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DEGREE OF APPROXIMATION OF FUNCTIONS OF CLASS $Zyg^{\omega}(\alpha,\gamma)$ BY CESARO MEANS OF FOURIER SERIES

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Abstract. In this paper, we investigate the degree of approximation of a function f belonging to the generalized Zygmund class $Zyg^{\omega}(\alpha,\gamma)$ by Cesaro means of its Fourier series.

1. Introduction

A lot of researchers like Khan [2], Chandra [1], Mittal et al. [6], Lal [3], Lal and Mishra [4] have studied the degree of approximation of a function belonging to the Lipschitz class $Lip(\alpha,\gamma)$ by certain means of its Fourier series.

We recall that the Zygmund class Zyg is defined as

$$Zyg = \{f \in C[-\pi,\pi] \,|\, |f(x+t)+f(x-t)-2f(x)| = O(|t|)\}$$

and that the Zygmund class Zyg includes the Lipschitz class Lip. Some forms of the generalized Zygmund classes have been defined and systematically investigated by Leindler [5], Moricz [7], Moricz and Nemeth [8]. In this note, we introduce a generalized Zygmund class $Zyg^{\omega}(\alpha,\gamma)$ defined as

$$Zyg^{\omega}(\alpha,\gamma) = \{f \in C[-\pi,\pi] \,|\, (\int_{-\pi}^{\pi} |f(x+t)+f(x-t)-2f(x)|^{\gamma} dx)^{\frac{1}{\gamma}} = O(|t|^{\alpha}\omega(|t|))\},$$

where $\alpha \geq 0$, $\gamma \geq 1$ and ω is a continuous, nonnegative and nondecreasing function. Note that if we take $\alpha = 1$, $\omega =$ constant and $\gamma \to \infty$, then $Zyg^{\omega}(\alpha,\gamma)$ class reduces to the Zyg class. Throughout this paper, only 2π-periodic continuous function f is considered. The sequence $\sigma_n(f;x)$ of Cesaro means of the sequence $s_n(f;x)$ of partial sums of the Fourier series of f is defined as

$$\sigma_n(f;x) = \frac{1}{n+1}(s_o(f;x)+,...,+s_n(f;x)).$$

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Notice that under convolution $*$, one has $\sigma_n(f;x) = (F_n * f)(x)$ and the Fejer kernel F_n [10] also has the closed form of

$$F_n(t) = \frac{1}{n+1}(\frac{sin^2(n+1)\frac{t}{2}}{sin^2\frac{t}{2}}).$$

In the present paper, we consider the degree of approximation of a function f belonging to $Zyg^{\omega}(\alpha,\gamma)$ by Cesaro means of its Fourier series.

2. Main results

First of all, the following lemmas are needed:

Lemma 2.1. *For $0 < t \leq \frac{1}{n+1}$, we have $F_n(t) = O(n+1)$.*

Proof. For $0 < t \leq \frac{1}{n+1}$, we have

$$sin(n+1)\frac{t}{2} \leq (n+1)sin\frac{t}{2},$$

which leads to

$$F_n(t) = \frac{1}{n+1}(\frac{sin^2(n+1)\frac{t}{2}}{sin^2\frac{t}{2}}) \leq \frac{1}{n+1}(\frac{(n+1)^2 sin^2\frac{t}{2}}{sin^2\frac{t}{2}}) = n+1.$$

This completes the proof. □

Lemma 2.2. *For $\frac{1}{n+1} < t \leq \pi$, we obtain $F_n(t) = O(\frac{1}{t^2(n+1)})$.*

Proof. For $\frac{1}{n+1} < t \leq \pi$, we get $sin\frac{t}{2} \geq (\frac{t}{\pi})$ and $sin(n+1)\frac{t}{2} \leq 1$, which imply

$$F_n(t) = \frac{1}{n+1}(\frac{sin^2(n+1)\frac{t}{2}}{sin^2\frac{t}{2}}) \leq \frac{1}{n+1}(\frac{\pi^2}{t^2}).$$

This completes the proof. □

Now we establish a theorem on the degree of approximation of a function of class $Zyg^{\omega}(\alpha,\gamma)$ by Cesaro means of its Fourier series. More precisely, we prove the following theorem:

Theorem 2.3. *Let f be a 2π-periodic continuous function belonging to $Zyg^{\omega}(\alpha,\gamma)$. Then the degree of approximation of f by Cesaro means of its Fourier series is given by $||\sigma_n - f||_{\gamma} = O\big(\frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k})\big)$.*

Proof. It follows from [9,10] that

$$\sigma_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \Phi(x,t)F_n(t)dt,$$

where $\Phi(x,t) = f(x+t) + f(x-t) - 2f(x)$.

