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SASAKIAN 3-MANIFOLDS SATISFYING SOME CURVATURE CONDITIONS ASSOCIATED TO $\mathcal{Z}$-TENSOR

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Abstract. In this paper, we study some curvature properties of Sasakian 3-manifolds associated to $\mathcal{Z}$-tensor. It is proved that if a Sasakian 3-manifold (M, g) satisfies one of the conditions (1) the $\mathcal{Z}$-tensor is of Codazzi type, (2) M is $\mathcal{Z}$-semisymmetric, (3) M satisfies $Q(\mathcal{Z}, R) = 0$, (4) M is projectively $\mathcal{Z}$-semisymmetric, (5) M is $\mathcal{Z}$-recurrent, then (M, g) is of constant curvature 1. Several consequences are drawn from these results.

1. Introduction

An odd dimensional differentiable manifold M together with a (φ, ξ, η, g) structure is called an almost contact metric manifold if (see [2, 3])

$$\varphi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \varphi\xi = 0,\ \eta \circ \varphi = 0, \tag{1.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.2}$$

for any vector fields X, Y on M, where φ is a $(1, 1)$-tensor field, ξ is a unit vector field, η is a 1-form defined by $\eta(X) = g(X, \xi)$ and g is the Riemannian metric. It can be easily seen from (1.2) that

$$g(\varphi X, Y) = -g(X, \varphi Y). \tag{1.3}$$

A contact metric manifold is an almost contact metric manifold M with

$$d\eta(X, Y) = g(X, \varphi Y).$$

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The notion of generalized $\mathcal{Z}$-tensor is introduced by Mantica and Molinari [8] and it is given by

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y), \tag{1.4}$$

where S is the $(0,2)$-symmetric Ricci tensor and ψ is a scalar function. Note that the $\mathcal{Z}$-tensor is of type $(0,2)$ and it is symmetric. The classical $\mathcal{Z}$-tensor can be obtained from the $(0,2)$-symmetric generalized $\mathcal{Z}$-tensor by setting $\psi = -\frac{r}{n}$ in (1.4), where r is the scalar curvature of the manifold of dimension n. Hereafter, we call the generalized $\mathcal{Z}$-tensor simply as the $\mathcal{Z}$-tensor. In [8], the authors studied weakly $\mathcal{Z}$-symmetric manifolds under several constraints. In [9], Mantica and Suh introduced a new type of Riemannian manifolds called pseudo $\mathcal{Z}$-symmetric manifolds and studied this notion on Riemannian manifolds with harmonic curvature tensors.

Let $T(M)$ be the Lie algebra of all vector fields on M. The endomorphism $X \wedge_A Y$ of $T(M)$ is defined by (see [12])

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \tag{1.5}$$

where A is a symmetric $(0,2)$-tensor and $X,\ Y,\ Z \in T(M)$. For a $(1,3)$-tensor field K and a $(0,2)$-tensor field A on M, we define $Q(A,K)$ as follows:

$$\begin{aligned} Q(A,K)(U,V,W;X,Y) &= ((X \wedge_A Y) \cdot K)(U,V)W \\ &= (X \wedge_A Y)K(U,V)W - K((X \wedge_A Y)U,V)W \\ &\quad -K(U,(X \wedge_A Y)V)W - K(U,V)(X \wedge_A Y)W. \end{aligned}$$

The well known projective curvature tensor on a 3-dimensional Riemannian manifold is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2}[S(Y,Z)X - S(X,Z)Y], \tag{1.6}$$

where R is the Riemann curvature tensor. The curvature properties associated to $Q(S,R)$ was studied in [4, 5]. The curvature property of the form $R(\xi,X) \cdot \mathcal{Z} = 0$, $P(\xi,X) \cdot \mathcal{Z} = 0$ on $N(k)$-quasi-Einstein manifolds was studied in [7].

