J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2021.28.2.155 ISSN(Online) 2287-6081
Volume 28, Number 2 (May 2021), Pages 155–185

SUM AND PRODUCT THEOREMS RELATING TO GENERALIZED RELATIVE ORDER (α, β) AND GENERALIZED RELATIVE TYPE (α, β) OF ENTIRE FUNCTIONS

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Abstract. Orders and types of entire functions have been actively investigated by many authors. In this paper, we investigate some basic properties in connection with sum and product of generalized relative order (α, β), generalized relative type (α, β) and generalized relative weak type (α, β) of entire functions with respect to another entire function where α, β are continuous non-negative functions on $(-\infty, +\infty)$.

1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined as $M_f = \max\limits_{|z|=r} |f(z)|$. Moreover, if f is non-constant entire then $M_f(r)$ is also strictly increasing and continuous functions of r. Therefore its inverse $M_f^{-1} : (M_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s\to+\infty} M_f^{-1}(s) = \infty$. Further a non-constant entire function f is said to have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds (see [2]). We use the standard notations and definitions of the theory of entire functions which are available in [6] and [7], and therefore we do not explain those in details.

Let L be a class of continuous non-negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly

Received by the editors March 27, 2020. Accepted May 07, 2021.
2010 *Mathematics Subject Classification.* 30D35, 30D30, 30D20.
Key words and phrases. entire function, growth, composition, generalized relative order (α, β), generalized relative type (α, β), generalized relative weak type (α, β).
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$L^0 \subset L$. Moreover we assume that throughout the present paper α and β always denote the functions belonging to L^0 unless otherwise specifically stated. The value

$$\varrho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} (\alpha \in L, \beta \in L)$$

is called [5] generalized order (α, β) of f .

Now we introduce the definition of the generalized order (α, β) and generalized lower order (α, β) of an entire function after giving a minor modification to the original definition of generalized order (α, β) of an entire function (e.g. see, [5]).

Definition 1.1. The generalized order (α, β) denoted by $\varrho_{(\alpha,\beta)}[f]$ and generalized lower order (α, β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:

$$\varrho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

Mainly the growth investigation of entire functions has usually been done through their maximum moduli function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. Now in order to make some progress in the study of relative order, one may introduce the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of an entire function with respect to another entire function in the following way:

Definition 1.2. The generalized relative order (α, β) denoted by $\varrho_{(\alpha,\beta)}[f]_g$ and generalized relative lower order (α, β) denoted by $\lambda_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to an entire function g are defined as:

$$\varrho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}.$$

If fact some works on generalized relative order (α, β) related to the growth of entire Dirichlet series have been explored by Mulyava et al. (see, e.g., [3], [4]).

Further if generalized relative order (α, β) and the generalized relative lower order (α, β) of an entire function f with respect to an entire function g are the same, then f is called a function of regular generalized relative (α, β) growth with respect to g. Otherwise, f is said to be irregular generalized relative (α, β) growth.with respect to g.

Now in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as generalized relative type (α,β) and generalized relative lower type (α,β) of entire function with respect to an entire function which are as follows:

Definition 1.3. The generalized relative type (α,β) denoted by $\sigma_{(\alpha,\beta)}[f]_g$ and generalized relative lower type (α,β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to an entire function g having non-zero finite generalized relative order (α,β) are defined as :

$$\begin{aligned}\sigma_{(\alpha,\beta)}[f]_g &= \limsup_{r\to+\infty}\frac{\exp(\alpha(M_g^{-1}M_f(r))))}{(\exp(\beta(r)))^{\varrho_{(\alpha,\beta)}[f]_g}}\\ \text{and } \overline{\sigma}_{(\alpha,\beta)}[f]_g &= \liminf_{r\to+\infty}\frac{\exp(\alpha(M_g^{-1}M_f(r))))}{(\exp(\beta(r)))^{\varrho_{(\alpha,\beta)}[f]_g}}.\end{aligned}$$

Analogously, to determine the relative growth of an entire function f having same non-zero finite generalized relative lower order (α,β) with respect to an entire function g, one can introduce the definition of generalized relative upper weak type (α,β) and generalized relative weak type (α,β) of f with respect to g of finite positive generalized relative lower order (α,β) in the following way:

Definition 1.4. The generalized relative upper weak type (α,β) denoted by $\tau_{(\alpha,\beta)}[f]_g$ and generalized relative weak type (α,β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to an entire function g having non-zero finite generalized relative lower order (α,β) are defined as :

$$\begin{aligned}\tau_{(\alpha,\beta)}[f]_g &= \limsup_{r\to+\infty}\frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}\\ \text{and } \overline{\tau}_{(\alpha,\beta)}[f]_g &= \liminf_{r\to+\infty}\frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}.\end{aligned}$$

In this connection, we finally remind the following definition which is needed in the sequel.

Definition 1.5. A pair of entire functions f and g are said to have *mutually Property* (X) if for all sufficiently large r,

$$M_{f\cdot g}(r) > M_f(r) \quad \text{and} \quad M_{f\cdot g}(r) > M_g(r)$$

hold simultaneously.

Here, in this paper, we aim at investigating some basic properties of generalized relative order (α, β), generalized relative type (α, β) and generalized relative weak type (α, β) of a entire function with respect to another entire function under somewhat different conditions. Throughout this paper, we assume that all the growth indicators are all nonzero finite.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([2])**.** *Suppose that f be an entire function, $a > 1, 0 < b < a$. Then*

$$M_f(ar) > bM_f(r).$$

Lemma 2.2 ([2])**.** *Let f be an entire function which satisfies the Property (A) then for any positive integer n and for all sufficiently large r,*

$$[M_f(r)]^n \leq M_f(r^\delta)$$

holds where $\delta > 1$.

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let f_1, f_2 and g_1 be three entire functions such that at least f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1. Then*

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}.$$

The equality holds when $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1}$ with at least f_j is of regular generalized relative growth (α, β) with respect to g_1 where $i, j = 1, 2$ and $i \neq j$.

Proof. If $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = 0$ then the result is obvious. So we suppose that $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} > 0$. We can clearly assume that $\lambda_{(\alpha,\beta)}[f_k]_{g_1}$ is finite for $k = 1, 2$. Also let $\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\} = \Delta$ and f_2 is of regular generalized relative growth (α, β) with respect to g_1. Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, we have for a sequence values of r tending to infinity that

$$M_{f_1}(r) \leq M_{g_1}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)\beta(r)])$$

$$\text{i.e., } M_{f_1}(r) \leq M_{g_1}(\alpha^{-1}[(\Delta + \varepsilon)\beta(r)]). \tag{3.1}$$

Also for any arbitrary $\varepsilon > 0$ from the definition of $\varrho_{(\alpha,\beta)}[f_2]_{g_1} (= \lambda_{(\alpha,\beta)}[f_2]_{g_1})$, we obtain for all sufficiently large values of r that

$$M_{f_2}(r) \le M_{g_1}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[f_2]_{g_1} + \varepsilon)\beta(r)])$$

$$i.e., \; M_{f_2}(r) \le M_{g_1}(\alpha^{-1}[(\Delta + \varepsilon)\beta(r)]). \tag{3.2}$$

So in view of (3.1) and (3.2), we obtain for a sequence values of r tending to infinity that

$$M_{f_1 \pm f_2}(r) < 2M_{g_1}(\alpha^{-1}[(\Delta + \varepsilon)\beta(r)]). \tag{3.3}$$

Therefore in view of Lemma 2.1, we obtain from (3.3) for a sequence values of r tending to infinity that

$$\frac{1}{2}M_{f_1 \pm f_2}(r) < M_{g_1}(\alpha^{-1}[(\Delta + \varepsilon)\beta(r)])$$

$$i.e., \; M_{f_1 \pm f_2}(\frac{r}{3}) < M_{g_1}(\alpha^{-1}[(\Delta + \varepsilon)\beta(r)])$$

$$i.e., \; \frac{\alpha(M_{g_1}^{-1}(M_{f_1 \pm f_2}(\frac{r}{3})))}{\beta(r)} < (\Delta + \varepsilon).$$

Hence

$$\begin{aligned} \liminf_{r \to +\infty} \frac{\alpha(M_{g_1}^{-1}(M_{f_1 \pm f_2}(\frac{r}{3})))}{\beta(r)} &= \liminf_{r \to +\infty} \frac{\alpha(M_{g_1}^{-1}(M_{f_1 \pm f_2}(\frac{r}{3})))}{\beta(\frac{r}{3})} . \limsup_{r \to +\infty} \frac{\beta(\frac{r}{3})}{\beta(r)} \le \Delta + \varepsilon. \\ i.e., \; \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} &\le \Delta + \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary,

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \le \Delta = \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}. \tag{3.4}$$

Similarly, if we consider that f_1 is of regular generalized relative growth (α, β) with respect to g_1 or both f_1 and f_2 are of regular generalized relative growth (α, β) with respect to g_1, then one can easily obtain (3.4).

