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COMMON FIXED POINTS OF w-COMPATIBLE MAPS IN MODULAR A-METRIC SPACES

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Abstract. The aim of this paper is to prove a common fixed point theorem for two w-compatible maps in modular A-metric spaces. The main result is also illustrated by an example to demonstrate the degree of validity of our hypothesis.

1. Introduction

Modular metric spaces are a natural generalization of metric spaces. The introduction of this new concept was given by V. V. Chistyakov [2], [3]. In the last decade, there has been an enormous progress in modular metric.

In 2017, Aydin and Kutukcu [1] introduced a new structure of generalized metric space and called it modular A-metric space.

Now, we give some definitions and results which are used in this paper.

Definition 1.1 ([1]). The modular A-metric on X where X is non-empty is defined by a mapping $A_\lambda : (0,\infty) \times X^n \to [0,\infty]$ that satisfying following conditions for all $x_i,\, a \in X$ and $\lambda, \lambda_i > 0$ for $i = \overline{1,\, n}$:

(A1) $A_\lambda\,(x_1,\, x_2,\, x_3, ..., \, x_{n-1},\, x_n) \geq 0,$

(A2) $A_\lambda\,(x_1,\, x_2,\, x_3, ..., \, x_{n-1},\, x_n) = 0$ if and only if $x_1 = x_2 = ... = x_{n-1} = x_n$

(A3)
$$\begin{aligned} A_{\lambda_1+\lambda_2+...+\lambda_n}\,(x_1,\, x_2,\, x_3, ..., \, x_{n-1},\, x_n)) \leq\, & A_{\lambda_1}(x_1,\, x_1, ..., (x_1)_{n-1},\, a) \\ &+ A_{\lambda_2}(x_2,\, x_2, ..., (x_2)_{n-1},\, a) \\ &\vdots \\ &+ A_{\lambda_n}(x_n,\, x_n, ..., \, (x_n)_{n-1}\,,\, a) \end{aligned}$$

The pair $(X,\, A)$ is said to be a *modular A-metric space*.

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Lemma 1.2 ([1]). *Let (X, A) be a modular A-metric space. If for each $x_1, ..., x_n \in X$, the mapping $A(\cdot, x_1, x_2, ..., x_n) : (0, \infty) \to [0, \infty]$ is continuous, then the following equality is true*

$$A_\lambda(x, x, ..., x, y) = A_\lambda(y, y, ..., y, x)$$

for each $x, y \in X$ and $\lambda > 0$.

Theorem 1.3 ([1]). *Let (X, A) be a modular A-metric space and the mapping $A(\cdot, x_1, x_2, ..., x_n) : (0, \infty) \to [0, \infty]$ is continuous for each $x_1, x_2, ..., x_n \in X$. Then, there are the following inequalities such that*

$$A_\lambda(x,\, x,\, x, ...,\, x,\, z) \leq (n-1)A_{\frac{\lambda}{n}}(x,\, x,\, x, ...,\, x,\, y) + A_{\frac{\lambda}{n}}(z,\, z,\, z, ...,\, z,\, y)$$

and

$$A_\lambda(x,\, x,\, x, ...,\, x,\, z) \leq (n-1)A_{\frac{\lambda}{n}}(x,\, x,\, x, ...,\, x,\, y) + A_{\frac{\lambda}{n}}(y,\, y,\, y, ...,\, y,\, z)$$

for each $x, y, z \in X$.

Proposition 1.4 ([1]). *Let (X, A) be a modular A-metric space and the mapping $A(\cdot, x_1, x_2, ..., x_n) : (0, \infty) \to [0, \infty]$ is continuous for each $x_1, x_2, ..., x_n \in X$. Then, the following inequality*

$$A_\lambda(x, x, ..., x, y) \leq A_{\frac{\lambda}{n}}(x, x, ..., x, y) \leq A_{\frac{\lambda}{n^2}}(x, x, ..., x, y)$$

is satisfied for $\frac{\lambda}{n^2} \leq \frac{\lambda}{n} \leq \lambda$.

