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LIE BRACKET JORDAN DERIVATIONS IN BANACH JORDAN ALGEBRAS

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Abstract. In this paper, we introduce Lie bracket Jordan derivations in Banach Jordan algebras. Using the direct method and the fixed point method, we prove the Hyers-Ulam stability of Lie bracket Jordan derivations in complex Banach Jordan algebras.

1. Introduction and Preliminaries

Let A be a complex Banach Jordan algebra with Jordan product $\odot$ and $Der(A)$ be the set of $\mathbb{C}$-linear (bounded) Jordan derivations on A. For $\delta_1, \delta_2 \in Der(A)$,

$$\begin{aligned}\delta_1 \circ \delta_2(a \odot b) &= (\delta_1 \circ \delta_2(a)) \odot b + \delta_2(a) \odot \delta_1(b) + \delta_1(a) \odot \delta_2(b) + a \odot (\delta_1 \circ \delta_2(b)),\\ \delta_2 \circ \delta_1(a \odot b) &= (\delta_2 \circ \delta_1(a)) \odot b + \delta_1(a) \odot \delta_2(b) + \delta_2(a) \odot \delta_1(b) + a \odot (\delta_2 \circ \delta_1(b))\end{aligned}$$

for all $a, b \in A$. Let $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$. Then

$$[\delta_1, \delta_2](a \odot b) = [\delta_1, \delta_2](a) \odot b + a \odot [\delta_1, \delta_2](b)$$

for all $a, b \in A$. Since $[\delta_1, \delta_2] : A \to A$ is $\mathbb{C}$-linear, $[\delta_1, \delta_2] \in Der(A)$ for all $\delta_1, \delta_2 \in Der(A)$. Thus $Der(A)$ is a Lie algebra with Lie bracket $[\delta_1, \delta_2]$, since $\delta_1 + \delta_2$ and $\alpha\delta_1$ are $\mathbb{C}$-linear derivations on A for all $\delta_1, \delta_2 \in Der(A)$ and all $\alpha \in \mathbb{C}$. One can easily show that $Der(A)$ is a Banach space, since A is complete.

In this paper, we introduce and investigate Lie bracket derivations in a complex Banach Jordan algebra.

Definition 1.1. Let A be a complex Banach Jordan algebra and $G, H : A \to A$ be $\mathbb{C}$-linear mappings. Let $[G, H](a) = G(H(a)) - H(G(a))$ for all $a \in A$. A $\mathbb{C}$-linear

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mapping $[G, H] : A \to A$ is called a *Lie bracket Jordan derivation* in A if $[G, H]$ is a Jordan derivation in A, i.e.,

$$[G, H](a \odot b) = [G, H](a) \odot b + a \odot [G, H](b)$$

for all $a, b \in A$.

Since $[\delta_1, \delta_2] \in Der(A)$ for $\delta_1, \delta_2 \in Der(A)$, $[\delta_1, \delta_2]$ is a Lie bracket Jordan derivation.

The stability problem of functional equations originated from a question of Ulam [22] concerning the stability of group homomorphisms. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [19] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [20] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Park [14, 15] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors (see [8, 21, 23]).

We recall a fundamental result in fixed point theory.

Theorem 1.2 ([2, 6])**.** *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [11] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4, 16, 17, 18]).

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of Lie bracket Jordan derivations in a complex Banach Jordan algebra by using the fixed point method. In Section 3, we prove the Hyers-Ulam stability of Lie bracket Jordan derivations in a Banach Jordan algebra by using the direct method.

Throughout this paper, let A be a complex Banach Jordan algebra and p be a nonzero complex number with $|p| < 1$.

2. Hyers-Ulam Stability of Lie Bracket Derivations in Complex Banach Jordan Algebras: Fixed Point Method

In this section, we prove the Hyers-Ulam stability of Lie bracket Jordan derivations in complex Banach Jordan algebras by using the fixed point method.