Moreover without loss of any generality, let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and $f = f_1 \pm f_2$. Then in view of (3.4) we get that $\lambda_{(\alpha,\beta)}[f]_{g_1} \le \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. As, $f_2 = \pm(f - f_1)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[f_2]_{g_1} \le \max\{\lambda_{(\alpha,\beta)}[f]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_1}\}$. As we assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, therefore we have $\lambda_{(\alpha,\beta)}[f_2]_{g_1} \le \lambda_{(\alpha,\beta)}[f]_{g_1}$ and hence $\lambda_{(\alpha,\beta)}[f]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}$. Therefore, $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_i]_{g_1} \mid i = 1, 2$ provided $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \ne \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Thus the theorem follows. □

Theorem 3.2. *Let f_1, f_2, g_1 be three entire functions such that $\varrho_{(\alpha,\beta)}[f_1]_{g_1}$ and $\varrho_{(\alpha,\beta)}[f_2]_{g_1}$ exists. Then*

$$\varrho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \le \max\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_2]_{g_1}\}.$$

The equality holds when $\varrho_{(\alpha,\beta)}[f_1]_{g_1} \ne, \varrho_{(\alpha,\beta)}[f_2]_{g_1}$.

We omit the proof of Theorem 3.2 as it can easily be carried out in the line of Theorem 3.1.

Theorem 3.3. *Let f_1, g_1, g_2 be three entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ exists. Then*

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \ge \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}.$$

The equality holds when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \ne \lambda_{(\alpha,\beta)}[f_1]_{g_2}$.

Proof. If $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \infty$ then the result is obvious. So we suppose that $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} < \infty$. We can clearly assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_k}$ is finite for $k = 1, 2$. Also let $\Psi = \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}$. Now for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{(\alpha,\beta)}[f_1]_{g_k}$ where $k = 1, 2$, we have for all sufficiently large values of r that

$$M_{g_k}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[f_1]_{g_k} - \varepsilon)\beta(r)]) \le M_{f_1}(r)$$
$$i.e,\ M_{g_k}(\alpha^{-1}[(\Psi - \varepsilon)\beta(r)]) \le M_{f_1}(r)$$

Now in view of Lemma 2.1, we obtain from above for all sufficiently large values of r that

$$M_{g_1 \pm g_2}(\alpha^{-1}[(\Psi - \varepsilon)\beta(r)])$$
$$< M_{g_1}(\alpha^{-1}[(\Psi - \varepsilon)\beta(r)]) + M_{g_2}(\alpha^{-1}[(\Psi - \varepsilon)\beta(r)])$$
$$i.e.,\ M_{g_1 \pm g_2}(\alpha^{-1}[(\Psi - \varepsilon)\beta(r)]) < 2M_{f_1}(r)$$
$$i.e.,\ M_{g_1 \pm g_2}(\alpha^{-1}[(\Psi - \varepsilon)\beta(r)]) < M_{f_1}(3r)$$
$$i.e.,\ \frac{\alpha(M^{-1}_{g_1 \pm g_2}(M_{f_1}(3r)))}{\beta(r)} > \Psi - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \ge \Psi = \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}. \tag{3.5}$$

Now without loss of any generality, we may consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $g = g_1 \pm g_2$. Then in view of (3.5) we get that $\lambda_{(\alpha,\beta)}[f_1]_g \ge \lambda_{(\alpha,\beta)}[f_1]_{g_1}$. Further, $g_1 = (g \pm g_2)$ and in this case we obtain that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \ge \min\{\lambda_{(\alpha,\beta)}[f_1]_g, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}$.

As we assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, therefore we have $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_1]_g$ and hence $\lambda_{(\alpha,\beta)}[f_1]_g = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}$. Therefore, $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_i} \mid i = 1,2$ provided $\lambda_{(\alpha,\beta)}[f_1]_{g_1} \neq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Thus the theorem follows. □

Theorem 3.4. *Let f_1, g_1, g_2 be three entire functions such that f_1 is of regular generalized relative growth (α, β) with respect to at least any one of g_1 or g_2. Then*

$$\varrho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \min\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_1]_{g_2}\}.$$

The equality holds when $\varrho_{(\alpha,\beta)}[f_1]_{g_i} < \varrho_{(\alpha,\beta)}[f_1]_{g_j}$ with at least f_1 is of regular generalized relative growth (α, β) with respect to g_j where $i = j = 1, 2$ and $i \neq j$.

We omit the proof of Theorem 3.4 as it can easily be carried out in the line of Theorem 3.3.

Theorem 3.5. *Let f_1, f_2, g_1, g_2 be entire functions, then*

$$\varrho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} \leq \max[\min\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\varrho_{(\alpha,\beta)}[f_2]_{g_1}, \varrho_{(\alpha,\beta)}[f_2]_{g_2}\}]$$

when the following two conditions holds:
(i) $\varrho_{(\alpha,\beta)}[f_1]_{g_i} < \varrho_{(\alpha,\beta)}[f_1]_{g_j}$ with at least f_1 is of regular generalized relative growth (α, β) with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and
(ii) $\varrho_{(\alpha,\beta)}[f_2]_{g_i} < \varrho_{(\alpha,\beta)}[f_2]_{g_j}$ with at least f_2 is of regular generalized relative growth (α, β) with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.
The equality holds when $\varrho_{(\alpha,\beta)}[f_i]_{g_1} < \varrho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\varrho_{(\alpha,\beta)}[f_i]_{g_2} < \varrho_{(\alpha,\beta)}[f_j]_{g_2}$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Proof. Suppose that the conditions (i) and (ii) of the theorem holds. Therefore in view of Theorem 3.2 and Theorem 3.4 we get that

$$\begin{aligned} &\max[\min\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\varrho_{(\alpha,\beta)}[f_2]_{g_1}, \varrho_{(\alpha,\beta)}[f_2]_{g_2}\}] \\ &= \max[\varrho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2}, \varrho_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}] \\ &\geq \varrho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} \ . \end{aligned} \tag{3.6}$$

As $\varrho_{(\alpha,\beta)}[f_i]_{g_1} < \varrho_{(\alpha,\beta)}[f_j]_{g_1}$ and $\varrho_{(\alpha,\beta)}[f_i]_{g_2} < \varrho_{(\alpha,\beta)}[f_j]_{g_2}$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we obtain that

$$\text{either } \min\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_1]_{g_2}\} > \min\{\varrho_{(\alpha,\beta)}[f_2]_{g_1}, \varrho_{(\alpha,\beta)}[f_2]_{g_2}\} \text{ or}$$

$$\min\{\varrho_{(\alpha,\beta)}[f_2]_{g_1}, \varrho_{(\alpha,\beta)}[f_2]_{g_2}\} > \min\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_1]_{g_2}\} \text{ holds.}$$

Therefore in view of the conditions (i) and (ii) of the theorem, it follows from above that

$$\text{either } \varrho_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} > \varrho_{(\alpha,\beta)}[f_2]_{g_1\pm g_2} \text{ or } \varrho_{(\alpha,\beta)}[f_2]_{g_1\pm g_2} > \varrho_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}$$

which is the condition for holding equality in (3.6).

