J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2021.28.2.119 ISSN(Online) 2287-6081
Volume 28, Number 2 (May 2021), Pages 119–142

A NEW ANALYTIC FOURIER-FEYNMAN TRANSFORM W.R.T. SUBORDINATE BROWNIAN MOTION

Mohamed El Koufi

Abstract. In this paper, we first introduce a new L_p analytic Fourier-Feynman transform with respect to subordinate Brownian motion (AFFTSB), which extends the Fourier-Feynman transform in the Wiener space. We next examine several relationships involving the L_p-AFFTSB, the convolution product, and the gradient operator for several types of functionals.

1. Introduction and Preliminaries

The study of an L_1 analytic Fourier-Feynman transformation on a classical Wiener space was initiated by Brue in [3]. In [4], Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform on classical Wiener space. In [10], Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \le p \le 2$ that extended the results in [4] and gave various relationships between the L_1 and L_2 theories. In [5, 6], Chang, Choi, and Skoug developed a generalized Fourier-Feynman transform and established several relationships involving convolution product and first variation on function space. For an elementary introduction to the analytic Fourier-Feynman transform, see [12] and the references cited therein.

Since the introduction of the Fourier-Feynman transform many researches on this theory focused on the Wiener measure which is the measure associated to a Brownian motion $(B_t)_{t\geq 0}$ or on the generalized Wiener measure which is the measure associated to stochastic process $(a(t) + B_{b(t)})_{t\geq 0}$ where a and b are a deterministic functions, see [6, 7, 8, 11]. In this paper we introduce a new analytic Fourier-Feynman transform with respect to subordinate Brownian motion which can be seen as a natural extension of this transform.

Received by the editors March 16, 2021. Accepted April 15, 2021.
2010 *Mathematics Subject Classification.* 28C20, 60G51, 60J65 and 42B10.
Key words and phrases. subordinate Brownian motion, analytic Fourier-Feynman transform.

Next, we introduce some notations, some definitions and some basic facts related to subordinate Brownian motion, which are needed to understand the contents of the subsequent sections.

Throughout this paper, let $\mathbb{C}_+$ and $\tilde{\mathbb{C}}_+$ denote the set of the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part. Given a real number $T > 0$ and a probability space Ω, we recall that a subordinator $(S_t)_{t\in[0,T]}$ is an increasing Lévy process (see [1, 2]). Such process has stationary and independent increments, and its trajectories are cadlag (i.e. right-continuous with left limits). The Laplace transform of a subordinator $(S_t)_{t\in[0,T]}$ can be expressed in the form

$$\mathbb{E}[\exp(-uS_t)] = \exp(-t\varphi(u)), \quad u \geq 0, \tag{1.1}$$

where $\varphi:\ [0,\infty[\longmapsto [0,\infty[$ is called the Laplace exponent of $(S_t)_{t\in[0,T]}$. The function φ is an example of a Bernstein function with $\varphi(0+) = 0$, it is known by the Lvy-Khintchine formula that there exist a unique nonnegative real number δ and a unique measure Π on $]0,\infty[$ with $\int_0^\infty (1\wedge x)\Pi(dx) < \infty$, such that for every $u \geq 0$

$$\varphi(u) = \delta u + \int_0^\infty \left(1 - e^{-ux}\right)\Pi(dx).$$

By [13, Proposition 3.6, p.25], the Laplace exponent φ of a subordinator admits an extension which is continuous on $\tilde{\mathbb{C}}_+$ and analytic on $\mathbb{C}_+$. We will still denote by φ this extension. It should be clear that

$$\mathbb{E}[\exp(-zS_t)] = \exp(-t\varphi(z)), \quad z \in \tilde{\mathbb{C}}_+.$$

Let μ be the distribution of $(S_t)_{t\in[0,T]}$, which is a probability measure on the path space

$$\mathbb{S} = \{\ell:\ [0,T] \to (0,\infty):\ \ell \ \text{ increasing and cdlg},\ \ell_0 = 0\},$$

equipped with the Skorokhod topology $\tilde{\mathfrak{B}}(\mathbb{S})$. Thus, the subordinator $(S_t)_{t\in[0,T]}$ can be realized as a canonical process on $\left(\mathbb{S}, \tilde{\mathfrak{B}}(\mathbb{S}), \mu\right)$ defined by

$$S_t(\ell) = \ell_t, \quad (t,\ell) \in [0,T]\times\mathbb{S}.$$

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion starting from zero. The Wiener measure W, that is, the distribution of $(B_t)_{t\geq 0}$, is a probability measure on the path space

$$\mathcal{C}_0 = \{x:\ [0,\infty) \to \mathbb{R}:\ x \ \text{ is continuous and},\ x(0) = 0\},$$

which is endowed with the topology of locally uniform convergence $\mathfrak{B}(\mathcal{C}_0)$. Note that $(B_t)_{t\geq 0}$ can be regarded as a process on the classical Wiener space $(\mathcal{C}_0, \mathfrak{B}(\mathcal{C}_0), W)$

defined by

$$B_t(x) = x(t), \quad (t, x) \in [0, M] \times \mathcal{C}_0.$$

Throughout this article, we assume that $(S_t)_{t\in[0,T]}$ is independent of the standard Brownian motion $(B_t)_{t\geq 0}$. The process $(B_{S_t})_{t\in[0,T]}$ is called a subordinate Brownian motion. This process is a Lévy process. Since S and B are independent, $(B_{S_t})_{t\in[0,T]}$ is the canonical process on the product space $(\mathcal{C}_0 \times \mathbb{S}, \mathfrak{B}(\mathcal{C}_0) \otimes \tilde{\mathfrak{B}}(\mathbb{S}), W \times \mu)$:

$$B_{S_t}(x, \ell) = B_{S_t(\ell)}(x) = x(\ell_t), \quad (t, x, \ell) \in [0, T] \times \mathcal{C}_0 \times \mathbb{S}.$$

Let W^μ be the distributions of $(B_{S_t})_{t\in[0,T]}$, then W^μ is a probability measure on the path space

$$\Omega_0 = \{x \circ \ell : \quad (x, \ell) \in \mathcal{C}_0 \times \mathbb{S}\},$$

equipped with the Skorokhod topology $\tilde{\mathfrak{B}}(\Omega_0)$.