Proof. If it is taken $a = x$ in the condition (A3) and used the inequality in Theorem (1.3), the following inequality is written:

$$\begin{aligned} A_\lambda(x,\, x,\, ...,\, x,\, y) &\leq (n-1)A_{\frac{\lambda}{n}}(x,\, x,\, ...,\, x,\, x) + A_{\frac{\lambda}{n}}(y,\, y,\, ...y,\, x) \\ &= A_{\frac{\lambda}{n}}(y,\, y,\, ...y,\, x) \\ &\leq (n-1)\, A_{\frac{\lambda}{n^2}}(y,\, y,\, ...y,\, y) + A_{\frac{\lambda}{n^2}}(x, x, ..., x, y) \\ &= A_{\frac{\lambda}{n^2}}(x, x, ..., x, y) \end{aligned}$$

Thus,

$$A_\lambda(x, x, ..., x, y) \leq A_{\frac{\lambda}{n}}(x, x, ..., x, y) \leq A_{\frac{\lambda}{n^2}}(x, x, ..., x, y)$$

is obtained with Lemma (1.2). □

Example 1.5 ([1])**.** Let $X = \mathbf{R}$. Define a function $A_\lambda : (0,\infty) \times X^n \to [0,\infty]$ by

$$\begin{aligned} & A_\lambda(x_1, x_2, x_3, ..., x_{n-1}, x_n) \\ &= \frac{\lambda}{n} |x_1 - x_2| + |x_1 - x_3| + ... + |x_1 - x_n| \\ &\quad + |x_2 - x_3| + |x_2 - x_4| + ... + |x_2 - x_n| \\ &\qquad \vdots \\ &\quad + |x_{n-2} - x_{n-1}| + |x_{n-2} - x_n| \\ &\quad + |x_{n-1} - x_n| \\ &= \frac{\lambda}{n} \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j| \end{aligned}$$

for all $\lambda > 0$ and $x_1, x_2, ..., x_n \in X$.

Then, (X, A) is a usual modular A-metric space on X.

Definition 1.6 ([1])**.** Let (X, A) be a modular A-metric space and let $x_0 \in X$. Then, for any $r > 0$, the set

$$B_{A_\lambda}(x_0, r) = \{y \in X : A_\lambda(y, y, y, ..., y, x_0) < r\}$$

is defined as an open ball with center x_0 and radius r.

Definition 1.7 ([1])**.** Let (X, A) be a modular A-metric space and $Y \subset X$.

(1) If there exists a $r > 0$ such that $B_{A_\lambda}(x, r) \subset Y$ for each $x \in Y$ and $\lambda > 0$, then Y is called be an *open set*.

(2) Let

$$\tau_A := \{Y \subset X : x \in Y \text{ iff there exists a } r > 0 \text{ such that } B_{A_\lambda}(x, r) \subset Y\}.$$

In this case, (X, τ_A) is a topological space.

Theorem 1.8 ([1])**.** *Let A be a modular A-metric on X. In this case, (X, τ) is a Hausdorff space.*

Definition 1.9 ([1])**.** Let A be a modular A-metric on X, $\{x_k\}_{k\in IN} \subset X$ and $x \in X$.

(1) $\{x_k\}$ converges to x if $A_\lambda(x_k, x_k, x_k, ..., x_k, x) \to 0$ as $k \to \infty$ for all $\lambda > 0$. In other words, for each $\varepsilon > 0$, there exists a natural number $k_0(\varepsilon) \in IN$ such that for all $k \geq k_0$, $A_\lambda(x_k, x_k, x_k, ..., x_k, x) \leq \varepsilon$.

(2) $\{x_k\}$ is said to be a *Cauchy sequence* if $A_\lambda(x_k, x_k, x_k, ..., x_k, x_m) \to 0$ as $k, m \to \infty$ for all $\lambda > 0$. In other words, for each $\varepsilon > 0$, there exists a natural number $k_0(\varepsilon) \in IN$ such that for all $k, m \geq k_0$,

$$A_\lambda(x_k, x_k, x_k, ..., x_k, x_m) \leq \varepsilon.$$

(3) (X, A) is said to be *complete modular A-metric space* if every Cauchy sequence in X is convergent.

Theorem 1.10 ([1]). *Let A be a modular A-metric on X. If the sequence $\{x_k\}_{k\in IN} \in X$ converges to x in X, in this case x is unique.*

Theorem 1.11 ([1]). *Let A be a modular A-metric on X. If $\{x_k\}_{k\in IN} \subset X$ is a convergent sequence in X, then $\{x_k\}_{k\in IN}$ is a Cauchy sequence.*

2. w-Compatible Maps in Modular A-Metric Spaces

In 1986, Jungck [4] introduced the concept of compatible maps in metric spaces as follows:

Definition 2.1. Let (M, d) be a metric space and $f, g : M \to M$. The mappings f and g are said to be *compatible* if $\lim_{k\to\infty} d(fgx_k, gfx_k) = 0$, whenever $\{x_k\}$ is a sequence in M such that $\lim_{k\to\infty} fx_k = \lim_{k\to\infty} gx_k = z$ for some $z \in M$.