The paper is organized as follows: Some preliminary results on Sasakian 3-manifolds are given in Section 2. Section 3 deals with Codazzi type $\mathcal{Z}$-tensor on Sasakian 3-manifolds. Section 4 is devoted to the study of $\mathcal{Z}$-semisymmetric Sasakian 3-manifolds. In Section 5, the curvature condition $Q(\mathcal{Z},R) = 0$ is discussed. Section 6 contains a study of projectively $\mathcal{Z}$-semisymmetric Sasakian 3-manifolds. Finally, in Section 7, we discuss about $\mathcal{Z}$-recurrent Sasakian 3-manifolds.

2. Preliminaries

A contact metric manifold with a normal (φ, ξ, η, g) structure is called a Sasakian Manifold. Also, an almost contact meteic manifold is Sasakian if and only if

$$(\nabla_X \varphi Y) = g(X, Y)\xi - \eta(Y)X \tag{2.1}$$

for any vector fields X, Y on M. A contact metric manifold is said to be K-contact if the vector field ξ is Killing, that is, $\mathcal{L}_\xi g = 0$, where $\mathcal{L}$ denotes the Lie differentiation. A Sasakian manifold is K-contact but the converse holds only in dimension 3. It may not be true for higher dimension (see [6]). On a Sasakian 3-manifold, the following relations are well known:

$$(\nabla_X \eta)Y = g(X, \varphi Y), \tag{2.2}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.3}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.4}$$

$$S(X, \xi) = 2\eta(X), \quad Q\xi = 2\xi, \tag{2.5}$$

where R denotes the Riemann curvature tensor, Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$. The curvature tensor of a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X, Y)Z &= [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &\quad -\frac{r}{2}[g(Y, Z)X - g(X, Z)Y]. \end{aligned} \tag{2.6}$$

The scalar curvature r is not constant in general. Contracting X in (2.6), the Ricci tensor for a Sasakian 3-manifold can be obtained as

$$S(X, Y) = \frac{1}{2}[(r-2)g(X, Y) + (6-r)\eta(X)\eta(Y)]. \tag{2.7}$$

3. Sasakian 3-manifolds admitting Codazzi Type $\mathcal{Z}$-tensor

In this section, we characterize a Sasakian 3-manifold admitting Codazzi type $\mathcal{Z}$-tensor. We first recall the definition of a Codazzi type tensor on a Riemannian manifold given as follows:

Definition 3.1. A $(0,2)$-symmetric tensor T on a Riemannian manifold (M,g) is said to be of *Codazzi type* if

$$(\nabla_X T)(Y,Z) = (\nabla_Y T)(X,Z)$$

holds for all vector fields X, Y and Z on M.

Theorem 3.2. *The $\mathcal{Z}$-tensor of a Sasakian 3-manifold (M,φ,ξ,η,g) is of Codazzi type if and only if (M,g) is of constant curvature 1 and ψ is constant.*

Proof. We first consider that the $\mathcal{Z}$-tensor of a Sasakian 3-manifold M is of Codazzi type. Then we have

$$(\nabla_X \mathcal{Z})(Y,Z) = (\nabla_Y \mathcal{Z})(X,Z) \tag{3.1}$$

for all vector fields X, Y and Z on M. Equations (1.4) and (3.1) together implies

$$(\nabla_X S)(Y,Z) + X(\psi)g(Y,Z) = (\nabla_Y S)(X,Z) + Y(\psi)g(X,Z). \tag{3.2}$$

Now, from (2.7), we can obtain

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \frac{1}{2}[X(r)g(Y,Z) - X(r)\eta(Y)\eta(Z) \\ &\quad +(6-r)\{\eta(Y)(\nabla_X\eta)Z + \eta(Z)(\nabla_X\eta)Y\}].\end{aligned} \tag{3.3}$$

Applying (3.3) in (3.2) yields

$$\begin{aligned}&\frac{1}{2}[X(r)g(Y,Z) - X(r)\eta(Y)\eta(Z) + (6-r)\{\eta(Y)(\nabla_X\eta)Z + \eta(Z)(\nabla_X\eta)Y\}] \\ &+X(\psi)g(Y,Z) = \frac{1}{2}[Y(r)g(X,Z) - Y(r)\eta(X)\eta(Z) + (6-r)\{\eta(X)(\nabla_Y\eta)Z \\ &+\eta(Z)(\nabla_Y\eta)X\}] + Y(\psi)g(X,Z).\end{aligned}$$