Lemma 2.1 ([13, Theorem 2.1])**.** *Let X be a complex normed space and Y be a complex Banach space. Let $f : X \to Y$ be a mapping such that*

$$f(\lambda(a+b)) = \lambda f(a) + \lambda f(b)$$

for all $\lambda \in \mathbb{T}^1 := \{\xi \in \mathbb{C} : |\xi| = 1\}$ and all $a, b \in X$. Then $f : X \to Y$ is $\mathbb{C}$-linear.

For given mappings $\phi, \psi : A \to A$, we define

$$\begin{aligned} E_\lambda \phi(x,y) &:= \phi(\lambda(x+y)) - \lambda\phi(x) - \lambda\phi(y), \\ F_\lambda \psi(x,y) &:= 2\psi\left(\lambda \frac{x+y}{2}\right) - \lambda\psi(x) - \lambda\psi(y) \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$.

Lemma 2.2. *Let $g, h : X \to Y$ be mappings satisfying*

$$\|E_\lambda g(a,b)\| + \|E_\lambda h(a,b)\| \le \|pF_\lambda g(a,b)\| + \|pF_\lambda h(a,b)\| \tag{2.1}$$

for all $\lambda \in \mathbb{T}^1$ and all $a, b \in X$. Then $g, h : X \to Y$ are $\mathbb{C}$-linear.

Proof. Letting $b = a$ and $\lambda = 1$ in (2.1), $g(2a) = 2g(a)$ and $h(2a) = 2h(a)$ for all $a \in X$. So

$$\begin{aligned} \|E_\lambda g(a,b)\| + \|E_\lambda h(a,b)\| &\le \|pF_\lambda g(a,b)\| + \|pF_\lambda h(a,b)\| \\ &= \|pE_\lambda g(a,b)\| + \|pE_\lambda h(a,b)\| \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $a, b \in X$. Thus

$$g(\lambda(a+b)) = \lambda g(a) + \lambda g(b),$$
$$h(\lambda(a+b)) = \lambda h(a) + \lambda h(b)$$

for all $\lambda \in \mathbb{T}^1$ and all $a, b \in X$, since $|p| < 1$. By Lemma 2.1, $g, h : X \to Y$ are $\mathbb{C}$-linear. $\square$

Theorem 2.3. *Let $\varphi : A^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x,y) \le \frac{L}{2}\varphi(x,y) \tag{2.2}$$

for all $x, y \in A$. Let $g, h : A \to A$ be mappings satisfying

$$\|E_\lambda g(x,y)\| + \|E_\lambda h(x,y)\| \le \|pF_\lambda g(x,y)\| + \|pF_\lambda h(x,y)\| + \varphi(x,y), \tag{2.3}$$

$$\|[g,h](x \odot y) - [g,h](x) \odot y - x \odot [g,h](y)\| \le \varphi(x,y) \tag{2.4}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. Then there exist unique $\mathbb{C}$-linear mappings $G, H : A \to A$ such that

$$\|g(x) - G(x)\| + \|h(x) - H(x)\| \le \frac{L}{2(1-L)}\varphi(x,x) \tag{2.5}$$

for all $x \in A$. Furthermore, the $\mathbb{C}$-linear mapping $[G, H] : A \to A$ is a Lie bracket Jordan derivation in A.

Proof. Letting $y = x$ and $\lambda = 1$ in (2.3), we get

$$\|g(2x) - 2g(x)\| + \|h(2x) - 2h(x)\| \le \varphi(x,x) \tag{2.6}$$

for all $x \in A$.

Consider the set

$$S := \{(\phi, \psi) : \phi, \psi : A \to A\}$$

and introduce the generalized metric on S:

$$d((\phi_1, \psi_1), (\phi_2, \psi_2)) = \inf\{\mu \in \mathbb{R}_+ : \|\phi_1(x) - \phi_2(x)\| + \|\psi_1(x) - \psi_2(x)\| \le \mu\varphi(x,x),\ \forall x \in A\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [12]).

