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A NEW CLASS OF DOUBLE INTEGRALS

Aravind K. Anil [a], Prathima J. [b] and Insuk Kim [c,*]

Abstract. In this paper we aim to establish a new class of six definite double integrals in terms of gamma functions. The results are obtained with the help of some definite integrals obtained recently by Kim and Edward equality. The results established in this paper are simple, interesting, easily established and may be useful potentially.

1. Introduction

Hypergeometric functions form an important class of special functions. Almost all the elementary functions of mathematics are either hypergeometric or ratios of hypergeometric functions.The Gauss hypergeometric function is defined by [2,4]

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.1}$$

given $\mid z \mid< 1, c \neq 0, -1, -2, \cdots$ and the confluent hypergeometric function is defined by [2,4]

$$ {}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}. \tag{1.2}$$

Both the above functions are the special cases of the generalizesd hypergeometric function [2,4] defined by

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ c_1, c_2, \ldots, c_q; \end{matrix}\; z\right] = \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (c_j)_n} \frac{z^n}{n!}, \tag{1.3}$$

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*Corresponding author.

where

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{1.4}$$

and $(a)_0 = 1$.

The series given by (1.3) is convergent for all $p \le q$ by the ratio test. For $p < q+1$, it converges everywhere and converges nowhere for $p > q+1$. Further, if $p = q+1$, it converges absolutely for $|z| = 1$ provided $\delta = \Re\left(\sum_{j=1}^{q} c_j - \sum_{i=1}^{p} a_i\right) > 0$ holds and is conditionally convergent for $|z| = 1$ and $z \neq -1$ if $-1 < \delta \le 0$ and diverges for $|z| = 1$ and $z \neq 1$ if $\delta \le -1$.

In the theory of hypergeometric series, classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey play a key role. In 2011, Rakha and Rathie [5] generalized classical summation theorems such as those of Gauss second, Bailey and Kummer in the most general form for any $i = 0, 1, \cdots$ as follows:

Generalization of Gauss second summation theorem:

$$ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+i+1) & \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+i+1))\Gamma(\frac{1}{2}(a-b-i+1))}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}(b+1))\Gamma(\frac{1}{2}(a-b+i+1))} \times \sum_{r=0}^{i} \binom{i}{r} \frac{(-1)^r\Gamma(\frac{1}{2}(b+r))}{\Gamma(\frac{1}{2}(a-i+r+1))} \tag{1.5}$$

$$ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b-i+1) & \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b-i+1))}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}(b+1))} \times \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma(\frac{1}{2}(b+r))}{\Gamma(\frac{1}{2}(a-i+r+1))} \tag{1.6}$$

Generalizations of Bailey summation theorem:

$$ {}_2F_1\left[\begin{matrix} a, & 1-a+i \\ b & \end{matrix}; \frac{1}{2}\right] = \frac{2^{1+i-b}\Gamma(\frac{1}{2})\Gamma(b)\Gamma(a-i)}{\Gamma(a)\Gamma(\frac{1}{2}(b-a))\Gamma(\frac{1}{2}(b-a+1))} \times \sum_{r=0}^{i} \binom{i}{r} \frac{(-1)^r\Gamma(\frac{1}{2}(b-a+r))}{\Gamma(\frac{1}{2}b+\frac{1}{2}a+\frac{1}{2}r-i)} \tag{1.7}$$

$$ {}_2F_1\left[\begin{matrix} a, & 1-a-i \\ b & \end{matrix}; \frac{1}{2}\right] = \frac{2^{1-i-b}\Gamma(\frac{1}{2})\Gamma(b)}{\Gamma(\frac{1}{2}(b-a))\Gamma(\frac{1}{2}(b-a+1))} \times \sum_{r=0}^{i} \binom{i}{r} \frac{\Gamma(\frac{1}{2}(b-a+r))}{\Gamma(\frac{1}{2}(b+a+r))} \tag{1.8}$$

Generalizations of Kummer summation theorem:

$$(1.9)\quad {}_2F_1\left[\begin{matrix} a, & b \\ 1+a-b+i & \end{matrix};-1\right] = \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(b-i)\Gamma(1+a-b+i)}{\Gamma(b)\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+1)} \times \sum_{r=0}^{i}\binom{i}{r}\frac{(-1)^r\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}(a-i+r+1))}$$

$$(1.10)\quad {}_2F_1\left[\begin{matrix} a & b \\ 1+a-b-i & \end{matrix};-1\right] = \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1+a-b+i)}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+1)} \times \sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma(\frac{1}{2}a-b-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}(a-i+r+1))}$$

With the help of above mentioned results, in 2017, Kim [3] established the following six general definite integrals in terms of gamma function for $i=0,1,\cdots$.