Since a Sasakian 3-manifold is K-contact, then the characteristic vector field ξ of a Sasakian 3-manifold is Killing and hence $\xi(r) = 0$. Substituting $X = \xi$ in the foregoing equation and using $\xi(r) = 0$ and (2.2), we obtain

$$\xi(\psi)g(Y,Z) = (6-r)g(Y,\varphi Z) + Y(\psi)\eta(Z). \tag{3.4}$$

Replacing ξ for Z in the preceding equation yields

$$Y(\psi) = \xi(\psi)\eta(Y). \tag{3.5}$$

Let $\{e_i\}$, $i = 1,2,3$ be any orthonormal frame for M. Substituting $Y = Z = e_i$ in (3.4) and applying (1.3) yields $\xi(\psi) = 0$, where summation convention is used. Therefore, equation (3.5) gives $Y(\psi) = 0$ for all vector field Y on M. This shows that ψ is a constant. Hence, from (3.4), we get $r = 6$. Substituting $r = 6$ in (2.6), we get $S(X,Y) = 2g(X,Y)$, that is, the manifold is Einstein. Substituting this in (2.6) yields

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y \tag{3.6}$$

for all vector fields X, Y and Z on M. This shows that the manifold is of constant curvature 1.

Conversely, let (M,φ,ξ,η,g) is of constant curvature 1 and ψ is constant. Then the manifold M is Einstein and hence, the equation (3.2) holds trivially. This shows that the $\mathcal{Z}$-tensor is of Codazzi type. □

Since ψ is constant and the manifold is Einstein, then from (1.4), we get $\nabla\mathcal{Z} = 0$, which means that the manifold is $\mathcal{Z}$-parallel. Thus we can state the following:

Corollary 3.3. *The $\mathcal{Z}$-tensor of a Sasakian 3-manifold (M,φ,ξ,η,g) is of Codazzi type if and only if (M,φ,ξ,η,g) is $\mathcal{Z}$-parallel.*

4. $\mathcal{Z}$-semisymmetric Sasakian 3-manifolds

A Riemannian manifold (M,g) is said to be $\mathcal{Z}$-semisymmetric if $R\cdot\mathcal{Z} = 0$ holds on M, that is, $(R(X,Y)\cdot\mathcal{Z})(U,V) = 0$ holds for all vector fields X, Y, U and V on M. We first consider a weaker condition $R(\xi,X)\cdot\mathcal{Z} = 0$ than the $\mathcal{Z}$-semisymmetry on a Sasakian 3-manifold and prove the following:

Theorem 4.1. *A Sasakian 3-manifold (M,φ,ξ,η,g) satisfies the condition $R(\xi,X)\cdot\mathcal{Z} = 0$ if and only if (M,φ,ξ,η,g) is of constant curvature 1.*

Proof. Let a Sasakian 3-manifold M satisfies the condition

$$(R(\xi,X)\cdot\mathcal{Z})(U,V) = 0$$

for all vector fields X, U and V on M. Then we have

$$\mathcal{Z}(R(\xi,X)U,V) + \mathcal{Z}(U,R(\xi,X)V) = 0, \tag{4.1}$$

which implies

$$\begin{aligned} &S(R(\xi,X)U,V) + \psi g(R(\xi,X)U,V) \\ &+S(U,R(\xi,X)V) + \psi g(U,R(\xi,X)V) = 0. \end{aligned} \tag{4.2}$$

Since $g(R(X,Y)U,V) = -g(R(X,Y)V,U)$, then the foregoing equation reduces to

$$S(R(\xi,X)U,V) + S(U,R(\xi,X)V) = 0.$$

Applying (2.4) and (2.5) in the preceding equation yields

$$2g(X,U)\eta(V) - \eta(U)S(X,V) + 2g(X,V)\eta(U) - \eta(V)S(X,U) = 0. \tag{4.3}$$

Substituting ξ for V in the foregoing equation and using (2.5), we obtain

$$S(X,U) = 2g(X,U). \tag{4.4}$$

This shows that the manifold is Einstein and hence, from (2.6), we get

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$$

This proves that the manifold is of constant curvature 1.