Now we consider the linear mapping $J : S \to S$ such that

$$J(\phi, \psi)(x) := \left(2\phi\left(\frac{x}{2}\right), 2\psi\left(\frac{x}{2}\right)\right)$$

for all $x \in A$.

Let $(\phi_1, \psi_1), (\phi_2, \psi_2) \in S$ be given such that $d((\phi_1, \psi_1), (\phi_2, \psi_2)) = \varepsilon$. Then

$$\|\phi_1(x) - \phi_2(x)\| + \|\psi_1(x) - \psi_2(x)\| \leq \varepsilon\varphi(x, x)$$

for all $x \in A$. Since

$$\begin{aligned} &\left\|2\phi_1\left(\frac{x}{2}\right) - 2\phi_2\left(\frac{x}{2}\right)\right\| + \left\|2\psi_1\left(\frac{x}{2}\right) - 2\psi_2\left(\frac{x}{2}\right)\right\| \\ &\leq 2\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq 2\varepsilon\frac{L}{2}\varphi(x, x) = L\varepsilon\varphi(x, x) \end{aligned}$$

for all $x \in A$, $d(J(\phi_1, \psi_1), J(\phi_2, \psi_2)) \leq L\varepsilon$. This means that

$$d(J(\phi_1, \psi_1), J(\phi_2, \psi_2)) \leq Ld((\phi_1, \psi_1), (\phi_2, \psi_2))$$

for all $(\phi_1, \psi_1), (\phi_2, \psi_2) \in S$.

It follows from (2.6) that

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| + \left\|h(x) - 2h\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2}\varphi(x, x)$$

for all $x \in A$. So $d((g, h), J(g, h)) \leq \frac{L}{2}$.

By Theorem 1.2, there exist mappings $G, H : A \to A$ satisfying the following:

(1) (G, H) is a fixed point of J, i.e.,

$$(G(x), H(x)) = \left(2G\left(\frac{x}{2}\right), 2H\left(\frac{x}{2}\right)\right) \tag{2.7}$$

for all $x \in A$. The pair (G, H) is a unique fixed point of J. This implies that the pair (G, H) is a unique pair satisfying (2.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|g(x) - G(x)\| + \|h(x) - H(x)\| \leq \mu\varphi(x, x)$$

for all $x \in A$;

(2) $d(J^l(g, h), (G, H)) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^l g\left(\frac{x}{2^l}\right) = G(x), \qquad \lim_{l\to\infty} 2^l h\left(\frac{x}{2^l}\right) = H(x)$$

for all $x \in A$;

(3) $d((g, h), (G, H)) \leq \frac{1}{1-L} d((g, h), J(g, h))$, which implies

$$\|g(x) - G(x)\| + \|h(x) - H(x)\| \leq \frac{L}{2(1-L)}\varphi(x, x)$$

for all $x \in A$.

It follows from (2.3) and (2.7) that

$$\begin{aligned}\|E_\lambda G(x,y)\| + \|E_\lambda H(x,y)\| &= \lim_{n\to\infty} 2^n \left(\left\|E_\lambda g\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| + \left\|E_\lambda h\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\|\right)\\ &\le \lim_{n\to\infty} 2^n \left(\left\|pF_\lambda g\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| + \left\|pF_\lambda h\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\|\right) + \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\\ &= \|pF_\lambda G(x,y)\| + \|pF_\lambda H(x,y)\|\end{aligned}$$

for all $x, y \in A$, since $\lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right) \le \lim_{n\to\infty}\frac{2^nL^n}{2^n}\varphi(x,y) = 0$. So

$$\|E_\lambda G(x,y)\| + \|E_\lambda H(x,y)\| \le \|pF_\lambda G(x,y)\| + \|pF_\lambda H(x,y)\|$$

for all $x, y \in A$. By Lemma 2.2, the mappings $G, H : A \to A$ are $\mathbb{C}$-linear. So there exist unique $\mathbb{C}$-linear mappings $G, H : A \to A$ satisfying (2.5).