$$(1.11)\quad \int_0^1 t^{\frac{1}{2}(a-b+i+1)-1}(1-t)^{b-1}(1+t)^{-a}dt = 2^{b-a-1}\Gamma(\frac{1}{2}(a-b-i+1))\sum_{r=0}^{i}\binom{i}{r}\frac{(-1)^r\Gamma(\frac{1}{2}(b+r))}{\Gamma(\frac{1}{2}(a-i+r+1))},$$

provided $\Re(b)>0$ and $\Re(1+a-b+i)>0$.

$$(1.12)\quad \int_0^1 t^{\frac{1}{2}(a-b-i+1)-1}(1-t)^{b-1}(1+t)^{-a}dt = 2^{b-a-1}\Gamma(\frac{1}{2}(a-b-i+1))\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma(\frac{1}{2}(b+r))}{\Gamma(\frac{1}{2}(a-i+r+1))},$$

provided $\Re(b)>0$ and $\Re(1+a-b-i)>0$.

(1.13)

$$\int_0^1 t^{b+a-2-i}(1-t)^{-a+i}(1+t)^{-a}dt = 2^{i-2a}\frac{\Gamma(1-a+i)\Gamma(b+a-1-i)\Gamma(a-i)}{\Gamma(a)\Gamma(b-a)}\sum_{r=0}^{i}\binom{i}{r}\frac{(-1)^r\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}r)}{\Gamma(\frac{1}{2}b+\frac{1}{2}a+\frac{1}{2}r-i)},$$

provided $\Re(1-a+i)>0$ and $\Re(a+b-i-1)>0$.

$$(1.14)\qquad \int_0^1 t^{b+a+i-2}(1-t)^{-a-i}(1+t)^{-a}dt = \frac{2^{-i-2a}\Gamma(1-a-i)\Gamma(b+a-1+i)}{\Gamma(b-a)}\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma(\frac{1}{2}(b-a+r))}{\Gamma(\frac{1}{2}(b+a+r))},$$

provided $\Re(1-a+i)>0$ and $\Re(a+b+i-1)>0$.

$$(1.15)\qquad \int_0^1 t^{b-1}(1-t)^{a-2b+i}(1+t)^{-a}dt = 2^{i-2b}\Gamma(b-i)\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})},$$

provided $\Re(b)>0$ and $\Re(a-2b+i+1)>0$.

$$(1.16)\qquad \int_0^1 t^{b-1}(1-t)^{a-2b-i}(1+t)^{-a}dt = 2^{-i-2b}\Gamma(b)\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma(\frac{1}{2}a-b-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})},$$

provided $\Re(b)>0$ and $\Re(a-2b-i+1)>0$.

The aim of this paper is to establish a new class of six general double integrals in terms of gamma function by employing the results (1.11) to (1.16). For this we shall use the following well known result recorded in Edward [1].

$$(1.17)\qquad \int_0^1\int_0^1 f(xy)y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\times\int_0^1 f(t)(1-t)^{\alpha+\beta-1}dt$$

provided $\Re(\alpha)>0$, $\Re(\beta)>0$.

2. Main Results

The new class of six double integrals established in the paper are given in the following theorems.

Theorem 2.1. *For* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(b)>0$ *and* $\Re(a-b-i+1)>0$, $i=0,1,2,\cdots$, *the following result holds true.*

$$\text{(2.1)}\quad \int_0^1\int_0^1 x^{\frac{1}{2}(a-b+i+1)-1}y^{\alpha+\frac{1}{2}(a-b+i+1)-1}(1-xy)^{b-\alpha-\beta}(1+xy)^{-a} \times(1-x)^{\alpha-1}(1-y)^{\beta-1}dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}2^{(b-a-1)}\Gamma(\tfrac{1}{2}(a-b-i+1))\sum_{r=0}^{i}\binom{i}{r}\frac{(-1)^r\Gamma(\frac{1}{2}(b+r))}{\Gamma(\frac{1}{2}(a-i+r+1))}.$$

Theorem 2.2. *For* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(b)>0$ *and* $\Re(a-b-i+1)>0$, $i=1,2,\cdots$, *the following result holds true.*

$$\text{(2.2)}\quad \int_0^1\int_0^1 x^{\frac{1}{2}(a-b-i+1)-1}y^{\alpha+\frac{1}{2}(a-b-i+1)-1}(1-xy)^{b-\alpha-\beta}(1+xy)^{-a} \times(1-x)^{\alpha-1}(1-y)^{\beta-1}dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}2^{(b-a-1)}\Gamma(\tfrac{1}{2}(a-b-i+1))\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma(\frac{1}{2}(b+r))}{\Gamma(\frac{1}{2}(a-i+r+1))}.$$