A subset M of Ω_0 is said to be scale-invariant measurable [5,16] provided ρM is $\tilde{\mathfrak{B}}(\Omega_0)$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null set provided $W^\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (SI-a.e.)

Next we give the definitions of the analytic Feynman integral with respect to subordinate Brownian motion.

Let F be a measurable functional on Ω_0 such that for each $\lambda > 0$, the function space integral

$$\mathbb{E}_{\Omega_0}[F(\lambda^{-1/2}\cdot)] \equiv J_F(\lambda) := \int_{\Omega_0} F(\lambda^{-1/2} x \circ \ell)\, W^\mu(dx \circ \ell),$$

exists as a finite number. If there exists a function $J_F^*(\lambda)$ analytic in the half-plane $\mathbb{C}_+$ such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda > 0$ then $J_F^*(\lambda)$ is defined to be the analytic function space integral of F over Ω_0 with parameter λ, and for $\lambda \in \mathbb{C}_+$, we write

$$\mathbb{E}_{\Omega_0}^{anw_\lambda}[F] = J_F^*(\lambda).$$

For $q \in \mathbb{R} - \{0\}$, if the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

$$\mathbb{E}_{\Omega_0}^{anf_q}[F] = \lim_{\lambda \to -iq} \mathbb{E}_{\Omega_0}^{anw_\lambda}[F], \tag{1.2}$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$. Now we are ready to state the definition of the Lp analytic Fourier-Feynman transform with respect to the measure W^μ on Ω_0.

Definition 1.1. Let F be a measurable functional on Ω_0 such that for all a.e.-W^μ $y \circ \ell$ in Ω_0, $T_\lambda F(y) = \mathbb{E}_{\Omega_0}^{anw_\lambda}[F(\cdot + y \circ \ell)]$ exists. For $q \in \mathbb{R} - \{0\}$ and $p \in (1,2]$, the L_p-AFFTSB is defined by

$$(T_q^{(p)} F)(y \circ \ell) = \underset{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}}{\text{l. i. m.}} (W_s^{p'}) T_\lambda F(y \circ \ell),$$

if it exists; that is, for each $\varrho > 0$,

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} \mathbb{E}_{\Omega_0} \left[\left| (T_\lambda F)(\varrho y \circ \ell) - (T_q^{(p)} F)(\varrho y \circ \ell) \right|^{p'} \right] = 0,$$

where $1/p + 1/p' = 1$. We define the L_1-AFFTSB by the formula (if it exists)

$$(T_q^{(1)} F)(y \circ \ell) = \lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}_+}} T_\lambda F(y \circ \ell), \tag{1.3}$$

for SI-a.e. $y \circ \ell \in \Omega_0$.

We note that for $p \in [1,2]$, $(T_q^{(p)} F)$ is defined only SI-a.e.

Schilling in [9] has defined a gradient operator with respect to subordinate Brownian motion. For $h \in \mathcal{C}_0$, the directional derivative (first variation) of a function F on Ω_0 in direction h is defined as

$$D_h F(x \circ \ell) := \lim_{\epsilon \to 0} \frac{F(x \circ \ell + \epsilon h \circ \ell) - F(x \circ \ell)}{\epsilon}, \qquad x \circ \ell \in \Omega_0, \tag{1.4}$$

whenever the limit exists. Denote by $AC([0,\infty[;\mathbb{R})$ the family of all absolutely continuous functions from $[0,\infty[$ to $\mathbb{R}$. The following Cameron-Martin type space will be important $\mathbb{H}^{(k)}$ $(k \in \mathbb{R})$:

$$\mathbb{H}^{(k)} := \left\{ h \in \mathcal{C} \cap AC([0,\infty[;\mathbb{R}) : \int_0^\infty |h'(t)| [\mathbb{P}(S_T \geq t)]^k dt < \infty \right\} \tag{1.5}$$

which becomes a Hilbert space with the inner product

$$\langle g, h \rangle = \int_0^\infty g'(t) h'(t) [\mathbb{P}(S_T \geq t)]^k dt, \qquad g, h \in \mathbb{H}^{(k)}$$

An important class of functions on Ω_0 for which the above definition of $D_h F$ makes sense are the smooth cylinder functions, denoted by $\mathfrak{F}_b^\infty$, that is, the set of all functions having the form

$$F(x \circ \ell) = f(x \circ \ell_{t_1}, \dots, x \circ \ell_{t_n}), \qquad x \circ \ell \in \Omega_0, \tag{1.6}$$

where $n \in \mathbb{N}$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ and $0 < t_1 < \ldots < t_n < T$. Then it is clear that for all $x \circ \ell \in \Omega_0$, $h \in \mathbb{H}^{(k)}$

$$D_h F(x \circ \ell) = \sum_{j=1}^{n} \partial_j f(x \circ \ell_{t_1}, \ldots, x \circ \ell_{t_n}) h(\ell_{t_j}). \tag{1.7}$$

Moreover Schilling in [9] has proved that for $F \in \mathfrak{F}_b^\infty$, $x \in \mathcal{C}_0$, and μ-almost all $\ell \in \mathbb{S}$, the map $h \in\mapsto D_h F(x \circ \ell)$ is a bounded linear functional on $\mathbb{H}^{(k)}$.