It follows from (2.4) that

$$\begin{aligned}&\|[G,H](x\odot y) - [G,H](x)\odot y - x\odot[G,H](y)\|\\ &= \lim_{n\to\infty} 4^n \left\|[g,h]\left(\frac{x\odot y}{4^n}\right) - [g,h]\left(\frac{x}{2^n}\right)\odot\frac{y}{2^n} - \frac{x}{2^n}\odot[g,h]\left(\frac{y}{2^n}\right)\right\|\\ &\le \lim_{n\to\infty} 4^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right) \le \lim_{n\to\infty}\frac{4^nL^n}{4^n}\varphi(x,y) = 0\end{aligned}$$

for all $x, y \in A$. So

$$[G,H](x\odot y) = [G,H](x)\odot y + x\odot[G,H](y)$$

for all $x, y \in A$.

Therefore, the $\mathbb{C}$-linear mapping $[G,H] : A \to A$ is a Lie bracket Jordan derivation in A. $\square$

Corollary 2.4. *Let $r > 2$ and θ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying*

$$\|E_\lambda g(x,y)\| + \|E_\lambda h(x,y)\| \le \|pF_\lambda g(x,y)\| + \|pF_\lambda h(x,y)\| + \theta(\|x\|^r + \|y\|^r), \tag{2.8}$$

$$\|[g,h](x\odot y) - [g,h](x)\odot y - x\odot[g,h](y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{2.9}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y \in A$. Then there exist unique $\mathbb{C}$-linear mappings $G, H : A \to A$ such that

$$\|g(x) - G(x)\| + \|h(x) - H(x)\| \le \frac{4\theta}{2^r - 4}\|x\|^r \tag{2.10}$$

for all $x \in A$. Furthermore, the $\mathbb{C}$-linear mapping $[G,H] : A \to A$ is a Lie bracket Jordan derivation in A.

Proof. The proof follows from Theorem 2.3 by taking $L = 2^{2-r}$ and $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. $\square$

Theorem 2.5. *Let $\varphi : A^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right) \tag{2.11}$$

for all $x, y \in A$. Let $g, h : A \to A$ be mappings satisfying (2.3) *and* (2.4)*. Then there exist unique $\mathbb{C}$-linear mappings $G, H : A \to A$ such that*

$$\|g(x) - G(x)\| + \|h(x) - H(x)\| \le \frac{1}{2(1-L)}\varphi(x,x)$$

for all $x \in A$. Furthermore, the $\mathbb{C}$-linear mapping $[G,H] : A \to A$ is a Lie bracket Jordan derivation in A.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $J : S \to S$ such that

$$J(\phi,\psi)(x) := \left(\frac{1}{2}\phi(2x), \frac{1}{2}\psi(2x)\right)$$

for all $x \in A$.

It follows from (2.6) that

$$\left\|g(x) - \frac{1}{2}g(2x)\right\| + \left\|h(x) - \frac{1}{2}h(2x)\right\| \le \frac{1}{2}\varphi(x,x)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let $r < 1$ and θ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying* (2.8) *and* (2.9)*. Then there exist unique $\mathbb{C}$-linear mappings $G, H : A \to A$ such that*

$$\|g(x) - G(x)\| + \|h(x) - H(x)\| \le \frac{4\theta}{4-2^r}\|x\|^r \tag{2.12}$$

for all $x \in A$. Furthermore, the $\mathbb{C}$-linear mapping $[G,H] : A \to A$ is a Lie bracket Jordan derivation in A.

Proof. The proof follows from Theorem 2.5 by taking $L = 2^{r-2}$ and $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. □

3. Hyers-Ulam Stability of Lie Bracket Jordan Derivations in Banach Jordan Algebras: Direct Method

In this section, we prove the Hyers-Ulam stability of Lie bracket Jordan derivations in complex Banach Jordan algebras by using the direct method.