Theorem 2.3. *For* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(1-a+i)>0$ *and* $\Re(a+b-i-1)>0$, $i=0,1,2,\cdots$, *the following result holds true.*

$$\text{(2.3)}\quad \int_0^1\int_0^1 x^{(b+a-2-i)}y^{\alpha+(b+a-2-i)}(1-xy)^{-a+i-\alpha-\beta+1}(1+xy)^{-a} \times(1-x)^{\alpha-1}(1-y)^{\beta-1}dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}2^{i-2a}\frac{\Gamma(1-a+i)\Gamma(b+a-1-i)\Gamma(a-i)}{\Gamma(a)\Gamma(b-a)} \times\sum_{r=0}^{i}\binom{i}{r}\frac{(-1)^r\Gamma(\frac{1}{2}(b-a+r))}{\Gamma(\frac{1}{2}b+\frac{1}{2}a+\frac{1}{2}r-i)}.$$

Theorem 2.4. *For* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(1-a+i)>0$ *and* $\Re(a+b+i-1)>0$, $i=1,2,\cdots$, *the following result holds true.*

$$\text{(2.4)}\quad \int_0^1\int_0^1 x^{(b+a-2+i)}y^{\alpha+(b+a-2+i)}(1-xy)^{-a-i-\alpha-\beta+1}(1+xy)^{-a} \times(1-x)^{\alpha-1}(1-y)^{\beta-1}dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}2^{-i-2a}\frac{\Gamma(1-a-i)\Gamma(b+a-1+i)}{\Gamma(b-a)} \times\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma(\frac{1}{2}(b-a+r))}{\Gamma(\frac{1}{2}(b+a+r))}.$$

Theorem 2.5. *For* $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(b-i) > 0$ *and* $\Re(a-2b+i+1) > 0$, $i = 0, 1, 2, \cdots$, *the following result holds true.*

$$\begin{aligned}(2.5)\quad \int_0^1\int_0^1 & x^{(b-1)}y^{\alpha+(b-1)}(1-xy)^{a-2b+i-\alpha-\beta+1}(1+xy)^{-a} \\ & \times (1-x)^{\alpha-1}(1-y)^{\beta-1}dxdy \\ & = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}2^{i-2b}\Gamma(b-i)\sum_{r=0}^{i}(-1)^r\binom{i}{r}\frac{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})}.\end{aligned}$$

Theorem 2.6. *For* $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(b-i) > 0$ *and* $\Re(a-2b-i+1) > 0$, $i = 1, 2, \cdots$, *the following result holds true.*

$$\begin{aligned}(2.6)\quad \int_0^1\int_0^1 & x^{(b-1)}y^{\alpha+(b-1)}(1-xy)^{a-2b-i-\alpha-\beta+1}(1+xy)^{-a} \\ & \times (1-x)^{\alpha-1}(1-y)^{\beta-1}dxdy \\ & = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,2^{-i-2b}\,\Gamma(b)\sum_{r=0}^{i}\binom{i}{r}\frac{\Gamma(\frac{1}{2}a-b\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}i+\frac{1}{2}r+\frac{1}{2})}.\end{aligned}$$

Proof. In order to prove the result given in theorem (2.1), we proceed as follows.

If we set

$$f(t) = t^{\frac{1}{2}(a-b+i+1)-1}(1-t)^{b-\alpha-\beta}(1+t)^{-a},$$

then clearly

$$f(xy) = (xy)^{\frac{1}{2}(a-b+i+1)-1}(1-xy)^{b-\alpha-\beta}(1+xy)^{-a}.$$

Then equation (1.17) takes the following form

$$\begin{aligned}(2.7)\quad \int_0^1\int_0^1 & x^{\frac{1}{2}(a-b+i+1)-1}y^{\alpha+\frac{1}{2}(a-b+i+1)-1}(1-xy)^{b-\alpha-\beta}(1+xy)^{-a} \\ & \times (1-x)^{\alpha-1}(1-y)^{\beta-1}dxdy \\ & = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^1 t^{\frac{1}{2}(a-b+i+1)-1}(1-t)^{b-1}(1+t)^{-a}dt.\end{aligned}$$

We now observe that integral appearing on the right hand side of (2.7) can be evaluated with the help of the result (1.11) and we easily arrive at the result (2.1) asserted in Theorem (2.1). This complets the proof of Theorem (2.1). □

In exactly the same manner Theorems 2.2 to Theorem 2.6 can be established by selecting appropriate $f(xy)$ and applying the results (1.12) to (1.16). So we prefer to omit the details.

3. Concluding Remark

In this paper, we have obtained a new class of six double integrals in terms of gamma function. The results are established with the help of six definite integrals given by Kim. The results established in this paper may be useful in Mathematics, Engineering Mathematics and Mathematical Physics.

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[a]Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India
Email address: `1996aravind007@gmail.com`

[b]Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India
Email address: `prathima.amrutharaj@manipal.edu`

[c]Department of Mathematics Education, Wonkwang University, Iksan, 54538, Korea
Email address: `iki@wku.ac.kr`