Taking account of generalized Minkowski inequality for integrals [10], we get

$$\begin{aligned}
||\sigma_n - f||_r &= (\int_{-\pi}^{\pi} |\sigma_n(f;x) - f(x)|^r dx)^{\frac{1}{r}} \\
&= \frac{1}{2\pi}(\int_{-\pi}^{\pi} |\int_0^\pi \Phi(x,t)F_n(t)dt|^r dx)^{\frac{1}{r}} \\
&\leq \frac{1}{2\pi}\int_0^\pi (\int_{-\pi}^{\pi} |\Phi(x,t)F_n(t)|^r dx)^{\frac{1}{r}} dt \\
&= \frac{1}{2\pi}\int_0^\pi (\int_{-\pi}^{\pi} |\Phi(x,t)|^r dx)^{\frac{1}{r}} |F_n(t)|dt \\
&= \frac{1}{2\pi}\int_0^\pi O(t^\alpha \omega(t))|F_n(t)|dt
\end{aligned}$$

$$= O(\int_0^{\frac{1}{n+1}} t^\alpha \omega(t)|F_n(t)|dt) + O(\int_{\frac{1}{n+1}}^{\pi} t^\alpha \omega(t)|F_n(t)|dt)$$

$$= I_1 + I_2.$$

By virtue of Lemma 2.1, we have

$$I_1 = O(\int_0^{\frac{1}{n+1}} t^\alpha \omega(t)(n+1)dt).$$

Taking account of non-negative, increasing function $t^\alpha \omega(t)$ and mean value theorem for integral, we obtain

$$\begin{aligned}
I_1 &= O((\frac{1}{n+1})^\alpha \omega(\frac{1}{n+1})\int_\varepsilon^{\frac{1}{n+1}} (n+1)dt) \\
&= O((\frac{1}{n+1})^\alpha \omega(\frac{1}{n+1})).
\end{aligned}$$

Again, using non-negative, increasing function $t^\alpha \omega(t)$, we have

$$(\frac{1}{n+1})^\alpha \omega(\frac{1}{n+1}) \leq \frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^\alpha \omega(\frac{1}{k}),$$

which yields from the last identity

$$I_1 = O(\frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^\alpha \omega(\frac{1}{k})).$$

On the other hand, by virtue of Lemma 2.2, we obtain

$$I_2 = O(\int_{\frac{1}{n+1}}^{\pi} t^{\alpha}\omega(t)\frac{1}{t^2(n+1)}dt)$$

$$= O(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi} t^{\alpha-2}\omega(t)dt).$$

Putting $t = \frac{1}{u}$ in the last integral, we get

$$I_2 = O(\frac{1}{n+1}\int_{\frac{1}{\pi}}^{n+1} u^{-\alpha}\omega(\frac{1}{u})du).$$

Since $t^{\alpha}\omega(t)$ is an increasing function, $(\frac{1}{u})^{\alpha}\omega(\frac{1}{u})$ is now a decreasing function. Hence we have

$$O(\frac{1}{n+1}\int_{\frac{1}{\pi}}^{n+1} u^{-\alpha}\omega(\frac{1}{u})du) = O(\frac{1}{n+1}\pi^{\alpha}\omega(\pi) + \frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k})).$$

It follows from the continuity of $t^{\alpha}\omega(t)$ that we can choose a positive constant c such that

$$c(1^{\alpha}\omega(1)) \geq \pi^{\alpha}\omega(\pi).$$

By virtue of a non-negative property of $t^{\alpha}\omega(t)$ and the last inequality,

$$O(\frac{1}{n+1}\pi^{\alpha}\omega(\pi)+\frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k})) = O(\frac{1}{n+1}c(\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k}))+\frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k}))$$

$$= O(\frac{c+1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k})) = O(\frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k})).$$

Therefore, combining I_1 and I_2, we obtain

$$||\sigma_n - f||_{\gamma} = O(\frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k}))+O(\frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k})) = O(\frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k})).$$

This completes the proof. □

As a consequence, the following corollary can be derived from Theorem 2.3:

Corollary 2.4. *Let f be a 2π-periodic continuous function belonging to $Zyg^{\omega}(\alpha,\gamma)$ with $\omega(t) = t^m (m > 1)$. Then the degree of approximation by Cesaro means of its Fourier series is given by*

$$||\sigma_n - f||_{\gamma} = O(\frac{1}{n+1}).$$

Proof. Taking account of $\omega(t) = t^m$ $(m > 1)$ and $\alpha \geq 0$, we have from Theorem 2.3 that

$$\begin{aligned}\|\sigma_n - f\|_r &= O\Big(\frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{\alpha}\omega(\frac{1}{k})\Big)\\ &= O\Big(\frac{1}{n+1}\sum_{k=1}^{n+1}(\frac{1}{k})^{m+\alpha}\Big)\\ &= O\Big(\frac{1}{n+1}\Big).\end{aligned}$$

This completes the proof. □

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