Hence the theorem follows. □

Theorem 3.6. *Let f_1, f_2, g_1, g_2 be entire functions, then*

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1\pm g_2} \geq \min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}]$$

when the following two conditions holds:
(i) $\varrho_{(\alpha,\beta)}[f_i]_{g_1} > \varrho_{(\alpha,\beta)}[f_j]_{g_1}$ with at least f_j is of regular generalized relative growth (α,β) with respect to g_1 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and
(ii) $\varrho_{(\alpha,\beta)}[f_i]_{g_2} > \varrho_{(\alpha,\beta)}[f_j]_{g_2}$ with at least f_j is of regular generalized relative growth (α,β) with respect to g_2 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.
The sign of equality holds when $\varrho_{(\alpha,\beta)}[f_1]_{g_i} < \varrho_{(\alpha,\beta)}[f_1]_{g_j}$ and $\varrho_{(\alpha,\beta)}[f_2]_{g_i} < \varrho_{(\alpha,\beta)}[f_2]_{g_j}$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Proof. Let us consider that the conditions (i) and (ii) of the theorem holds. Therefore in view of Theorem 3.1 and Theorem 3.3, we obtain that

$$\begin{aligned} &\min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}] \\ &= \min[\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}, \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}] \\ &\geq \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1\pm g_2} \ . \end{aligned} \tag{3.7}$$

Since $\varrho_{(\alpha,\beta)}[f_1]_{g_i} < \varrho_{(\alpha,\beta)}[f_1]_{g_j}$ and $\varrho_{(\alpha,\beta)}[f_2]_{g_i} < \varrho_{(\alpha,\beta)}[f_2]_{g_j}$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we get that

$$\text{either } \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\} < \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\} \text{ or}$$

$$\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\} < \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\} \text{ holds.}$$

Since condition (i) and (ii) of the theorem holds, it follows from above that

$$\text{either } \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2} \text{ or } \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2} < \lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}$$

which is the condition for holding equality in (3.7).

Hence the theorem follows. □

Theorem 3.7. *Let f_1, f_2, g_1 be three entire functions such that at least f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1 where g_1 satisfy the Property (A) and f_1 , f_2 satisfy the Property (X), then*

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}.$$

Proof. Suppose that $\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} > 0$. Otherwise if $\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = 0$ then the result is obvious. Let us consider that f_2 is of regular generalized relative growth (α, β) with respect to g_1. Also let that $\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\} = \Delta$. We can clearly assume that $\lambda_{(\alpha,\beta)}[f_k]_{g_1}$ is finite for $k = 1, 2$. Now for any arbitrary $\frac{\varepsilon}{2} > 0$, it follows from the definition of $\varrho_{(\alpha,\beta)}[f_1]_{g_1}$, for a sequence values of r tending to infinity that

$$M_{f_1}(r) \le M_{g_1}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})\beta(r)])$$

$$\text{(3.8)} \qquad i.e., \; M_{f_1}(r) \le M_{g_1}(\alpha^{-1}[(\Delta + \frac{\varepsilon}{2})\beta(r)]).$$

Also for any arbitrary $\frac{\varepsilon}{2} > 0$, we obtain from the definition of $\varrho_{(\alpha,\beta)}[f_2]_{g_1}$ $(= \lambda_{(\alpha,\beta)}[f_2]_{g_1})$, for all sufficiently large values of r that

$$M_{f_2}(r) \le M_{g_1}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[f_2]_{g_1} + \frac{\varepsilon}{2})\beta(r)])$$

$$\text{(3.9)} \qquad i.e., \; M_{f_2}(r) \le M_{g_1}(\alpha^{-1}[(\Delta + \frac{\varepsilon}{2})\beta(r)]).$$

Observe that

$$\frac{\Delta + \varepsilon}{\Delta + \frac{\varepsilon}{2}} > 1.$$

Therefore we consider the expression $\frac{\log[\alpha^{-1}[(\Delta+\varepsilon)\beta(r)]]}{\log[\alpha^{-1}[(\Delta+\frac{\varepsilon}{2})\beta(r)]]}$ for all sufficiently large values of r. Thus for any $\delta > 1$, it follows from the above expression for all sufficiently large values of r, say $r \ge r_1 \ge r_0$ that

$$\text{(3.10)} \qquad \frac{\log[\alpha^{-1}[(\Delta + \varepsilon)\beta(r_0)]]}{\log[\alpha^{-1}[(\Delta + \frac{\varepsilon}{2})\beta(r_0)]]} = \delta.$$

Hence from (3.8) and (3.9), we have for a sequence values of r tending to infinity that

$$M_{f_1 \cdot f_2}(r) < [M_{g_1}(\alpha^{-1}[(\Delta + \frac{\varepsilon}{2})\beta(r)])]^2.$$

Now in view of Lemma 2.2, we obtain from above for a sequence values of r tending to infinity that

$$M_{f_1 \cdot f_2}(r) < M_{g_1}(\alpha^{-1}[(\Delta + \frac{\varepsilon}{2})\beta(r)]^{\delta}),$$

since g_1 has the Property (A) and $\delta > 1$. Therefore in view of (3.10), it follows from above for a sequence values of r tending to infinity that

$$M_{f_1 \cdot f_2}(r) < M_{g_1}(\alpha^{-1}[(\Delta + \varepsilon)\beta(r)]).$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \Delta = \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}.$$

Similarly, if we consider that f_1 is of regular generalized relative growth (α, β) with respect to g_1 or both f_1 and f_2 are of regular generalized relative growth (α, β) with respect to g_1, then also one can easily verify that

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \leq \Delta = \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}.$$

Let us now show that $\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \geq \Delta$. Since f_1 , f_2 satisfy the Property (X), then we have $M_{f_1 \cdot f_2}(r) > M_{f_1}(r)$ for all sufficiently large values of r and therefore

$$\frac{\alpha(M_{g_1}^{-1}(M_{f_1 \cdot f_2}(r)))}{\beta(r)} > \frac{\alpha(M_{g_1}^{-1}(M_{f_1}(r)))}{\beta(r)}$$

since $M_{g_1}^{-1}(r)$ is an increasing function of r. So $\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and similarly, $\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} \geq \lambda_{(\alpha,\beta)}[f_2]_{g_1}$.

Hence the theorem follows. □

Now we state the following theorem which can easily be carried out in the line of Theorem 3.7 and therefore its proof is omitted.

Theorem 3.8. *Let f_1, f_2, g_1 be three entire functions such that $\varrho_{(\alpha,\beta)}[f_1]_{g_1}$, $\varrho_{(\alpha,\beta)}[f_1]_{g_2}$ exists where g_1 satisfy the Property (A) and f_1 , f_2 satisfy the Property (X), then*

$$\varrho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \max\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_2]_{g_1}\}.$$

Theorem 3.9. *Let f_1, g_1, g_2 be three entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ exists where f_1 satisfy the Property (A) and g_1, g_2 satisfy the Property (X), then*

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}.$$

Proof. Suppose that $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} < \infty$. Otherwise if $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \infty$ then the result is obvious. Also let $\min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\} = \Psi$ .We can clearly assume that $\lambda_{(\alpha,\beta)}[f_1]_{g_k}$ is finite for $k = 1, 2$. Now for any arbitrary $\varepsilon > 0$, with $\varepsilon < \Psi$, we obtain for all sufficiently large values of r that

$$M_{g_k}(\alpha^{-1}[(\lambda_{(\alpha,\beta)}[f_1]_{g_k} - \frac{\varepsilon}{2})\beta(r)]) \leq M_{f_1}(r)$$

$$i.e., \ M_{g_k}(\alpha^{-1}[(\Psi - \frac{\varepsilon}{2})\beta(r)]) \leq M_{f_1}(r)$$

$$i.e., \ M_{g_k}(r) \leq M_{f_1}[\beta^{-1}[\frac{\alpha(r)}{(\Psi - \frac{\varepsilon}{2})}]] \tag{3.11}$$

Observe that

$$\frac{\Psi - \frac{\varepsilon}{2}}{\Psi - \varepsilon} > 1.$$

Now we consider the expression $\frac{\log[\beta^{-1}[\frac{\alpha(r)}{(\Psi-\varepsilon)}]]}{\log[\beta^{-1}[\frac{\alpha(r)}{(\Psi-\frac{\varepsilon}{2})}]]}$ for all sufficiently large values of r. Thus for any $\delta > 1$, it follows from the above expression for all sufficiently large values of r, say $r \geq r_1 \geq r_0$ that

$$\frac{\log[\beta^{-1}[\frac{\alpha(r_0)}{(\Psi-\varepsilon)}]]}{\log[\beta^{-1}[\frac{\alpha(r_0)}{(\Psi-\frac{\varepsilon}{2})}]]} = \delta. \tag{3.12}$$

Now from (3.11) we have for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) < M_{f_1}[\beta^{-1}(\frac{\alpha(r)}{(\Psi - \frac{\varepsilon}{2})})]^2.$$

Now in view of Lemma 2.2, we obtain from above for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) < M_{f_1}[\beta^{-1}(\frac{\alpha(r)}{(\Psi - \frac{\varepsilon}{2})})^{\delta}],$$

since f_1 has the Property (A) and $\delta > 1$.