Conversely, if the manifold is of constant curvature 1, then equation (4.4) holds. From (4.3), we can say that the condition $R(\xi,X)\cdot\mathcal{Z} = 0$ holds on M. This completes the proof. □

Since $R\cdot\mathcal{Z} = 0$ implies $R(\xi,X)\cdot\mathcal{Z} = 0$, then we can say the following:

Corollary 4.2. *A Sasakian 3-manifold (M,φ,ξ,η,g) is $\mathcal{Z}$-semisymmetric if and only if (M,φ,ξ,η,g) is of constant curvature 1.*

Proof. If a Sasakian 3-manifold is $\mathcal{Z}$-semisymmetric, then from theorem 4.1 it follows that the manifold is of constant curvature 1. We only need to show the converse part. If the manifold is of constant curvature 1, then we have $S(X,Y) = 2g(X,Y)$. Now,

$$\begin{aligned}
(R(X,Y)\cdot\mathcal{Z})(U,V) &= -\mathcal{Z}(R(X,Y)U,V) - \mathcal{Z}(U,R(X,Y)V)\\
&= -S(R(X,Y)U,V) - \psi g(R(X,Y)U,V)\\
&\quad -S(U,R(X,Y)V) - \psi g(U,R(X,Y)V)\\
&= -2g(R(X,Y)U,V) - \psi g(R(X,Y)U,V)\\
&\quad -2g(U,R(X,Y)V) - \psi g(U,R(X,Y)V)\\
&= 0,
\end{aligned}$$

since the manifold is Einstein and $g(R(X,Y)U,V) = -g(R(X,Y)V,U)$. □

5. SASAKIAN 3-MANIFOLDS SATISFYING $Q(\mathcal{Z},R) = 0$

Suppose that a Sasakian 3-manifold satisfies the curvature condition $Q(\mathcal{Z},R) = 0$. Then we have

$$((X \wedge_{\mathcal{Z}} Y)\cdot R)(U,V)W = 0$$

for all vector fields X, Y, U, V and W on M. This means

$$\begin{aligned}&(X \wedge_{\mathcal{Z}} Y)R(U,V)W - R((X \wedge_{\mathcal{Z}} Y)U,V)W\\ &-R(U,(X \wedge_{\mathcal{Z}} Y)V)W - R(U,V)(X \wedge_{\mathcal{Z}} Y)W = 0,\end{aligned} \tag{5.1}$$

where

$$(X \wedge_{\mathcal{Z}} Y)W = \mathcal{Z}(Y,W)X - \mathcal{Z}(X,W)Y. \tag{5.2}$$

Applying (5.2) in (5.1), we get

$$\begin{aligned}&\mathcal{Z}(Y,R(U,V)W)X - \mathcal{Z}(X,R(U,V)W)Y\\ &-R(\mathcal{Z}(Y,U)X - \mathcal{Z}(X,U)Y,V)W\\ &-R(U,\mathcal{Z}(Y,V)X - \mathcal{Z}(X,V)Y)W\\ &-R(U,V)(\mathcal{Z}(Y,W)X - \mathcal{Z}(X,W)Y) = 0.\end{aligned} \tag{5.3}$$

Using (2.5) in (1.4), we get $\mathcal{Z}(X,\xi) = (2+\psi)\eta(X)$. Now, substituting $W = Y = \xi$ in the (5.3) and using (2.3)-(2.5), we get

$$\begin{aligned}(2+\psi)R(U,V)X \;&=\; \eta(U)[S(X,V)+\psi g(X,V)]\xi\\ &\quad -[S(X,U)V+\psi g(X,U)V].\end{aligned} \tag{5.4}$$

Replacing ξ for X and V in the preceding equation and using (2.3)-(2.5) yields

$$(2+\psi)[U-\eta(U)\xi] = 0$$

for any vector field U on M. This shows that $\psi = -2$. Then (5.4) reduces to

$$\eta(U)[S(X,V)-2g(X,V)]\xi - [S(X,U)V-2g(X,U)V] = 0.$$

First taking inner product of the above equation with ξ, then putting $U = \xi$ and applying (2.5), we obtain

$$S(X,V) = 2g(X,V). \tag{5.5}$$

Then from (2.6), we have

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{5.6}$$

This shows that the manifold is of constant curvature 1.