Next we state the definition of the convolution product in the subordinate Brownian motion space.

Definition 1.2. Let F and G be measurable functionals on Ω_0. We define their convolution product if it exists by

$$(F \tilde{*} G)_\lambda (y \circ \ell) = \begin{cases} \mathbb{E}_{\Omega_0}^{anw_\lambda}\left[F\left(\frac{y\circ\ell + \cdot}{2}\right) G\left(\frac{y\circ\ell - \cdot}{2}\right)\right], & \text{if } \lambda \in \mathbb{C}_+ \\ \mathbb{E}_{\Omega_0}^{anw_q}\left[F\left(\frac{y\circ\ell + \cdot}{2}\right) G\left(\frac{y\circ\ell - \cdot}{2}\right)\right], & \text{if } \lambda = -iq \end{cases} \tag{1.8}$$

Remark 1.3. When $\lambda = -iq$ we denote $(F \tilde{*} G)_\lambda$ by $(F \tilde{*} G)_q$.

We next describe two classes of spaces of functionals on Ω_0 that we will be working with in this paper.

Definition 1.4. Let $\mathcal{E}$ be the space of functional F that can be expressed in the form

$$F(x \circ \ell) = \int_0^T \exp\left(i x \circ \ell_t\right) \alpha(dt), \quad x \circ \ell \in \Omega_0, \tag{1.9}$$

where α is a finite Borel measure on $[0, T]$.

Definition 1.5. We denote by $\mathcal{A}(n, p)$ the space of functional F expressed in the form

$$F(x \circ \ell) = f(\pi_{\vec{t}}(x \circ \ell)) := f(x \circ \ell_{t_1}, \ldots, x \circ \ell_{t_n}), \quad x \circ \ell \in \Omega_0, \tag{1.10}$$

where $0 < t_1 < \ldots < t_n \leq T$, $\vec{t} = (t_1, \ldots, t_n)$, and $f \in L^p(\mathbb{R}^n)$.

2. An L_p-AFFTSB applied to Functional $F \in \mathcal{E}$

In this subsection we establish the existence and give the expression of the L_p-AFFTSB of functionals F form $\mathcal{E}$. It is clear that F is measurable on Ω_0 with respect to W^μ.

The following lemma gives the expression of the analytic Feynman integral of F.

Lemma 2.1. *Let* $q \in \mathbb{R} - \{0\}$ *and* F *of the form* (1.9). *For SI-a.e.* $y \circ \ell \in \Omega_0$. *Then the analytic Feynman integral* $\mathbb{E}_{\Omega_0}^{anf_q}[F]$ *exists and has the form*

$$\mathbb{E}_{\Omega_0}^{anf_q}[F] = \int_0^T \exp\left(-t\varphi(\frac{1}{-2iq})\right) \alpha(dt), \tag{2.1}$$

where φ *is the Laplace exponent of* S.

Proof. Let $\lambda > 0$, since B and S are independents and using the fact that B_{ℓ_t} is normally distributed with mean 0 and variance ℓ_t, then we have

$$\begin{aligned}
\mathbb{E}_{\Omega_0}[F(\lambda^{-1/2}\cdot)] &= \int_{\Omega_0} F(\lambda^{-1/2} x \circ \ell) W^{\mu}(dx \circ \ell) \\
&= \iint_{\mathbb{S}\times\mathcal{C}_0} F(\lambda^{-1/2} x \circ \ell) W(dx)\mu(d\ell) \\
&= \int_{\mathbb{S}} \int_{\mathcal{C}_0} \int_0^T \exp(i\lambda^{-1/2} x(\ell_t))\ \alpha(dt)\ W(dx)\mu(d\ell) \\
&= \int_0^T \int_{\mathbb{S}} \int_{\mathcal{C}_0} \exp(i\lambda^{-1/2} x(\ell_t))\ W(dx)\mu(d\ell)\alpha(dt) \\
&= \int_0^T \int_{\mathbb{S}} \int_{\mathbb{R}} \exp\left(i\lambda^{-1/2} u\right) \frac{1}{\sqrt{2\pi\ell_t}} \exp\left(-\frac{u^2}{2\ell_t}\right) du\mu(d\ell)\alpha(dt) \\
&= \int_0^T \int_{\mathbb{S}} \exp\left(-\frac{\ell_t}{2\lambda}\right) \mu(d\ell)\alpha(dt) \\
&= \int_0^T \mathbb{E}_{\mathbb{S}}\left[\exp\left(-\frac{S_t}{2\lambda}\right)\right] \alpha(dt) \\
&= \int_0^T \exp\left(-t\varphi(\frac{1}{2\lambda})\right) \alpha(dt).
\end{aligned}$$

Since the Laplace exponent φ of a subordinator can be continued analytically on $\tilde{\mathbb{C}}_+$, then $\lambda \mapsto \varphi(\frac{1}{2\lambda})$ is analytic on $\mathbb{C}_+$. It is easy to see that $\lambda \mapsto \int_0^1 \exp\left(-t\varphi(\frac{1}{2\lambda})\right) \alpha(dt)$ is continuous on $\mathbb{C}_+$. Let Δ be a rectifiable contour in $\mathbb{C}_+$, then by the Fubini theorem and the Cauchy theorem we get that

$$\int_{\Delta} \int_0^T \exp\left(-t\varphi(\frac{1}{2\lambda})\right) \alpha(dt)\ d\lambda = \int_0^T \int_{\Delta} \exp\left(-t\varphi(\frac{1}{2\lambda})\right)\ d\lambda\ \alpha(dt) = 0.$$