Therefore in view of (3.12), it follows from above for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) < M_{f_1}[\beta^{-1}(\frac{\alpha(r)}{(\Psi - \varepsilon)})].$$

Since $\varepsilon > 0$ is arbitrary, therefore from above we get that

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \geq \Psi = \min\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2}\}.$$

Let us now show that $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \leq \Psi$. Since g_1, g_2 satisfy the Property (X), then we have $M_{g_1 \cdot g_2}(r) > M_{g_1}(r)$ for all sufficiently large values of r and therefore $M^{-1}_{g_1 \cdot g_2}(r) < M^{-1}_{g_1}(r)$. Hence

$$\frac{\alpha(M^{-1}_{g_1 \cdot g_2}(M_{f_1}(r)))}{\beta(r)} < \frac{\alpha(M^{-1}_{g_1}(M_{f_1}(r)))}{\beta(r)}.$$

So $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \leq \lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and similarly, $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \leq \lambda_{(\alpha,\beta)}[f_1]_{g_2}$.

Hence the theorem follows. □

Theorem 3.10. *Let f_1, g_1, g_2 be three entire functions such that f_1 is of regular generalized relative growth (α, β) with respect to at least any one of g_1 or g_2 and f_1 satisfy the Property (A) and g_1, g_2 satisfy the Property (X), then*

$$\varrho_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \min\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_1]_{g_2}\}.$$

We omit the proof of Theorem 3.10 as it can easily be carried out in the line of Theorem 3.9.

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 3.5 and Theorem 3.6 respectively.

Theorem 3.11. *Let f_1, f_2, g_1, g_2 be four entire functions such that $g_1 \cdot g_2$, f_1 , f_2 be satisfy the Property (A), f_1, f_2 satisfy the Property (X) and g_1, g_2 satisfy the Property (X), then,*

$$\varrho_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \max[\min\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\varrho_{(\alpha,\beta)}[f_2]_{g_1}, \varrho_{(\alpha,\beta)}[f_2]_{g_2}\}],$$

when the following two conditions holds:
(i) f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2; and
(ii) f_2 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2.

Theorem 3.12. *Let f_1, f_2, g_1, g_2 be four entire functions such that $g_1 \cdot g_2$, f_1 , f_2 be satisfy the Property (A), f_1, f_2 satisfy the Property (X) and g_1, g_2 satisfy the Property (X), then,*

$$\lambda_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}]$$

when the following two conditions holds:
(i) At least f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_1; and
(ii) At least f_1 or f_2 is of regular generalized relative growth (α,β) with respect to g_2.

Next we intend to find out the sum and product theorems of generalized relative type (α,β) (respectively generalized relative lower type (α,β)) and generalized relative waek type (α,β) of an entire function with respect to another entire function taking into consideration of the above theorems.

Theorem 3.13. *Let f_1, f_2, g_1, $g_2$be four entire functions such that $\varrho_{(\alpha,\beta)}[f_1]_{g_1}$, $\varrho_{(\alpha,\beta)}[f_2]_{g_1}$, $\varrho_{(\alpha,\beta)}[f_1]_{g_2}$ and $\varrho_{(\alpha,\beta)}[f_2]_{g_2}$ are all non-zero and finite.*
(A) *If $\varrho_{(\alpha,\beta)}[f_i]_{g_1} > \varrho_{(\alpha,\beta)}[f_j]_{g_1}$ for i, $j = 1,2$ and $i \neq j$, then*

$$\sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_i]_{g_1}.$$

(B) *If* $\varrho_{(\alpha,\beta)}[f_1]_{g_i} < \varrho_{(\alpha,\beta)}[f_1]_{g_j}$ *with at least* f_1 *is of regular generalized relative growth* (α,β) *with respect to* g_j *for* $i,\ j = 1,2$ *and* $i \neq j$, *then*

$$\sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_i}.$$

(C) *Assume the functions* f_1, f_2, g_1 *and* g_2 *satisfy the following conditions:*
(i) $\varrho_{(\alpha,\beta)}[f_1]_{g_i} < \varrho_{(\alpha,\beta)}[f_1]_{g_j}$ *with at least* f_1 *is of regular generalized relative growth* (α,β) *with respect to* g_j *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$;
(ii) $\varrho_{(\alpha,\beta)}[f_2]_{g_i} < \varrho_{(\alpha,\beta)}[f_2]_{g_j}$ *with at least* f_2 *is of regular generalized relative growth* (α,β) *with respect to* g_j *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$;
(iii) $\varrho_{(\alpha,\beta)}[f_i]_{g_1} < \varrho_{(\alpha,\beta)}[f_j]_{g_1}$ *and* $\varrho_{(\alpha,\beta)}[f_i]_{g_2} < \varrho_{(\alpha,\beta)}[f_j]_{g_2}$ *holds simultaneously for* $i = 1,2;\ j = 1,2$ *and* $i \neq j$;
(iv)$\varrho_{(\alpha,\beta)}[f_l]_{g_m} = \max[\min\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\varrho_{(\alpha,\beta)}[f_2]_{g_1}, \varrho_{(\alpha,\beta)}[f_2]_{g_2}\}] \mid l = m = 1,2$;
then we have

$$\sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2$$

and

$$\overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2.$$

Proof. From the definition of generalized relative type (α,β) and generalized relative lower type (α,β), we have for all sufficiently large values of r that

$$M_{f_k}(r) \leq M_{g_l}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_k]_{g_l} + \varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_k]_{g_l}}\}), \tag{3.13}$$

$$M_{f_k}(r) \geq M_{g_l}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_k]_{g_l}}\}]$$

$$i.e.,\ M_{g_l}(r) \leq M_{f_k}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_k]_{g_l}}}))), \tag{3.14}$$

and for a sequence of values of r tending to infinity, we obtain that

$$M_{f_k}(r) \geq M_{g_l}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_k]_{g_l}}\})$$

$$i.e.,\ M_{g_l}(r) \leq M_{f_k}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_k]_{g_l}}}))), \tag{3.15}$$

and

$$M_{f_k}(r) \leq M_{g_l}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_k]_{g_l} + \varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_k]_{g_l}}\}), \tag{3.16}$$

where $\varepsilon > 0$ is any arbitrary positive number, $k = 1,\ 2$ and $l = 1,2$.

Case I: Suppose that $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Also let $\varepsilon(>0)$ be arbitrary. Now in view of (3.13), we get for all sufficiently large values of r that

$$M_{f_1 \pm f_2}(r)(r) \leq M_{g_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\})(1+A)\,. \tag{3.17}$$

where

$$A = \frac{M_{g_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_2]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_2]_{g_1}}\})}{M_{g_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\})},$$

and in view of $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$, and for all sufficiently large values of r, we can make the term A sufficiently small .

Hence for any $\xi = 1+\varepsilon_1$, where $\varepsilon_1 = A$, it follows from (3.17) for all sufficiently large values of r that

$$M_{f_1 \pm f_2}(r)(r) \leq M_{g_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\}) \cdot (1+\varepsilon_1)$$

$$i.e.,\ M_{f_1 \pm f_2}(r)(r) \leq M_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\}] \cdot \xi.$$

Hence making $\xi \to 1+$, we get in view of Theorem 3.2, $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$ and above for all sufficiently large values of r that

$$\limsup_{r \to +\infty} \frac{\exp(\alpha(M_{g_1}^{-1}(M_{f_1 \pm f_2}(r))))}{[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}}} \leq \sigma_{(\alpha,\beta)}[f_1]_{g_1}$$

$$i.e.,\ \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \sigma_{(\alpha,\beta)}[f_1]_{g_1}. \tag{3.18}$$

Now we may consider that $f = f_1 \pm f_2$. Since $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Then $\sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \sigma_{(\alpha,\beta)}[f_1]_{g_1}$. Further, let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 3.2 and $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$, we obtain that $\varrho_{(\alpha,\beta)}[f]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$ holds. Hence in view of (3.18) $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \leq \sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1}$. Therefore $\sigma_{(\alpha,\beta)}[f]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1} \Rightarrow \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly, if we consider $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_2]_{g_1}$, then one can easily verify that $\sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$.

Case II: Let us consider that $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$ hold. Also let $\varepsilon(>0)$ are arbitrary. Now(3.13) and (3.16), we get for a sequence of values of r tending to infinity that

$$M_{f_1 \pm f_2}(r)(r) \leq M_{g_1}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\})(1+B). \tag{3.19}$$

where

$$B = \frac{M_{g_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_2]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_2]_{g_1}}\})}{M_{g_1}(\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\})},$$

and in view of $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$, we can make the term B sufficiently small by taking n sufficiently large and therefore using the similar technique for as executed in the proof of Case I we get from (3.19) that $\overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}$ when $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$ hold.