Now, we consider a Sasakian 3-manifold of constant curvature 1. Then the equations (5.5) and (5.6) are satisfied. Now

$$\begin{aligned} Q(\mathcal{Z},R)(U,V,W;X,Y) &= \mathcal{Z}(Y,R(U,V)W)X - \mathcal{Z}(X,R(U,V)W)Y \\ &\quad -R(\mathcal{Z}(Y,U)X - \mathcal{Z}(X,U)Y,V)W \\ &\quad -R(U,\mathcal{Z}(Y,V)X - \mathcal{Z}(X,V)Y)W \\ &\quad -R(U,V)(\mathcal{Z}(Y,W)X - \mathcal{Z}(X,W)Y). \end{aligned}$$

Applying (5.6) in the preceding equation yields

$$\begin{aligned} Q(\mathcal{Z},R)(U,V,W;X,Y) &= \mathcal{Z}(Y,U)g(X,W)V - \mathcal{Z}(X,U)g(Y,W)V \\ &\quad -\mathcal{Z}(Y,V)g(X,W)U + \mathcal{Z}(X,V)g(Y,W)U \\ &\quad -\mathcal{Z}(Y,W)g(X,V)U + \mathcal{Z}(Y,W)g(U,X)V \\ &\quad +\mathcal{Z}(X,W)g(V,Y)U - \mathcal{Z}(X,W)g(U,Y)V. \end{aligned}$$

Since (5.5) holds, then from (1.4), we have $\mathcal{Z}(X,Y) = (2+\psi)g(X,Y)$. Applying this in the above equation, we will obtain

$$Q(\mathcal{Z},R)(U,V,W;X,Y) = 0$$

for all vector fields X, Y, U, V and W on M.
From the above discussion, we arrive to the following:

Theorem 5.1. *A Sasakian 3-manifold (M,φ,ξ,η,g) satisfies the curvature condition $Q(\mathcal{Z},R) = 0$ if and only if (M,φ,ξ,η,g) is of constant curvature 1.*

Also, for a Sasakian 3-manifold (M,φ,ξ,η,g) satisfying the curvature condition $Q(\mathcal{Z},R) = 0$, we have obtained $\psi = -2$. Further (M,φ,ξ,η,g) is Einstein (see equation (5.5)). Then from (1.4), we get $\mathcal{Z} = 0$. Therefore, we can also state the following:

Theorem 5.2. *A Sasakian 3-manifold (M,φ,ξ,η,g) satisfies the curvature condition $Q(\mathcal{Z},R) = 0$ if and only if the manifold (M,φ,ξ,η,g) is $\mathcal{Z}$-flat.*

6. PROJECTIVELY $\mathcal{Z}$-SEMISYMMETRIC SASAKIAN 3-MANIFOLDS

We start this section with the following definition:

Definition 6.1. *A Riemannian manifold* (M, g) *is said to be projectively* $\mathcal{Z}$*-semisymmetric if the condition* $P \cdot \mathcal{Z} = 0$ *holds on* M*, that is,*

$$(P(X,Y) \cdot \mathcal{Z})(U,V) = 0$$

holds for all vector fields X*,* Y*,* U *and* V *on* M*.*

We first consider that a Sasakian 3-manifold satisfies a weaker condition $P(\xi, Y) \cdot \mathcal{Z} = 0$ than the projectively $\mathcal{Z}$-semisymmetry. Then for all vector fields U, V on M, we have

$$\mathcal{Z}(P(\xi,Y)U,V) + \mathcal{Z}(U, P(\xi,Y)V) = 0. \tag{6.1}$$

With the help of (1.6), (2.4) and (2.5), we obtain

$$P(\xi,Y)U = g(Y,U)\xi - \frac{1}{2}S(Y,U)\xi. \tag{6.2}$$

Applying (6.2) in (6.1) and then using (1.4), we obtain

$$\begin{aligned}
&(2+\psi)g(Y,U)\eta(V) - \frac{1}{2}(2+\psi)S(Y,U)\eta(V) \\
&+(2+\psi)g(Y,V)\eta(U) - \frac{1}{2}(2+\psi)S(Y,V)\eta(U) = 0.
\end{aligned} \tag{6.3}$$