Using the Morera theorem, we deduce that $\lambda \mapsto \mathbb{E}_{\Omega_0}[F(\lambda^{-1/2}\cdot)]$ is analytic on $\mathbb{C}_+$. Hence the analytic function space integral $\mathbb{E}_{\Omega_0}^{anw_\lambda}[F]$ exists. Thus by the dominated convergence theorem and the fact that φ is continuous on $\tilde{\mathbb{C}}_+$ we obtain that

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}^+}}\int_0^T \exp\left(-t\varphi(\frac{1}{2\lambda})\right)\alpha(dt)=\int_0^T \exp\left(-t\varphi(\frac{1}{-2iq})\right)\alpha(dt). \tag{2.2}$$

Then (2.1) is proved, which completes the proof. □

Remark 2.2. Notice that the convergence in (2.2) can be obtained in $L^p([0,T],\alpha)$. Hence we have

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\left\|\exp\left(-\cdot\varphi(\frac{1}{2\lambda})\right)-\exp\left(-\cdot\varphi(\frac{1}{-2iq})\right)\right\|_{L^p([0,T],\alpha)}=0. \tag{2.3}$$

Lemma 2.3. *Let F be of the form* (1.9). *Then the analytic function space integral $T_\lambda F(y\circ\ell)$ exists for all $\lambda\in\mathbb{C}_+$ and has the form*

$$T_\lambda F(y\circ\ell)=\int_0^T \exp\left(iy(\ell_t)\right)\exp\left(-t\varphi(\frac{1}{2\lambda})\right)\alpha(dt), \tag{2.4}$$

for SI-a.e. $y\circ\ell\in\Omega_0$.

Proof. Let $\lambda>0$ and $y\circ\ell\in\Omega_0$, then by the Fubini theorem

$$\mathbb{E}_{\Omega_0}[F(\lambda^{-1/2}\cdot+y\circ\ell)]=\int_0^T \exp\left(iy(\ell_t)\right)\exp\left(-t\varphi(\frac{1}{2\lambda})\right)\alpha(dt).$$

By the same why as in the proof of Lemma 2.1, we obtain that $\mathbb{E}_{\Omega_0}[F(\lambda^{-1/2}\cdot+y\circ\ell)]$ admits an analytic extension on $\mathbb{C}_+$ given by (2.4), which completes the proof. □

The following theorem is the main theorem in this section.

Theorem 2.4. *Let F be of the form* (1.9) *and let $p\in[1,2]$. Then for all $q\in\mathbb{R}\setminus\{0\}$, the L_p-AFFTSB of F exists and is given by*

$$(T_q^{(p)}F)(y\circ\ell)=\int_0^T \exp\left(iy(\ell_t)\right)\exp\left(-t\varphi(\frac{1}{-2iq})\right)\alpha(dt), \tag{2.5}$$

for SI-a.e. $y\circ\ell\in\Omega_0$. Furthermore, $T_q^{(p)}F$ is an element of the class $\mathcal{E}$.

Proof. By Lemma 2.3, the analytic function space integral $T_\lambda F(y\circ\ell)]$ exists for SI-a.e. $y\circ\ell$ in Ω_0 and is given by (2.4). Clearly, by the dominated convergence theorem, Eq. (2.5) with $p=1$ holds for SI-a.e. $y\circ\ell\in\Omega_0$. In order to establish (2.5) with $p\in(1,2]$, it suffices to show that for each $\varrho>0$

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}^+}}\mathbb{E}_{\Omega_0}\left[|T_\lambda F(\varrho y\circ\ell)-T_{-iq}F(\varrho y\circ\ell)|^{p'}\right]=0.$$

By Hölder inequality and the Fubini theorem, it follows that for each $\varrho > 0$

$$
\begin{aligned}
&\mathbb{E}_{\Omega_0}\left[|T_\lambda F(\varrho y\circ\ell) - T_{-iq}F(\varrho y\circ\ell)|^{p'}\right]\\
&= \iint_{\mathbb{S}\times\mathcal{C}_0}\Big|\int_0^T \exp\left(i\varrho y(\ell_t)\right)\exp\left(-t\varphi(\frac{1}{2\lambda})\right)dt\\
&\qquad - \int_0^T \exp\left(i\varrho y(\ell_t)\right)\exp\left(-t\varphi(\frac{1}{-2iq})\right)\alpha(dt)\Big|^{p'} W(dy)\mu(d\ell)\\
&= \iint_{\mathbb{S}\times\mathcal{C}_0}\Big|\int_0^T \exp\left(i\varrho y(\ell_t)\right)\Big(\exp\left(-t\varphi(\frac{1}{2\lambda})\right)\\
&\qquad - \exp\left(-t\varphi(\frac{1}{-2iq})\right)\Big)\alpha(dt)\Big|^{p'} W(dy)\mu(d\ell)\\
&\le \iint_{\mathbb{S}\times\mathcal{C}_0}\|\exp\left(ip'\varrho y(\ell_t)\right)\|^{p'}_{L^{p'}([0,T],\alpha)}\, W(dy)\mu(d\ell)\\
&\qquad \cdot\left\|\exp\left(-\cdot\,\varphi(\frac{1}{2\lambda})\right) - \exp\left(-\cdot\,\varphi(\frac{1}{-2iq})\right)\right\|^{p'}_{L^p([0,T],\alpha)}\\
&= \left\|\exp\left(-\cdot\,\varphi(\frac{1}{2\lambda})\right) - \exp\left(-\cdot\,\varphi(\frac{1}{-2iq})\right)\right\|^{p'}_{L^p([0,T],\alpha)}.
\end{aligned}
$$

Then by Remark (2.3) we obtain the desired result.