Likewise, if we consider $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_2]_{g_1}$, then one can easily verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III: Let us consider that $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α, β) with respect to g_2. Therefore in view of (3.14) and (3.15), we obtain for a sequence of values of r tending to infinity that

$$M_{g_1 \pm g_2}(r) \leq M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))(1 + C), \tag{3.20}$$

where

$$C = \frac{M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_2} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_2}}})))}{M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))},$$

and since $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$, we can make the term C sufficiently small by taking r sufficiently large. Hence for any $\xi = 1 + \varepsilon_1$, where $\varepsilon_1 = C$, we get from (3.20) and Theorem 3.4, for a sequence of values of r tending to infinity that

$$M_{g_1 \pm g_2}(r) < M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))(1 + \varepsilon_1)$$

$$i.e., \ M_{g_1 \pm g_2}(r) < M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))\xi,$$

Hence, making $\xi \to 1+$, we obtain from above for a sequence of values of r tending to infinity that

$$(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2}} < \exp(\alpha(M^{-1}_{g_1 \pm g_2}(M_{f_1}(r))))$$

Since $\varepsilon > 0$ is arbitrary, we find that

$$\sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \sigma_{(\alpha,\beta)}[f_1]_{g_1} \ . \tag{3.21}$$

Now we may consider that $g = g_1 \pm g_2$. Also $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ and at least f_1 is of regular generalized relative growth (α, β) with respect to g_2. Then $\sigma_{(\alpha,\beta)}[f_1]_g = \sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \sigma_{(\alpha,\beta)}[f_1]_{g_1}$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 3.4 and $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$, we obtain that $\varrho_{(\alpha,\beta)}[f_1]_g < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ as at least f_1 is of regular generalized relative growth (α, β) with respect to g_2. Hence in view of (3.21), $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \geq \sigma_{(\alpha,\beta)}[f_1]_g = \sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2}$. Therefore $\sigma_{(\alpha,\beta)}[f_1]_g = \sigma_{(\alpha,\beta)}[f_1]_{g_1} \Rightarrow \sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly if we consider $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_2}$.

CASE IV: In this case suppose that $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_2. Hence from (3.14), we get for all sufficiently large values of r that

$$M_{g_1 \pm g_2}(r) \leq M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))(1+D), \tag{3.22}$$

where

$$D = \frac{M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_2}}})))}{M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))}$$

and in view of $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$, we can make the term D sufficiently small by taking r sufficiently large and therefore using the similar technique for as executed in the proof of Case III we get from (3.22) that $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}$ where $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ and at least f_1 is of regular generalized relative growth (α,β) with respect to g_2.

Likewise if we consider $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1, then $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem 3.5 and the first part and second part of the theorem. Hence its proof is omitted. □

Theorem 3.14. *Let f_1, f_2, g_1, g_2 be four entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_2]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_2}$ are all non-zero and finite.*

(A) *If $\lambda_{(\alpha,\beta)}[f_i]_{g_1} > \lambda_{(\alpha,\beta)}[f_j]_{g_1}$ with at least f_j is of regular generalized relative growth (α,β) with respect to g_1 for i, $j = 1,2$ and $i \neq j$, then*

$$\tau_{(\alpha,\beta)}[f_1+f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1+f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_i]_{g_1}.$$

(B) *If $\lambda_{(\alpha,\beta)}[f_1]_{g_i} < \lambda_{(\alpha,\beta)}[f_1]_{g_j}$ for $i = j = 1,2$ and $i \neq j$, then*

$$\tau_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_i}.$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*

(i) $\varrho_{(\alpha,\beta)}[f_i]_{g_1} > \varrho_{(\alpha,\beta)}[f_j]_{g_1}$ with at least f_j is of regular generalized relative growth (α,β) with respect to g_1 for i,j $= 1,2$ and $i \neq j$;

(ii) $\varrho_{(\alpha,\beta)}[f_i]_{g_2} > \varrho_{(\alpha,\beta)}[f_j]_{g_2}$ *with at least* f_j *is of regular generalized relative growth* (α,β) *with respect to* g_2 *for* $i,\ j = 1,2$ *and* $i \neq j$*;*
(iii) $\varrho_{(\alpha,\beta)}[f_1]_{g_i} < \varrho_{(\alpha,\beta)}[f_1]_{g_j}$ *and* $\varrho_{(\alpha,\beta)}[f_2]_{g_i} < \varrho_{(\alpha,\beta)}[f_2]_{g_j}$ *holds simultaneously for* $i,\ j = 1,2$ *and* $i \neq j$*;*
(iv)$\lambda_{(\alpha,\beta)}[f_l]_{g_m} = \min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}] \mid l = m = 1,2$*;*
then we have

$$\tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \tau_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2$$

and

$$\overline{\tau}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_l]_{g_m} \mid l,m = 1,2.$$

Proof. For any arbitrary positive number $\varepsilon(> 0)$, we have for all sufficiently large values of r that

$$M_{f_k}(r) \leq M_{g_l}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_k]_{g_l} + \varepsilon)[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_k]_{g_l}}\}), \tag{3.23}$$

$$M_{f_k}(r) \geq M_{g_l}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_k]_{g_l}}\}) \tag{3.24}$$

$$\text{i.e., } M_{g_l}(r) \leq M_{f_k}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_k]_{g_l}}})), \tag{3.25}$$

and for a sequence of values of r tending to infinity we obtain that

$$M_{f_k}(r) \geq M_{g_l}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_k]_{g_l}}\}) \tag{3.26}$$

$$\text{i.e., } M_{g_l}(r) \leq M_{f_k}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\tau}_{(\alpha,\beta)}[f_k]_{g_l} - \varepsilon)})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_k]_{g_l}}})), \tag{3.27}$$

and

$$M_{f_k}(r) \leq M_{g_l}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_k]_{g_l} + \varepsilon)[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_k]_{g_l}}\}), \tag{3.28}$$

where $k = 1,2$ and $l = 1,2$.

CASE I: Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_2 is of regular generalized relative growth (α,β) with respect to g_1. Also let $\varepsilon(> 0)$ be arbitrary. Now we get from (3.23) and (3.28), for a sequence of values of r tending to infinity that

$$M_{f_1 \pm f_2}(r)(r) \leq M_{g_1}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})(1 + E). \tag{3.29}$$

where

$$E = \frac{M_{g_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} + \varepsilon)[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_2]_{g_1}}\})}{M_{g_1}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1} + \varepsilon)[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})}$$

and in view of $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, we can make the term E sufficiently small by taking r sufficiently large. Now with the help of Theorem 3.1 and using the similar technique of Case I of Theorem 3.13, we get from (3.29) that

$$\tau_{(\alpha,\beta)}[f_1+f_2]_{g_1} \leq \tau_{(\alpha,\beta)}[f_1]_{g_1}. \tag{3.30}$$

Further, we may consider that $f = f_1 \pm f_2$. Also suppose that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and at least f_2 is of regular generalized relative growth (α,β) with respect to g_1. Then $\tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1+f_2]_{g_1} \leq \tau_{(\alpha,\beta)}[f_1]_{g_1}$. Now let $f_1 = (f \pm f_2)$. Therefore in view of Theorem 3.1, $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and at least f_2 is of regular generalized relative growth (α,β) with respect to g_1, we obtain that $\lambda_{(\alpha,\beta)}[f]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ holds. Hence in view of (3.30), $\tau_{(\alpha,\beta)}[f_1]_{g_1} \leq \tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1+f_2]_{g_1}$. Therefore $\tau_{(\alpha,\beta)}[f]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$ i.e., $\tau_{(\alpha,\beta)}[f_1+f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1 then one can easily verify that $\tau_{(\alpha,\beta)}[f_1+f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$.

Case II: Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_2 is of regular generalized relative growth (α,β) with respect to g_1. Also let $\varepsilon(>0)$ be arbitrary. Therefore we obtain from (3.23) for all sufficiently large values of r that

$$M_{f_1\pm f_2}(r)(r) \leq M_{g_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})(1+F)\,. \tag{3.31}$$

where

$$F = \frac{M_{g_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_2]_{g_1}}\})}{M_{g_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})},$$

and in view of $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, we can make the term F sufficiently small by taking r sufficiently large and therefore for similar reasoning of Case I we get from (3.31) that $\overline{\tau}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ and at least f_2 is of regular generalized relative growth (α,β) with respect to g_1.