Substituting $U = \xi$ in the foregoing equation yields

$$(2+\psi)[S(Y,V) - 2g(Y,V)] = 0,$$

which implies either $\psi = -2$ or $S(Y,V) = 2g(Y,V)$.
For the second case, the manifold is Einstein and is of constant curvature 1.
From the above discussion, we can state the following:

Theorem 6.2. *If a Sasakian 3-manifold* $(M, \varphi, \xi, \eta, g)$ *satisfies the curvature condition* $P(\xi, Y) \cdot \mathcal{Z} = 0$*, then either* $(M, \varphi, \xi, \eta, g)$ *is of constant curvature 1 or* $\psi = -2$*.*

Since $P(X,Y) \cdot \mathcal{Z} = 0$ implies $P(\xi,Y) \cdot \mathcal{Z} = 0$, then from the above theorem, we get

Theorem 6.3. *If a Sasakian 3-manifold* $(M, \varphi, \xi, \eta, g)$ *is projectively* $\mathcal{Z}$*-semisymmetric, then either* $(M, \varphi, \xi, \eta, g)$ *is of constant curvature 1 or* $\psi = -2$*.*

7. $\mathcal{Z}$-recurrent Sasakian 3-manifolds

A non-flat Riemannian manifold (M, g) is said to be $\mathcal{Z}$-recurrent (see [7, 10]) if the $\mathcal{Z}$-tensor satisfies the condition

$$(\nabla_X \mathcal{Z})(U, V) = \eta(X)\mathcal{Z}(U, V) \tag{7.1}$$

for all vector fields X, U and V on M, where η is a non-zero 1-form.
We now consider a $\mathcal{Z}$-recurrent Sasakian 3-manifold and prove the following theorem:

Theorem 7.1. *A $\mathcal{Z}$-recurrent Sasakian 3-manifold $(M, \varphi, \xi, \eta, g)$ is Ricci recurrent and $(M, \varphi, \xi, \eta, g)$ is of constant curvature 1.*

Proof. For a $\mathcal{Z}$-recurrent Sasakian 3-manifold (M, g) the equation (7.1) holds, where the non-zero 1-form η is given by $\eta(X) = g(X, \xi)$. From (1.4) and (2.7), we can easily obtain

$$\begin{aligned}(\nabla_X \mathcal{Z})(U, V) &= \frac{1}{2}[X(r)g(U, V) - X(r)\eta(U)\eta(V) + (6 - r)\{\eta(U)(\nabla_X \eta)V \\ &\quad + \eta(V)(\nabla_X \eta)U\}] + X(\psi)g(U, V).\end{aligned} \tag{7.2}$$

Substituting (7.2) in (7.1), then replacing ξ for X and using (2.2) and $\xi(r) = 0$, we obtain

$$\mathcal{Z}(U, V) = \xi(\psi)g(U, V).$$

Using (1.4) in the preceding equation yields

$$S(U, V) = (\psi + \xi(\psi))g(U, V). \tag{7.3}$$

Again substituting $U = V = \xi$ in (7.2) gives $X(\psi) = (2 + \psi)\eta(X)$, which gives $\xi(\psi) = 2 + \psi$. Using this in (7.4) yields

$$S(U, V) = (2\psi + 2)g(U, V). \tag{7.4}$$

This shows that the manifold is Einstein. Again from (2.7), M will be Einstein if and only if $r = 6$ and in that case $S(X, Y) = 2g(X, Y)$. Comapring this with (7.4), we get $\psi = 0$. Then from (2.6), we get that the manifold M is of constant curvature 1. Since $\psi = 0$, then the $\mathcal{Z}$-tensor is identical with the Ricci tensor S and hence, a $\mathcal{Z}$-recurrent Sasakian 3-manifold $(M, \varphi, \xi, \eta, g)$ is Ricci recurrent. This completes the proof. □

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