Next let

$$\tilde{\alpha}_q(dt) = \exp\left(-t\varphi(\frac{1}{-2iq})\right)\alpha(dt)$$

Then it is clear that $\tilde{\alpha}_q$ is a Borel measure, and so $T_q^{(p)}(F)$ is in $\mathcal{E}$. This completes the proof. □

3. OPERATOR GRADIENT AND CONVOLUTION PRODUCT APPLIED TO FUNCTIONAL $F \in \mathcal{E}$

In this section we establish several relationships involving the gradient operator, the convolution product, and the L_p-AFFTSB for functionals from $\mathcal{E}$.

Theorem 3.1. *Let F and G be functionals on Ω_0 of the form (1.9), then the L_2 AFFTSB of $(F\tilde{*}G)_q$ exists and for SI-a.e. $y\circ\ell\in\Omega_0$ we have*

$$(F\tilde{*}G)_q(y\circ\ell) = \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right)\alpha(dt)\beta(ds) \tag{3.1}$$

Proof. Let $\lambda > 0$ and $y \circ \ell \in \Omega_0$, then

$$\begin{aligned}
&\mathbb{E}_{\Omega_0}\left[F\left(\frac{y\circ\ell+\lambda^{-1/2}\cdot}{\sqrt{2}}\right)G\left(\frac{y\circ\ell-\lambda^{-1/2}\cdot}{\sqrt{2}}\right)\right]\\
&=\iint_{\mathbb{S}\times\mathcal{C}_0} F\left(\frac{y\circ\ell+\lambda^{-1/2}x\circ\sigma}{2}\right)G\left(\frac{y\circ\ell-\lambda^{-1/2}x\circ\sigma}{2}\right)W^{\mu}(dx\circ\sigma)\\
&=\int_{\mathbb{S}}\int_{\mathcal{C}_0}\int_0^T \exp\left(i\frac{y(\ell_t)+\lambda^{-1/2}x(\sigma_t)}{2}\right)\alpha(dt)\\
&\qquad\qquad\cdot\int_0^T \exp\left(i\frac{y(\ell_s)-\lambda^{-1/2}x(\sigma_s)}{2}\right)\beta(ds)W(dx)\mu(d\sigma)\\
&=\int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\\
&\qquad\qquad\cdot\int_{\mathbb{S}}\int_{\mathcal{C}_0}\exp\left(\frac{i\lambda^{-1/2}}{2}(x(\sigma_t)-x(\sigma_s))\right)W(dx)\mu(d\sigma)\alpha(dt)\beta(ds)\\
&=\int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\\
&\qquad\cdot\int_{\mathbb{S}}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi|\sigma_t-\sigma_s|}}\exp\left(\frac{i\lambda^{-1/2}}{2}sign(\sigma_t-\sigma_s)u\right)\exp\left(\frac{-u^2}{2|\sigma_t-\sigma_s|}\right)\\
&\qquad\qquad\cdot du\mu(d\sigma)\alpha(dt)\beta(ds)\\
&=\int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\int_{\mathbb{S}}\exp\left(\frac{-\lambda^{-1}|\sigma_t-\sigma_s|}{8}\right)\mu(d\sigma)\alpha(dt)\beta(ds)\\
&=\int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\mathbb{E}_{\Omega_0}\left[\exp\left(-\frac{|S_t-S_s|}{8\lambda}\right)\right]\alpha(dt)\beta(ds)\\
&=\int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}-|t-s|\varphi\left(\frac{1}{8\lambda}\right)\right)\alpha(dt)\beta(ds).
\end{aligned}$$

By the properties of the Laplace exponent φ, it is clear that

$$\lambda \mapsto \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}-|t-s|\varphi\left(\frac{1}{8\lambda}\right)\right)$$

is continuous on $\tilde{\mathbb{C}}_+$, analytic on $\mathbb{C}_+$, and $\mathcal{R}(\varphi(\lambda)) > 0$ whenever $\lambda \in \mathbb{C}_+$. Thus

$$\left|\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}-|t-s|\varphi\left(\frac{1}{8\lambda}\right)\right)\right| \le 1$$

for all $(t,s) \in [0,T]^2$ and $\lambda \in \mathbb{C}_+$. Then $\lambda \mapsto (F\tilde{*}G)_\lambda(y\circ\ell)$ is continuous on $\tilde{\mathbb{C}}_+$ for SI-a.e $y \circ \ell \in \Omega_0$. Moreover by the Morera theorem and the Cauchy theorem we obtain that $\mathbb{E}_{\Omega_0}\left[F\left(\frac{y\circ\ell+\lambda^{-1/2}\cdot}{\sqrt{2}}\right)G\left(\frac{y\circ\ell-\lambda^{-1/2}\cdot}{\sqrt{2}}\right)\right]$ admits an analytic extension on $\mathbb{C}_+$, and for SI-a.e $y\circ\ell\in\Omega_0$

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}(F\tilde{*}G)_\lambda(y\circ\ell)=(F\tilde{*}G)_{-iq}(y\circ\ell).$$