Likewise, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 is of regular generalized relative growth (α,β) with respect to g_1 then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1+f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III: Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Hence we get from (3.25) for all sufficiently large values of r that

$$M_{g_1\pm g_2}(r) \leq M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}})))(1+G), \tag{3.32}$$

where

$$G=\frac{M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_2}-\varepsilon)})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_2}}})))}{M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}})))},$$

and since $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, we can make the term G sufficiently small by taking r sufficiently large. Therefore in view of Theorem 3.3 and using the similar technique of Case III of Theorem 3.13, we get from (3.32) that

$$\tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} \geq \tau_{(\alpha,\beta)}[f_1]_{g_1}. \tag{3.33}$$

Further, we may consider that $g = g_1 \pm g_2$. As $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, so $\tau_{(\alpha,\beta)}[f_1]_g = \tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} \geq \tau_{(\alpha,\beta)}[f_1]_{g_1}$. Further let $g_1 = (g \pm g_2)$. Therefore in view of Theorem 3.3 and $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ we obtain that $\lambda_{(\alpha,\beta)}[f_1]_g < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ holds. Hence in view of (3.33) $\tau_{(\alpha,\beta)}[f_1]_{g_1} \geq \tau_{(\alpha,\beta)}[f_1]_g = \tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2}$. Therefore $\tau_{(\alpha,\beta)}[f_1]_g = \tau_{(\alpha,\beta)}[f_1]_{g_1}$i.e., $\tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$.

Likewise, if we consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\tau_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$.

Case IV: In this case further we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$.Therefore we obtain from (3.25) and (3.27), for a sequence $\{r\}$ of values of r tending to infinity that

$$M_{g_1\pm g_2}(r) \leq M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}})))(1+H), \tag{3.34}$$

where

$$H=\frac{M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_2}-\varepsilon)})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_2}}})))}{M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\varepsilon)})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}})))}.$$

Now in view of $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, we can make the term H sufficiently small by taking n sufficiently large and therefore using the similar technique for as executed in the proof of Case IV of Theorem 3.13, we get from (3.34) that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ when $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$.

Similarly, if we consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\pm g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 3.6 and the above cases. □

In the next two theorems we reconsider the equalities in Theorem 3.1 to Theorem 3.4 under somewhat different conditions.

Theorem 3.15. *Let f_1, f_2, g_1, g_2 be entire functions.*
(A) *The following condition is assumed to be satisfied:*
(i) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$ holds, then

$$\varrho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_2]_{g_1}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$ holds;
(ii) f_1 is of regular generalized relative growth (α, β) with respect to at least any one of g_1 or g_2, then

$$\varrho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_1]_{g_2}.$$

Proof. Case I: Suppose that $\varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_2]_{g_1}$ $(0 < \varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_2]_{g_1} < \infty)$. Now in view of Theorem 3.2 it is easy to see that $\varrho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_2]_{g_1}$. If possible let

$$\varrho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_2]_{g_1} \ . \tag{3.35}$$

Let $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$. Then in view of the first part of Theorem 3.13 and (3.35) we obtain that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}[f_1 \pm f_2 \mp f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Hence $\varrho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_2]_{g_1}$. Similarly with the help of the first part of Theorem 3.13, one can obtain the same conclusion under the hypothesis $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$. This proves the first part of the theorem.

Case II: Let us consider that $\varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ $(0 < \varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_1]_{g_2} < \infty)$ and f_1 is of regular generalized relative growth (α, β) with respect to at least any one of g_1 or g_2 and $(g_1 \pm g_2)$. Therefore in view of Theorem 3.4, it follows that $\varrho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ and if possible let

$$\varrho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} > \varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_1]_{g_2}. \tag{3.36}$$

Let us consider that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$. Then. in view of the proof of the second part of Theorem 3.13 and (3.36) we obtain that $\sigma_{(\alpha,\beta)}[f_1]_{g_1} = \sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2 \mp g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ which is a contradiction. Hence $\varrho_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_1]_{g_2}$. Also in view of the proof of second part of Theorem 3.13 one can derive the same conclusion for the condition $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$ and therefore the second part of the theorem is established. □

Theorem 3.16. *Let f_1, f_2, g_1, g_2 be entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) $(f_1 \pm f_2)$ is of regular generalized relative growth (α, β) with respect to at least any one of g_1 or g_2;
(ii) Either $\sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$;
(iii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$;
(iv) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_2} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\varrho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_2]_{g_1} = \varrho_{(\alpha,\beta)}[f_1]_{g_2} = \varrho_{(\alpha,\beta)}[f_2]_{g_2}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) f_1 and f_2 are of regular generalized relative growth (α, β) with respect to at least any one of g_1 or g_2;
(ii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$;
(iii) Either $\sigma_{(\alpha,\beta)}[f_1]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$;
(iv) Either $\sigma_{(\alpha,\beta)}[f_2]_{g_1} \neq \sigma_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1} \neq \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\varrho_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \varrho_{(\alpha,\beta)}[f_1]_{g_1} = \varrho_{(\alpha,\beta)}[f_2]_{g_1} = \varrho_{(\alpha,\beta)}[f_1]_{g_2} = \varrho_{(\alpha,\beta)}[f_2]_{g_2}.$$

We omit the proof of Theorem 3.16 as it is a natural consequence of Theorem 3.15.

Theorem 3.17. *Let f_1, f_2, g_1, g_2 be entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1;
(ii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$ holds, then

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) f_1, g_1 and g_2 be any three entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$ and $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ exists;
(ii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ holds, then

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}.$$

Proof. CASE I: Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ $(0 < \lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1} < \infty)$ and at least f_1 or f_2 and $(f_1 \pm f_2)$ are of regular generalized relative growth (α, β) with respect to g_1. Now, in view of Theorem 3.1, it is easy to see that $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} \leq \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. If possible let

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}. \tag{3.37}$$

Let $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$. Then in view of the proof of the first part of Theorem 3.14 and (3.37) we obtain that $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f_1 \pm f_2 \mp f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1}$. Similarly in view of the proof of the first part of Theorem 3.14 , one can establish the same conclusion under the hypothesis $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$. This proves the first part of the theorem.

Case II: Let us consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ $(0 < \lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_1]_{g_2} < \infty$. Therefore in view of Theorem 3.3, it follows that $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \geq \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and if possible let

$$\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} > \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}. \tag{3.38}$$

Suppose $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$. Then in view of the second part of Theorem 3.14 and (3.38), we obtain that $\tau_{(\alpha,\beta)}[f_1]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2 \mp g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$ which is a contradiction. Hence $\lambda_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2}$. Analogously with the help of the second part of Theorem 3.14, the same conclusion can also be derived under the condition $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ and therefore the second part of the theorem is established. □

Theorem 3.18. *Let f_1, f_2, g_1, g_2 be entire functions.*
(A) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1 and g_2;
(ii) Either $\tau_{(\alpha,\beta)}[f_1 + f_2]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1 + f_2]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1 \pm f_2]_{g_2}$;
(iii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$;
(iv) Either $\tau_{(\alpha,\beta)}[f_1]_{g_2} \neq \tau_{(\alpha,\beta)}[f_2]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_2}$; then

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2} = \lambda_{(\alpha,\beta)}[f_2]_{g_2}.$$

(B) *The following conditions are assumed to be satisfied:*
(i) At least any one of f_1 or f_2 are of regular generalized relative growth (α, β) with respect to $g_1 \pm g_2$;
(ii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \tau_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1 \pm g_2} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1 \pm g_2}$ holds;
(iii) Either $\tau_{(\alpha,\beta)}[f_1]_{g_1} \neq \tau_{(\alpha,\beta)}[f_1]_{g_2}$ or $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$ holds;

(iv) Either $\tau_{(\alpha,\beta)}[f_2]_{g_1} \neq \tau_{(\alpha,\beta)}[f_2]_{g_2}$ *or* $\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} \neq \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_2}$ *holds, then*

$$\lambda_{(\alpha,\beta)}[f_1 \pm f_2]_{g_1 \pm g_2} = \lambda_{(\alpha,\beta)}[f_1]_{g_1} = \lambda_{(\alpha,\beta)}[f_2]_{g_1} = \lambda_{(\alpha,\beta)}[f_1]_{g_2} = \lambda_{(\alpha,\beta)}[f_2]_{g_2}.$$

We omit the proof of Theorem 3.18 as it is a natural consequence of Theorem 3.17.