In modular A-metric spaces, the notion of w-compatible maps is given as follows:

Definition 2.2 ([5]). *Let M and N be two self maps on a modular A-metric space (X, A). If*

$$\lim_{k\to\infty} A_\lambda(MNx_k, ..., MNx_k, NMx_k) = 0$$

where $\{x_k\}$ is a sequence in X which satisfies $\lim_{k\to\infty} Mx_k = \lim_{k\to\infty} Nx_k = t$ for some point $t \in X$ and $\lambda > 0$, the maps M and N are said to be w-compatible.

Example 2.3. Let $X = \mathbf{R}$ and A be a function on X defined by

$$A_\lambda(x_1, x_2, ..., x_n) = \frac{\lambda}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n} |x_i - x_j|$$

for all $\lambda > 0$ and $x_1, x_2, ..., x_n \in X$. Then, (X, A) is a modular A-metric space. Let M and N be two self maps defined on X by $M(x) = x^2$ and $N(x) = x^3$ for each $x \in \mathbf{R}$. Take $\{x_k\}$ such that $\{x_k\} = \frac{1}{k}$, $k = 1, 2, \ldots$. In this case, the maps M and N are w-compatible maps.

Theorem 2.4. *Let (X, A) be a complete modular A-metric space. Let T and S be maps from X into itself such that*

(1) $T(X) \subset S(X)$

(2) T *or* S *is continuous*

(3) $A_\lambda (Tx_1, Tx_2, ..., Tx_k) \leq qA_\lambda (Sx_1, Sx_2, ..., Sx_k)$ *for each* $x_1, x_2, ..., x_k \in X$ *and* $0 \leq q < 1$

(4) T *and* S *are* w*-compatible maps.*

Then, T *and* S *have a unique common fixed point in* X.

Proof. Let x_0 be an arbitrary point in X. We can choose a point x_1 in X such that $Tx_0 = Sx_1$ with $T(X) \subset S(X)$. By generalizing this situation, we can construct a sequence $\{x_{k+1}\}$ in X as follows :

$$y_k = Tx_k = Sx_{k+1},\ k = 0, 1, 2, ...$$

From (3), we have

$$\begin{aligned}
&A_\lambda (Tx_k, Tx_k, ..., Tx_k, Tx_{k+1}) \\
&\leq qA_\lambda (Sx_k, Sx_k, ..., Sx_k, Sx_{k+1}) \\
&= qA_\lambda (Tx_{k-1}, Tx_{k-1}, ..., Tx_{k-1}, Tx_k) \\
&\leq q^2 A_\lambda (Sx_{k-1}, Sx_{k-1}, ..., Sx_{k-1}, Sx_k) \\
&= q^2 A_\lambda (Tx_{k-2}, Tx_{k-2}, ..., Tx_{k-2}, Tx_{k-1}) \\
&\qquad\vdots \\
&\leq q^k A_\lambda (Sx_1, Sx_1, ..., Sx_1, Sx_2) \\
&= q^k A_\lambda (Tx_0, Tx_0, ..., Tx_0, Tx_1)
\end{aligned}$$

Letting as $k \to \infty$, we obtain

$$\lim_{k\to\infty} A_\lambda (Tx_k, Tx_k, ..., Tx_{k+1}) \leq \lim_{k\to\infty} q^k A_\lambda (Tx_0, Tx_0, ..., Tx_1) = 0.$$

For all $k, m \in IN$ and $k < m$, we have by rectangle inequality that

$$\begin{aligned}
&A_\lambda (Tx_k, Tx_k, ..., Tx_m) \\
&\leq (n-1) \sum_{i=k}^{m-2} A_{\frac{\lambda}{n^{n-2}}} (Tx_i, ..., Tx_{i+1}) + A_{\frac{\lambda}{n^{n-2}}} (Tx_{m-1}, ..., Tx_m) \\
&\leq (n-1) \sum_{i=k}^{m-2} q^i A_{\frac{\lambda}{n^{n-2}}} (Tx_0, ..., Tx_1) + q^{m-1} A_{\frac{\lambda}{n^{n-2}}} (Tx_0, ..., Tx_1)
\end{aligned}$$

$$\leq (n-1)\left[q^k + q^{k+1} + ... + q^{m-2} + q^{m-1}\right] A_{\frac{\lambda}{n^{n-2}}}\left(Tx_0, Tx_0, ..., Tx_1\right)$$
$$\leq (n-1)q^k \left(\frac{1-q^{m-k}}{1-q}\right) A_{\frac{\lambda}{n^{n-2}}}\left(Tx_0, Tx_0, ..., Tx_1\right)$$
$$\leq (n-1)\left(\frac{q^k}{1-q}\right) A_{\frac{\lambda}{n^{n-2}}}\left(Tx_0, Tx_0, ..., Tx_1\right)$$