This completes the proof. □

For $(t,s)\in[0,T]$ and $\lambda\in\tilde{\mathbb{C}}_+$, let Φ be defined by

$$\Phi((t,s),\lambda)=\exp\left(-|t-s|\left(\varphi\left(\frac{1}{-8iq}\right)+\varphi\left(\frac{1}{2\lambda}\right)\right)-(t\wedge s)\varphi\left(\frac{2}{\lambda}\right)\right), \tag{3.2}$$

then we have the following lemma which will be helpful in the next theorem

Lemma 3.2. *Let Φ be defined by* (3.2)*, then for all $(t,s)\in[0,T]$, $\lambda\mapsto\Phi((t,s),\lambda)$ is continuous on $\tilde{\mathbb{C}}_+$ and analytic on $\mathbb{C}_+$. Moreover we have the following limit*

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\|\Phi(\cdot,\lambda)-\Phi(\cdot,-iq)\|_{L^{p'}([0,T]^2,\alpha\times\beta)}=0. \tag{3.3}$$

Proof. The continuity and the analyticity of $\lambda\mapsto\Phi((t,s),\lambda)$ are obvious. Thus for all $(t,s)\in[0,T]$

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\Phi((t,s),\lambda)=\Phi((t,s),-iq), \tag{3.4}$$

Note that if $z\in\mathbb{C}$ with $\mathcal{R}(z)>0$, then $\mathcal{R}(\varphi(z))>0$. We deduce that for all $(t,s)\in[0,T]$ and $\lambda\in\tilde{\mathbb{C}}_+$

$$|\Phi((t,s),\lambda)|\le 1, \tag{3.5}$$

then the dominated convergence theorem implies (3.3) and the lemma follows. □

Theorem 3.3. *Let F and G be functionals on Ω_0 of the form* (1.9) *then the L_2 AFFTSB of $(F\tilde{*}G)_q$ exists and for SI-a.e. $y\circ\ell\in\Omega_0$ we have*

$$T_q(F\tilde{*}G)_q(y\circ\ell)=\int_0^T\int_0^T\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\Phi((t,s),-iq)\alpha(dt)\beta(ds), \tag{3.6}$$

where Φ is given by (3.2).

Proof. Let $\lambda > 0$ and $y \circ \ell \in \Omega_0$, then

$$
\begin{aligned}
&\mathbb{E}_{\Omega_0}\left[(F\tilde{*}G)_q\left(y\circ\ell+\lambda^{-1/2}\cdot\right)\right] \\
&= \iint_{\mathbb{S}\times\mathcal{C}_0}\int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)+\lambda^{-1/2}(x(\sigma_t)+x(\sigma_s))}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right) \\
&\qquad\qquad \cdot\alpha(dt)\beta(ds)W^{\mu}(dx\circ\sigma) \\
&= \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right) \\
&\qquad\qquad \cdot\int_{\mathbb{S}}\int_{\mathcal{C}_0}\exp\left(i\lambda^{-1/2}(x(\sigma_t)+x(\sigma_s))\right)W(dx)\mu(d\sigma)\alpha(dt)\beta(ds) \\
&= \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right)\int_{\mathbb{S}}\int_{\mathbb{R}^2}\exp\left(i\lambda^{-1/2}(u+v)\right) \\
&\qquad \cdot\frac{1}{\sqrt{2\pi(\sigma_t\vee_{\Omega_0}\sigma_s-\sigma\wedge\sigma_s)}}\exp\left(-\frac{u^2}{2(\sigma_t\vee\sigma_s-\sigma\wedge\sigma_s)}\right)\frac{1}{\sqrt{2\pi 4\sigma_t\wedge\sigma_s}} \\
&\qquad \cdot\exp\left(-\frac{v^2}{8\sigma_t\wedge\sigma_s}\right)dudv\mu(d\sigma)\alpha(dt)\beta(ds) \\
&= \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right) \\
&\qquad\qquad \cdot\int_{\mathbb{S}}\exp\left(-\frac{\lambda^{-1}(\sigma_t+2\sigma_t\wedge\sigma_s+\sigma_s)}{2}\right)\mu(d\sigma)\alpha(dt)\beta(ds) \\
&= \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right) \\
&\qquad\qquad \cdot\mathbb{E}_{\Omega_0}\left[\exp\left(-\frac{\lambda^{-1}(S_t+2S_t\wedge S_s+S_s)}{2}\right)\right]\alpha(dt)\beta(ds) \\
&= \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right)\mathbb{E}_{\Omega_0}\left[\exp\left(-\frac{\lambda^{-1}(S_t-S_s)}{2}\right)\right] \\
&\qquad \cdot\mathbb{E}_{\Omega_0}\left[\exp\left(-\frac{\lambda^{-1}(2S_s)}{2}\right)\right]\alpha(dt)\beta(ds) \\
&= \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right)\exp\left(-|t-s|\varphi(\frac{1}{2\lambda})\right) \\
&\qquad\qquad\qquad \cdot\exp\left(-(t\wedge s)\varphi(\frac{2}{\lambda})\right)\alpha(dt)\beta(ds) \\
&= \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\left(\varphi\left(\frac{1}{-8iq}\right)+\varphi\left(\frac{1}{2\lambda}\right)\right)\right. \\
&\qquad\qquad\qquad \left. -(t\wedge s)\varphi\left(\frac{2}{\lambda}\right)\right)\alpha(dt)\beta(ds) \\
&= \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\Phi((t,s),\lambda)\alpha(dt)\beta(ds).
\end{aligned}
$$

Since $\lambda \mapsto \Phi((t,s),\lambda)$ is analytic on $\mathbb{C}_+$ and satisfy $|\Phi((t,s),\lambda)| \leq 1$ for all $(t,s) \in [0,T]$, then by the dominated convergence theorem, the Cauchy theorem, and the Morera theorem, we obtain that $\lambda \mapsto \mathbb{E}_{\Omega_0}\left[(F\tilde{*}G)_q(y\circ\ell+\lambda^{-1/2}\cdot)\right]$ admits an extension analytic on $\mathbb{C}_+$. Thus $T_\lambda(F\tilde{*}G)_q(y\circ\ell)$ is well defined for SI-a.e. $y\circ\ell \in \Omega_0$. to obtain (3.6) it remains to show that for all $\rho > 0$