Theorem 3.19. *Let* f_1, f_2, g_1, g_2 *be four entire functions such that* $\varrho_{(\alpha,\beta)}[f_1]_{g_1}$, $\varrho_{(\alpha,\beta)}[f_2]_{g_1}$, $\varrho_{(\alpha,\beta)}[f_1]_{g_2}$ *and* $\varrho_{(\alpha,\beta)}[f_2]_{g_2}$ *are all non-zero.*
(A) *Assume the function* g_1 *satisfy the following condition:*
(i) g_1 *satisfies the Property (A);*
(ii) f_1, f_2 *satisfy the Property (X), then*

$$\sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_i]_{g_1}.$$

(B) *Assume the functions* g_1, g_2 *and* f_1 *satisfy the following conditions:*
(i) f_1 *is of regular generalized relative growth* (α,β) *with respect to at least any one of* g_1 *or* g_2 *and* f_1 *satisfy the Property (A);*
(ii) g_1, g_2 *satisfy the Property (X), then,*

$$\sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_i}.$$

(C) *Assume the functions* f_1, f_2, g_1 *and* g_2 *satisfy the following conditions:*
(i) $g_1 \cdot g_2$, f_1 *and* f_2 *satisfy the Property (A);*
(ii) f_1, f_2 *satisfy the Property (X) and* g_1, g_2 *satisfy the Property (X);*
(iii) f_1 *is of regular generalized relative growth* (α,β) *with respect to at least any one of* g_1 *or* g_2*;*
(iv) f_2 *is of regular generalized relative growth* (α,β) *with respect to at least any one of* g_1 *or* g_2*;*
(v) $\varrho_{(\alpha,\beta)}[f_l]_{g_m} = \max[\min\{\varrho_{(\alpha,\beta)}[f_1]_{g_1}, \varrho_{(\alpha,\beta)}[f_1]_{g_2}\}, \min\{\varrho_{(\alpha,\beta)}[f_2]_{g_1}, \varrho_{(\alpha,\beta)}[f_2]_{g_2}\}] \mid l, m = 1, 2$*; then*

$$\sigma_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \sigma_{(\alpha,\beta)}[f_l]_{g_m} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_l]_{g_m}.$$

Proof. Case I: Suppose that $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$. Also let g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we have from (3.13) for all sufficiently large values of r that

$$M_{f_1 \cdot f_2}(r) \leq M_{g_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\}))$$

$$\times M_{g_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_2]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_2]_{g_1}}\})). \tag{3.39}$$

Since $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$, we get that

$$\lim_{r\to+\infty}\frac{(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}}{(\sigma_{(\alpha,\beta)}[f_2]_{g_1}+\frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_2]_{g_1}}}=\infty.$$

As $M_{g_1}(r)$ is an increasing function of r, therefore we get from (3.39) for all sufficiently large values of r that

$$M_{f_1\cdot f_2}(r) < [M_{g_1}(\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\})]^2. \tag{3.40}$$

Let us observe that

$$\frac{\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon}{\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\frac{\varepsilon}{2}} > 1$$

$$\Rightarrow \frac{\log(\alpha^{-1}(\log(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)))[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}}{\log(\alpha^{-1}(\log(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\frac{\varepsilon}{2})))[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}}=\delta(\text{say})>1. \tag{3.41}$$

Since g_1 satisfy the Property (A), in view of Lemma 2.2 and (3.41) we obtain from (3.40) for all sufficiently large values of r that

$$M_{f_1\cdot f_2}(r) < M_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\})]^{\delta})$$

$$\begin{aligned} i.e.,\ M_{f_1\cdot f_2}(r) &< M_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\})]. \\ \text{for } \delta &\to 1+ \end{aligned}$$

Now in view of Theorem 3.8, we get from above for all sufficiently large values of r that

$$M_{f_1\cdot f_2}(r) < M_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1}}\})].$$

$$i.e.,\ \frac{\exp(\alpha(M_{g_1}^{-1}(M_{f_1\cdot f_2}(r))))}{[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1}}} < (\sigma_{(\alpha,\beta)}[f_1]_{g_1}+\varepsilon)$$

$$i.e.,\ \sigma_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} \le \sigma_{(\alpha,\beta)}[f_1]_{g_1}. \tag{3.42}$$

Now we establish the equality of (3.42). Since f_1, f_2 satisfy the Property (X), then have $M_{f_1\cdot f_2}(r) > M_{f_1}$ for all sufficiently large values of r and therefore

$$\frac{\exp(\alpha(M_{g_1}^{-1}(M_{f_1\cdot f_2}(r))))}{[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1}}} > \frac{\exp(\alpha(M_{g_1}^{-1}(M_{f_1}(r))))}{[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}}$$

as $M_{g_1}^{-1}(r)$ is an increasing function of r. So $\sigma_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} \ge \sigma_{(\alpha,\beta)}[f_1]_{g_1}$. Hence $\sigma_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} \le \sigma_{(\alpha,\beta)}[f_1]_{g_1}$.

Similarly, if we consider $\varrho_{(\alpha,\beta)}[f_1] _{g_1} < \varrho_{(\alpha,\beta)}[f_2]_{g_1}$, then one can verify that $\sigma_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} = \sigma_{(\alpha,\beta)}[f_2]_{g_1}$.

Case II: Let $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$ and g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we have from (3.13) and (3.16) for a sequence of values of r tending to infinity that

$$\begin{aligned} M_{f_1\cdot f_2}(r) \leq\ & M_{g_1}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\})] \\ & \times M_{g_1}[\alpha^{-1}(\log\{(\sigma_{(\alpha,\beta)}[f_2]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_2]_{g_1}}\})]. \end{aligned} \tag{3.43}$$

Now in view of $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_2]_{g_1}$, we get that

$$\lim_{r\to+\infty} \frac{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}}{(\sigma_{(\alpha,\beta)}[f_2]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_2]_{g_1}}} = \infty.$$

As $M_{g_1}(r)$ is an increasing function of r, therefore it follows from (3.43) for a sequence of values of r tending to infinity that

$$M_{f_1\cdot f_2}(r) < [M_{g_1}[\alpha^{-1}(\log\{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}\})]]^2.$$

Now using the similar technique for a sequence of values of r tending to infinity as explored in the proof of Case I, one can easily verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1\cdot f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}$ under the conditions specified in the theorem.

Similarly, if we consider $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_2]_{g_1}$, then one can verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\sigma}_{(\alpha,\beta)}[f_2]_{g_1}$.

Therefore the first part of theorem follows from Case I and Case II.

Case III: Let f_1 satisfy the Property (A) and $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ with f_1 is of regular generalized relative growth (α,β) with respect to at least any one of g_1 or g_2. Therefore in view of (3.14) and (3.15), we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} M_{g_1\cdot g_2}(r) \leq\ & M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}}))) \\ & \times M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_2}}}))). \end{aligned} \tag{3.44}$$

Now in view of $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$, we obtain that

$$\lim_{r\to+\infty} \frac{(\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}}}{(\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_2}}}} = \infty.$$

As $M_{f_1}(r)$ is an increasing function of r, therefore it follows from (3.44) for a sequence of values of r tending to infinity that

$$M_{g_1 \cdot g_2}(r) < [M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))]^2. \tag{3.45}$$

Now we observe that

$$\frac{\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2}}{\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon} > 1$$

$$\Rightarrow \frac{\log(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))}{\log(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))} = \delta(\text{say}) > 1. \tag{3.46}$$

Since f_1 satisfy the Property (A), in view of Lemma 2.2 and (3.46) we obtain from (3.45) for a sequence of values of r tending to infinity that

$$M_{g_1 \cdot g_2}(r) < M_{f_1}[(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))^{\delta}]$$

$$i.e., \ M_{g_1 \cdot g_2}(r) < M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}}))).$$

Now we get in view of Theorem 3.10 and from above for a sequence of values of r tending to infinity that

$$M_{g_1 \cdot g_2}(r) < M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\sigma_{(\alpha,\beta)}[f_1]_{g_1} - \varepsilon)})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2}}})))$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \geq \sigma_{(\alpha,\beta)}[f_1]_{g_1} \ . \tag{3.47}$$

Now we establish the equality of (3.47). Since g_1, g_2 satisfy the Property (X), then we have $M_{g_1 \cdot g_2}(r) > M_{g_1}(r)$ for all sufficiently large values of r and therefore $M^{-1}_{g_1 \cdot g_2}(r) < M^{-1}_{g_1}(r)$. Hence

$$\frac{\exp(\alpha(M^{-1}_{g_1 \cdot g_2}(M_{f_1}(r))))}{[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2}}} < \frac{\exp(\alpha(M^{-1}_{g_1}(M_{f_1}(r)))}{[\exp(\beta(r))]^{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}}$$

as $M_{f_1}(r)$ is an increasing function of r. So $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} \leq \sigma_{(\alpha,\beta)}[f_1]_{g_1}$.
So $\sigma_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \sigma_{(\alpha,\beta)}[f_1]_{g_1}$.