Letting as $k \to \infty$, we have

$$\lim_{k\to\infty} A_\lambda\left(Tx_k, Tx_k, ..., Tx_m\right) \leq \lim_{k\to\infty}(n-1)\left(\frac{q^k}{1-q}\right) A_{\frac{\lambda}{n^{n-2}}}\left(Tx_0, Tx_0, ..., Tx_1\right) = 0.$$

Thus, $\{Tx_k\}$ is a Cauchy sequence in X. Since (X, A) is complete modular A-metric space, it has a limit in X such that

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} Tx_k = \lim_{k\to\infty} Sx_{k+1} = z.$$

Since the maps T or S is continuous (assume that S is continuous), $\lim\limits_{k\to\infty} STx_k = Sz$. Further, the maps S and T are w-compatible, therefore

$$\lim_{k\to\infty} A_\lambda\left(STx_k, ..., STx_k, TSx_k\right) = 0$$

implies $\lim\limits_{k\to\infty} TSx_k = Sz$. From (3), we have

$$\begin{aligned} A_\lambda\left(Sz, ..., Sz, z\right) &= A_\lambda\left(TSx_k, ..., TSx_k, Tx_k\right) \\ &\leq qA_\lambda\left(SSx_k, ..., SSx_k, Sx_k\right). \end{aligned}$$

Proceeding limit as $k \to \infty$,we have $Sz = z$. Again by (3), we obtain

$$A_\lambda\left(Tx_k, ..., Tx_k, Tz\right) \leq qA_\lambda\left(Sx_k, ..., Sx_k, Sz\right)$$

and taking limit as $k \to \infty$, we have $z = Tz$. Thus, we have $Tz = Sz = z$ and z is a common fixed point of T and S.

Finally, the uniqueness of z as the common fixed point of T and S shows easily as follows :

Suppose that $z_1 \,(\neq z)$ be another common fixed point of T and S. Then,

$$\begin{aligned} A_\lambda\left(z, z, ..., z, z_1\right) = A_\lambda\left(Tz, Tz, ..., Tz, Tz_1\right) &\leq qA_\lambda\left(Sz, Sz, ..., Sz, Sz_1\right) \\ &= qA_\lambda\left(z, z, ..., z, z_1\right) \\ &< A_\lambda\left(z, z, ..., z, z_1\right) \end{aligned}$$

for $0 \leq q < 1$, so this is a contradiction. Therefore, $z = z_1$. Hence, the common fixed point of T and S is unique. □

Example 2.5. Let $X = [-1, 1]$ and define $A_\lambda : (0, \infty) \times X^n \to [0, \infty]$ as follows :

$$A_\lambda (x_1, x_2, ..., x_n) = \frac{\lambda}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n} |x_i - x_j|$$

for all $x_1, x_2, ..., x_n \in X$ and all $\lambda > 0$. Clearly, (X, A) is a modular A-metric space. Let T and S be maps from X into itself defined as $T(x) = \frac{x}{7}$ and $S(x) = \frac{x}{3}$ for all $x \in X$. Then,

$$T(X) = \left[-\frac{1}{7}, \frac{1}{7}\right] \subset \left[-\frac{1}{3}, \frac{1}{3}\right] = S(X)$$

and the maps T, S are continuous. Also,

$$A_\lambda (Tx_1, Tx_2, ..., Tx_n) \leq q A_\lambda (Sx_1, Sx_2, ..., Sx_n)$$

holds for all $\overline{x_1, x_n} \in X$ and $\frac{3}{7} \leq q < 1$. Moreover, the maps T and S are w-compatible since

$$\lim_{k\to\infty} A_\lambda (TSx_k, ..., TSx_k, STx_k) = 0$$

where $\{x_k\} = \frac{1}{k}$ is a sequence for $k = 1, 2, ...$ in X such that

$$\lim_{k\to\infty} Sx_k = \lim_{k\to\infty} \tfrac{1}{3k} = 0$$
$$\lim_{k\to\infty} Tx_k = \lim_{k\to\infty} \tfrac{1}{7k} = 0$$

for $0 \in X$. Thus, 0 is the unique common fixed point of T and S.

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