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \mathbb{E}_{\Omega_0}\left[\left|T_\lambda(F\tilde{*}G)_q(\rho\cdot) - T_q(F\tilde{*}G)_q(\rho\cdot)\right|^{p'}\right]. \tag{3.7}$$

But for $\rho > 0$ we have by the Hölder inequality and the Fubini theorem

$$\begin{aligned}
&\mathbb{E}_{\Omega_0}\left[\left|T_\lambda(F\tilde{*}G)_q(\rho\cdot) - T_q(F\tilde{*}G)_q(\rho\cdot)\right|^{p'}\right]\\
&= \iint_{\mathbb{S}\times\mathcal{C}_0} \left|T_\lambda(F\tilde{*}G)_q(\rho y\circ\ell) - T_q(F\tilde{*}G)_q(\rho y\circ\ell)\right|^{p'} W^\mu(dy\circ\ell)\\
&= \iint_{\mathbb{S}\times\mathcal{C}_0} \left|\int_0^T\int_0^T \exp\left(i\frac{\rho y(\ell_t)+\rho y(\ell_s)}{2}\right)[\Phi((t,s),\lambda)-\Phi((t,s),-iq)]\,\alpha(dt)\beta(ds)\right|^{p'}\\
&\qquad\qquad \cdot W^\mu(dy\circ\ell)\\
&\leq \iint_{\mathbb{S}\times\mathcal{C}_0} \left\|\exp\left(ip\frac{\rho y(\ell_t)+\rho y(\ell_s)}{2}\right)\right\|^{p'}_{L^p([0,T]^2,\alpha\times\beta)}\\
&\qquad\qquad\qquad \cdot W^\mu(dy\circ\ell)\,\|\Phi(\cdot,\lambda)-\Phi(\cdot,-iq)\|^{p'}_{L^{p'}([0,T]^2,\alpha\times\beta)}\\
&= (\alpha([0,T])\beta([0,T]))^{p'/p}\,\|\Phi(\cdot,\lambda)-\Phi(\cdot,-iq)\|^{p'}_{L^2([0,T]^2,\alpha\times\beta)}.
\end{aligned}$$

Thus by Lemma 3.2 we obtain (3.7), which completes the proof. □

Theorem 3.4. *Let F and G be functionals on Ω_0 of the form* (1.9) *then the convolution product of T_q^pF and T_q^pG exists and for SI-a.e. $y\circ\ell \in \Omega_0$ and is given by*

(3.8)

$$\begin{aligned}
&(T_qF\tilde{*}T_qG)_q(y\circ\ell)\\
&= \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right) - (t+s)\varphi(\frac{1}{-2iq})\right)\alpha(dt)\beta(ds)
\end{aligned}$$

Proof. Let $\lambda > 0$ and $y\circ\ell \in \Omega_0$, then

$$
\begin{aligned}
&\mathbb{E}_{\Omega_0}\left[T_qF\left(\frac{y\circ\ell+\lambda^{-1/2}\cdot}{2}\right)T_qG\left(\frac{y\circ\ell-\lambda^{-1/2}\cdot}{2}\right)\right]\\
&=\iint\limits_{\mathbb{S}\times\mathcal{C}_0}\int_0^T\int_0^T\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\exp\left(-(t+s)\varphi(\frac{1}{-2iq})\right)\alpha(dt)\beta(ds)\\
&\qquad\cdot\exp\left(i\lambda^{-1/2}\frac{x(\sigma_t)-x(\sigma_s)}{2}\right)W^{\mu}(dx\circ\sigma)\\
&=\int_0^T\int_0^T\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\exp\left(-(t+s)\varphi(\frac{1}{-2iq})\right)\\
&\qquad\cdot\int_{\mathbb{S}}\int_{\mathcal{C}_0}\exp\left(i\lambda^{-1/2}\frac{x(\sigma_t)-x(\sigma_s)}{2}\right)W(dx)\mu(d\sigma)\alpha(dt)\beta(ds)\\
&=\int_0^T\int_0^T\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\exp\left(-(t+s)\varphi(\frac{1}{-2iq})\right)\\
&\qquad\cdot\int_{\mathbb{S}}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi|\sigma_t-\sigma_s|}}\exp\left(\frac{i\lambda^{-1/2}}{2}sign(\sigma_t-\sigma_s)u\right)\\
&\qquad\qquad\cdot\exp\left(\frac{-u^2}{2|\sigma_t-\sigma_s|}\right)du\mu(d\sigma)\alpha(dt)\beta(ds)\\
&=\int_0^T\int_0^T\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\exp\left(-(t+s)\varphi(\frac{1}{-2iq})\right)\\
&\qquad\cdot\int_{\mathbb{S}}\exp\left(\frac{-\lambda^{-1}|\sigma_t-\sigma_s|}{8}\right)\mu(d\sigma)\alpha(dt)\beta(ds)\\
&=\int_0^T\int_0^T\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\exp\left(-(t+s)\varphi(\frac{1}{-2iq})\right)\\
&\qquad\cdot\mathbb{E}_{\Omega_0}\left[\exp\left(-\frac{|S_t-S_s|}{8\lambda}\right)\right]\alpha(dt)\beta(ds)\\
&=\int_0^T\int_0^T\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}-|t-s|\varphi\left(\frac{1}{8\lambda}\right)-(t+s)\varphi(\frac{1}{-2iq})\right)\alpha(dt)\beta(ds).
\end{aligned}
$$