Theorem 3.1. *Let $\varphi : A^2 \to [0, \infty)$ be a function such that*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty \tag{3.1}$$

and $g, h : A \to A$ be mappings satisfying (2.3) *and* (2.4)*. Then there exist unique $\mathbb{C}$-linear mappings $G, H : A \to A$ such that*

$$\|g(x) - G(x)\| + \|h(x) - H(x)\| \leq \frac{1}{2}\Psi(x, x) \tag{3.2}$$

for all $x \in A$. Furthermore, the $\mathbb{C}$-linear mapping $[G, H] : A \to A$ is a Lie bracket Jordan derivation in A.

Proof. Letting $y = x$ and $\lambda = 1$ in (2.3), we get

$$\|g(2x) - 2g(x)\| + \|h(2x) - 2h(x)\| \leq \varphi(x, x) \tag{3.3}$$

for all $x \in A$. Thus

$$\left\| g(x) - \frac{1}{2} g(2x) \right\| + \left\| h(x) - \frac{1}{2} h(2x) \right\| \leq \frac{1}{2} \varphi(x, x)$$

for all $x \in A$. So

$$\begin{aligned} &\left\| \frac{1}{2^l} g\left(2^l x\right) - \frac{1}{2^m} g\left(2^m x\right) \right\| + \left\| \frac{1}{2^l} h\left(2^l x\right) - \frac{1}{2^m} h\left(2^m x\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \left(\left\| \frac{1}{2^j} g\left(2^j x\right) - \frac{1}{2^{j+1}} g\left(2^{j+1} x\right) \right\| + \left\| \frac{1}{2^j} h\left(2^j x\right) - \frac{1}{2^{j+1}} h\left(2^{j+1} x\right) \right\| \right) \\ &\leq \sum_{j=l}^{m-1} \frac{1}{2} \cdot \frac{1}{2^j} \varphi\left(2^j x, 2^j x\right) \end{aligned} \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (3.4) that the sequences $\{\frac{1}{2^k} g(2^k x)\}$ and $\{\frac{1}{2^k} h(2^k x)\}$ are Cauchy sequences for all $x \in A$. Since A is complete, the sequences $\{\frac{1}{2^k} g(2^k x)\}$ and $\{\frac{1}{2^k} h(2^k x)\}$ converge. So one can define the mappings $G, H : A \to A$ by

$$G(x) := \lim_{k \to \infty} \frac{1}{2^k} g\left(2^k x\right), \qquad H(x) := \lim_{k \to \infty} \frac{1}{2^k} h\left(2^k x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.2).

It follows from (2.3) that

$$\begin{aligned}
&\|E_\lambda G(x,y)\| + \|E_\lambda H(x,y)\| = \lim_{n\to\infty} 2^n \left(\left\| E_\lambda g\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| + \left\| E_\lambda h\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\|\right) \\
&\le \lim_{n\to\infty} 2^n \left(\left\| pF_\lambda g\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| + \left\| pF_\lambda h\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\|\right) + \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&= \|pF_\lambda G(x,y)\| + \|pF_\lambda H(x,y)\|
\end{aligned}$$

for all $x, y \in A$, since $\lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le \lim_{n\to\infty} \frac{2^n L^n}{2^n} \varphi(x,y) = 0$. So

$$\|E_\lambda G(x,y)\| + \|E_\lambda H(x,y)\| \le \|pF_\lambda G(x,y)\| + \|pF_\lambda H(x,y)\|$$

for all $x, y \in A$. By Lemma 2.2, the mappings $G, H : A \to A$ are $\mathbb{C}$-linear.

