z| < r_0$ .

Also, if we take

$$f(z) = 2z - \frac{z}{(1-z)^2} \quad (|z| < 1),$$

then from (2.2) we have

$$\operatorname{Re}\{f'(z) - \alpha\} \geq 0 \quad |z| \leq r_0 < 1,$$

and  $f'(r_0) = \alpha$ . Hence the radius  $r_0$  is sharp. This evidently completes the proof of Theorem 2.1.  $\square$

By the theorem of de Branges [2], we can restate Theorem 2.1 as follows:

**Corollary 2.2.** *If  $f(z) \in \mathcal{S}$ , then*

$$f(z) \in \mathcal{P}(\alpha) \quad (|z| < r_0),$$

where  $0 \leq \alpha < 1$  and  $r_0$  is defined in Theorem 2.1.

**Corollary 2.3.** *Let the function  $f(z)$  be defined by (1.1) and suppose that  $|a_n| \leq n$  for  $n = 2, 3, 4, \dots$ . If we put*

$$F(z) = \frac{1}{z} \int_0^z f(t) dt, \tag{2.4}$$

then

$$F(z) \in \mathcal{P}(\alpha) \quad \text{for } |z| < r_1,$$

where  $0 \leq \alpha < 1$  and  $r_1$  is the positive root of the equation

$$\frac{1}{(1-r)^2} - \frac{1}{r(1-r)} - \frac{1}{r^2} \log(1-r) + \alpha = 1.$$

*Proof.* From (2.4) we obtain

$$\begin{aligned} \operatorname{Re}\{F'(z) - \alpha\} &\geq \frac{1}{2} - \sum_{n=2}^{\infty} \frac{n^2}{n+1} r^{n-1} - \alpha \quad (|z| = r) \\ &= 1 - \alpha - \sum_{n=1}^{\infty} \frac{n^2}{n+1} r^{n-1}. \end{aligned} \tag{2.5}$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^2}{n+1} z^{n-1} &= \left( \frac{1}{z} \int_0^z \frac{t}{(1-t)^2} dt \right)' \\ &= \frac{1}{(1-z)^2} - \frac{1}{z(1-z)} - \frac{1}{z^2} \log(1-z), \end{aligned}$$

it follows from (2.5) that  $\operatorname{Re}\{F'(z) - \alpha\} > 0$  for  $|z| < r_1$ . Hence we conclude that  $F(z) \in \mathcal{P}(\alpha)$  for  $|z| < r_1$ , which completes the proof of Corollary 2.3.  $\square$

Next, we can prove the following theorem.

**Theorem 2.4.** *Let the function  $f(z)$  be defined by (1.1) and suppose that  $|a_n| \leq n$  for  $n = 2, 3, 4, \dots$ . Then*

$$\mathcal{L}_{m,\delta} f(z) \in \mathcal{P}(\alpha) \quad \text{for } |z| < r_2, \tag{2.6}$$

where  $m \in \mathbb{N}$ ,  $\delta > 0$ ,  $0 \leq \alpha < 1$  and  $r_2$  is the solution of the equation

$${}_{m+2}F_{m+1}(1 + \delta, \dots, 1 + \delta, 2, 2; 1, 2 + \delta, \dots, 2 + \delta; r) + \alpha = 2.$$

*Proof.* From (1.6) we have

$$\begin{aligned} & \operatorname{Re} \{ (\mathcal{L}_{m,\delta} f(z))' - \alpha \} \\ & \geq 1 - \sum_{n=2}^{\infty} \left( \frac{1+\delta}{n+\delta} \right)^m n^2 r^{n-1} - \alpha \quad (|z|=r) \\ & = 2 - \sum_{n=0}^{\infty} \left( \frac{1+\delta}{n+1+\delta} \right)^m (n+1)^2 r^n - \alpha \\ & = 2 - \alpha - {}_{m+2}F_{m+1}(1+\delta, \dots, 1+\delta, 2, 2; 1, 2+\delta, \dots, 2+\delta; r). \end{aligned}$$

Hence, if  $|z| < r_2$ , then we obtain (2.6). This completes the proof of Theorem 2.4.  $\square$

Taking  $\alpha = 0$  in Theorem 2.4, we have the following.

**Corollary 2.5.** *Let the function  $f(z)$  be defined by (1.1) and suppose that  $|a_n| \leq n$  for  $n = 2, 3, 4, \dots$ . Then*

$$\mathcal{L}_{m,\delta} f(z) \in \mathcal{P} \quad \text{for } |z| < r_3,$$

where  $m \in \mathbb{N}$ ,  $\delta > 0$  and  $r_3$  is the solution of the equation

$${}_{m+2}F_{m+1}(1+\delta, \dots, 1+\delta, 2, 2; 1, 2+\delta, \dots, 2+\delta; r) = 2.$$

**Remark 2.6.** If we put  $\delta = 1$  in Corollary 2.5, then it would immediately yield the result due to Kim and Nunokawa [8, Theorem 3].

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