Case IV: Suppose f_1 satisfy the Property (A). Also let $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ with f_1 is of regular generalized relative growth (α, β) with respect to at least any one of g_1 or g_2. Therefore in view of (3.14), we obtain for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) \leq M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))$$
$$\times M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_2}}}))). \tag{3.48}$$

Now in view of $\varrho_{(\alpha,\beta)}[f_1]_{g_1} < \varrho_{(\alpha,\beta)}[f_1]_{g_2}$, we get that

$$\lim_{r\to+\infty} \frac{(\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}}}{(\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_2}}}} = \infty.$$

As $M_{f_1}(r)$ is an increasing function of r, therefore it follows from (3.48) for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) < [M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\varrho_{(\alpha,\beta)}[f_1]_{g_1}}})))]^2.$$

Now using the similar technique for all sufficiently large values of r as explored in the proof of Case III, one can easily verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1}$ under the conditions specified in the theorem.

Likewise, if we consider $\varrho_{(\alpha,\beta)}[f_1]_{g_1} > \varrho_{(\alpha,\beta)}[f_1]_{g_2}$ with at least f_1 is of regular generalized relative growth (α, β) with respect to g_1, then one can verify that $\overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \overline{\sigma}_{(\alpha,\beta)}[f_1]_{g_2}$.

Therefore the second part of theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 3.11 and the above cases. □

Theorem 3.20. *Let f_1, f_2, g_1, g_2 be four entire functions such that $\lambda_{(\alpha,\beta)}[f_1]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_2]_{g_1}$, $\lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and $\lambda_{(\alpha,\beta)}[f_2]_{g_2}$ are all non-zero and finite.*

(A) *Assume the functions f_1, f_2 and g_1 satisfy the following conditions:*

(i) At least f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1 for i, $j = 1, 2$ and $i \neq j$;

(ii) g_1 satisfy the Property (A) and f_1, f_2 satisfy the Property (X), then

$$\tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_i]_{g_1} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_i]_{g_1}.$$

(B) *Assume the function f_1 satisfy the following condition:*

(i) f_1 satisfy the Property (A) and g_1, g_2 satisfy the Property (X), then

$$\tau_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_i} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1 \cdot g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_i}.$$

(C) *Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:*
(i) $g_1 \cdot g_2$, f_1 and f_2 are satisfy the Property (A);
(ii) f_1, f_2 satisfy the Property (X) and g_1, g_2 satisfy the Property (X);
(iii) At least f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$;
(iv) At least f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_2 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$;
(v) $\lambda_{(\alpha,\beta)}[f_l]_{g_m} = \min[\max\{\lambda_{(\alpha,\beta)}[f_1]_{g_1}, \lambda_{(\alpha,\beta)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta)}[f_1]_{g_2}, \lambda_{(\alpha,\beta)}[f_2]_{g_2}\}] \mid l, m = 1, 2$; then

$$\tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \tau_{(\alpha,\beta)}[f_l]_{g_m} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1 \cdot g_2} = \overline{\tau}_{(\alpha,\beta)}[f_l]_{g_m}.$$

Proof. Case I: Suppose $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1 and g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we obtain from (3.23) and (3.26) for a sequence values of r tending to infinity that

$$\begin{aligned} M_{f_1 \cdot f_2}(r) \leq\ & M_{g_1}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})) \\ & \times M_{g_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_2]_{g_1}}\})). \end{aligned} \tag{3.49}$$

Now in view of $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, we get that

$$\lim_{r \to +\infty} \frac{(\tau_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}}{(\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_2]_{g_1}}} = \infty.$$

As $M_{g_1}(r)$ is an increasing function of r, therefore we get from (3.49) for a sequence of values of r tending to infinity that

$$M_{f_1 \cdot f_2}(r) < [M_{g_1}(\alpha^{-1}(\log\{(\tau_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\}))]^2. \tag{3.50}$$

Now using the similar technique as explored in the proof of Case I of Theorem 3.19 we obtain from (3.50) that

$$\tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_1]_{g_1}.$$

Similarly, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1, then one can easily verify that $\tau_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \tau_{(\alpha,\beta)}[f_2]_{g_1}$.

Case II: Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1 and g_1 satisfy the Property (A). Now for any arbitrary $\varepsilon > 0$, we get from (3.23) for all sufficiently large values of r that

$$M_{f_1 \cdot f_2}(r) \le M_{g_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\}))$$
$$\times M_{g_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_2]_{g_1}}\})). \tag{3.51}$$

Now in view of $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_2]_{g_1}$, we get that

$$\lim_{r\to+\infty} \frac{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}}{(\overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_2]_{g_1}}} = \infty.$$

As $M_{g_1}(r)$ is an increasing function of r, therefore we get from (3.51) for all sufficiently large values of r that

$$M_{f_1 \cdot f_2}(r) < [M_{g_1}(\alpha^{-1}(\log\{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1} + \frac{\varepsilon}{2})[\exp(\beta(r))]^{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}\})]^2. \tag{3.52}$$

Now using the similar technique as explored in the proof of Case I of Theorem 3.20 we obtain from (3.52) that $\overline{\tau}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$ under the conditions specified in the theorem.

Likewise, if we consider $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_2]_{g_1}$ with at least f_1 or f_2 is of regular generalized relative growth (α, β) with respect to g_1, then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1 \cdot f_2]_{g_1} = \overline{\tau}_{(\alpha,\beta)}[f_2]_{g_1}$.

Therefore the first part of theorem follows Case I and Case II.

Case III: Let $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and f_1 satisfy the Property (A). Therefore in view of (3.25) we obtain for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) \le M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}})))$$
$$\times M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_2} - \frac{\varepsilon}{2})})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_2}}}))). \tag{3.53}$$

Now in view of $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, we get that

$$\lim_{r\to+\infty} \frac{(\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}}}{(\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_2} - \frac{\varepsilon}{2})})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_2}}}} = \infty.$$

As $M_{f_1}(r)$ is an increasing function of r, therefore it follows from (3.53) for all sufficiently large values of r that

$$M_{g_1 \cdot g_2}(r) < [M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_1} - \frac{\varepsilon}{2})})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}})))]^2. \tag{3.54}$$

Now using the similar technique as explored in the proof of Case III of Theorem 3.19 we obtain from (3.54) that $\tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_1}$. If $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\tau_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \tau_{(\alpha,\beta)}[f_1]_{g_2}$.

Case IV: Suppose $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$ and f_1 satisfy the Property (A). Therefore in view of (3.25) and (3.27) we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} M_{g_1\cdot g_2}(r) &\le M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\frac{\varepsilon}{2})})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}}))) \\ &\times M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_2}-\frac{\varepsilon}{2})})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_2}}}))). \end{aligned} \tag{3.55}$$

Now in view of $\lambda_{(\alpha,\beta)}[f_1]_{g_1} < \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, we get that

$$\lim_{r\to+\infty} \frac{(\frac{\exp(\alpha(r))}{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\frac{\varepsilon}{2})})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}}}{(\frac{\exp(\alpha(r))}{(\tau_{(\alpha,\beta)}[f_1]_{g_2}-\frac{\varepsilon}{2})})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_2}}}} = \infty.$$

As $M_{f_1}(r)$ is an increasing function of r, therefore it follows from (3.55) for a sequence of values of r tending to infinity that

$$M_{g_1\cdot g_2}(r) < [M_{f_1}(\beta^{-1}(\log((\frac{\exp(\alpha(r))}{(\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}-\frac{\varepsilon}{2})})^{\frac{1}{\lambda_{(\alpha,\beta)}[f_1]_{g_1}}})))]^2. \tag{3.56}$$

Now using the similar technique as explored in the proof of Case III of Theorem 3.20, we obtain from (3.56) that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1}$. Similarly if we consider that $\lambda_{(\alpha,\beta)}[f_1]_{g_1} > \lambda_{(\alpha,\beta)}[f_1]_{g_2}$, then one can easily verify that $\overline{\tau}_{(\alpha,\beta)}[f_1]_{g_1\cdot g_2} = \overline{\tau}_{(\alpha,\beta)}[f_1]_{g_2}$. Therefore the second part of the theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 3.12 and the above cases. □

Acknowledgment

The authors are thankful to the reviewers for his / her valuable comments towards the improvement of the paper.

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