Now, let $T, L : A \to A$ be another $\mathbb{C}$-linear mappings satisfying (3.2). Then we have

$$\begin{aligned}
&\|G(x) - T(x)\| + \|H(x) - L(x)\| \\
&= \left\| \frac{1}{2^q} G(2^q x) - \frac{1}{2^q} T(2^q x)\right\| + \left\| \frac{1}{2^q} H(2^q x) - \frac{1}{2^q} L(2^q x)\right\| \\
&\le \left\| \frac{1}{2^q} G(2^q x) - \frac{1}{2^q} g(2^q x)\right\| + \left\| \frac{1}{2^q} T(2^q x) - \frac{1}{2^q} g(2^q x)\right\| \\
&\quad + \left\| \frac{1}{2^q} H(2^q x) - \frac{1}{2^q} h(2^q x)\right\| + \left\| \frac{1}{2^q} L(2^q x) - \frac{1}{2^q} h(2^q x)\right\| \\
&\le \frac{1}{2^q} \Psi(2^q x, 2^q x),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in A$. So we can conclude that $G(x) = T(x)$ and $H(x) = L(x)$ for all $x \in A$. This proves the uniqueness of (G, H).

It follows from (2.4) that

$$\begin{aligned}
&\|[G,H](x \odot y) - [G,H](x) \odot y - x \odot [G,H](y)\| \\
&= \lim_{n\to\infty} \frac{1}{4^n} \|[g,h](4^n x \odot y) - [g,h](2^n x) \odot 2^n y - 2^n x \odot [g,h](2^n y)\| \\
&\le \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$[G,H](x \odot y) = [G,H](x) \odot y + x \odot [G,H](y)$$

for all $x, y \in A$.

Therefore, the $\mathbb{C}$-linear mapping $[G,H] : A \to A$ is a Lie bracket Jordan derivation in A. □

Corollary 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and $g, h : A \to A$ be mappings satisfying* (2.8) *and* (2.9)*. Then there exist unique $\mathbb{C}$-linear mappings $G, H : A \to A$ satisfying* (2.12)*. Furthermore, the $\mathbb{C}$-linear mapping $[G, H] : A \to A$ is a Lie bracket Jordan derivation in A.*

Similarly, we can obtain the following.

Theorem 3.3. *Let $\varphi : A^2 \to [0, \infty)$ be a function such that*

$$\sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.5}$$

for all $x, y \in A$ and $g, h : A \to A$ be mappings satisfying (2.3) *and* (2.4)*. Then there exist unique $\mathbb{C}$-linear mappings $G, H : A \to A$ such that*

$$\|g(x) - G(x)\| + \|h(x) - H(x)\| \le \frac{1}{2}\Psi(x, x) \tag{3.6}$$

for all $x \in A$, where

$$\Psi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$$

for all $x, y \in A$. Furthermore, the $\mathbb{C}$-linear mapping $[G, H] : A \to A$ is a Lie bracket Jordan derivation in A.

Proof. It follows from (3.3) that

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| + \left\|h(x) - 2h\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in A$.

By the same reasoning as in the proof of Theorem 3.1, there exist unique $\mathbb{C}$-mappings $G, H : A \to A$ satisfying (3.6). The $\mathbb{C}$-linear mappings $G, H : A \to A$ are defined by

$$G(x) = \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right), \qquad H(x) = \lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

It follows from (2.4) that

$$\begin{aligned}
&\|[G, H](x \odot y) - [G, H](x) \odot y - x \odot [G, H](y)\| \\
&\quad = \lim_{n\to\infty} 4^n \left\|[g, h]\left(\frac{x \odot y}{4^n}\right) - [g, h]\left(\frac{x}{2^n}\right) \odot \frac{y}{2^n} - \frac{x}{2^n} \odot [g, h]\left(\frac{y}{2^n}\right)\right\| \\
&\quad \le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$[G, H](x \odot y) = [G, H](x) \odot y + x \odot [G, H](y)$$

for all $x, y \in A$.

Therefore, the $\mathbb{C}$-linear mapping $[G, H] : A \to A$ is a Lie bracket Jordan derivation in A. $\square$

Corollary 3.4. *Let $r > 2$ and θ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying* (2.8) *and* (2.9)*. Then there exist unique $\mathbb{C}$-linear mappings $G, H : A \to A$ satisfying* (2.10)*. Furthermore, the $\mathbb{C}$-linear mapping $[G, H] : A \to A$ is a Lie bracket Jordan derivation in A.*

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