As in Theorem 3.1 we conclude that $\lambda\mapsto(T_qF\tilde{*}T_qG)_\lambda(y\circ\ell)$ admits an anaytic extension on $\mathbb{C}_+$ which is continuous on $\tilde{\mathbb{C}}_+$. Letting $\lambda\to -iq$ in $\mathbb{C}_+$ we get (3.8). This completes the proof. □

The next theorem allows to calculate $D_h\left[(F\tilde{*}G)_q\right]$. But to do this we have to put additional assumption on the subordinator $(S_t)_{t\in[0,T]}$. We suppose that

$$\int_0^T\mathbb{E}_{\mathbb{S}}\left[S_t\right](d\alpha(t)+d\beta(t))<\infty. \tag{3.9}$$

Theorem 3.5. *Let F be of the form* (1.9) *and $h \in \mathbb{H}$, then the analytic Feynman integral of D_hF exists and is given by*

$$\mathbb{E}_{\Omega_0}^{anf_q}[D_hF] = \int_0^T \mathbb{E}_{\mathbb{S}}\left[ih(S_t)\exp\left(-\frac{S_t}{2iq}\right)\right]\alpha(dt) \tag{3.10}$$

Proof. By the Cauchy Schwartz inequality we have

$$|h(\ell_t)\exp(ix(\ell_t))| = \left|\int_0^{\ell_t} h'(u)du\right| \leq \|h\|_{\mathbb{H}}\sqrt{\ell_t},$$

then by assumption we get

$$\mathbb{E}_{\Omega_0}\left[\int_0^T \sqrt{\ell_t}\alpha(dt)\right] \leq \int_0^T \mathbb{E}_{\mathbb{S}}\left[\sqrt{S_t}\right]\alpha(dt) < \infty.$$

Thus we have that

$$\int_0^T \sqrt{\ell_t}\alpha(dt) < \infty, \qquad a.s.$$

Using the Leibniz's rule for differentiation under the integral sign we obtain that

$$D_hF(x\circ\ell) = \int_0^T ih(\ell_t)\exp(ix(\ell_t))\,\alpha(dt).$$

Let $\lambda > 0$, then by Fubini theorem we obtain

$$\begin{aligned}
\mathbb{E}_{\Omega_0}\left[F(\lambda^{-1/2})\cdot\right] &= \iint\limits_{\mathbb{S}\times\mathcal{C}_0} D_hF(\lambda^{-1/2}x\circ\ell)W^{\mu}(dx\circ\ell) \\
&= \int_0^T \int_{\mathbb{S}}\int_{\mathcal{C}_0} ih(\ell_t)\exp\left(i\lambda^{-1/2}x(\ell_t)\right)W(dx)\mu(d\ell)\alpha(dt) \\
&= \int_0^T \int_{\mathbb{S}}\int_{\mathbb{R}} ih(\ell_t)\exp\left(i\lambda^{-1/2}u\right)\frac{1}{\sqrt{2\pi\ell_t}}\exp\left(-\frac{u^2}{2\ell_t}\right)du\mu(d\ell)\alpha(dt) \\
&= \int_0^T \int_{\mathbb{S}} ih(\ell_t)\exp\left(-\frac{\ell_t}{2\lambda}\right)\mu(d\ell)\alpha(dt) \\
&= \int_0^T \mathbb{E}_{\mathbb{S}}\left[ih(S_t)\exp\left(-\frac{S_t}{2\lambda}\right)\right]\alpha(dt).
\end{aligned}$$

The Morera theorem with assumption (3.9) entails the existence of $\mathbb{E}_{\Omega_0}^{anw_\lambda}[F]$, and by the convergence theorem we obtain (3.10). This completes the proof. $\square$

Theorem 3.6. *Let F and G be functionals on Ω_0 of the form* (1.9), *then the L_2 AFFTSB of $(F\tilde{*}G)_q$ exists and for SI-a.e. $y \circ \ell \in \Omega_0$ we have*

(3.11)

$$D_h\left[(F\tilde{*}G)_q\right](y\circ\ell) = \int_0^T\int_0^T i\frac{h(\ell_t)+h(\ell_s)}{2}\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right)\alpha(dt)\beta(ds)$$

Proof. By Theorem 3.1 we have that $(F\tilde{*}G)_q(y\circ\ell)$ exist for SI-a.e $y\circ\ell\in\Omega_0$, and is given by

$$(F\tilde{*}G)_q(y\circ\ell) = \int_0^T\int_0^T \exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right)\alpha(dt)\beta(ds).$$

Let $h\in\mathbb{H}$ and $(t,s)\in[0,T]^2$ then

$$\left|i\frac{h(\ell_t)+h(\ell_s)}{2}\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2} - |t-s|\varphi\left(\frac{1}{-8iq}\right)\right)\right| \le \|h'\|\frac{\ell_t+\ell_s}{2},$$

by assumption (3.9) it follows that

$$\mathbb{E}_{\mathbb{S}}\left[\int_0^T\int_0^T\frac{\ell_t+\ell_s}{2}\alpha(dt)\beta(ds)\right] = \frac{\beta([0,T])}{2}\mathbb{E}_{\mathbb{S}}\left[\int_0^T S_t\alpha(dt)\right] + \frac{\alpha([0,T])}{2}\mathbb{E}_{\mathbb{S}}\left[\int_0^T S_s\beta(ds)\right] < \infty.$$

Thus we get that for a.e. $\ell\in\mathbb{S}$

$$\int_0^T\int_0^T\frac{\ell_t+\ell_s}{2}\alpha(dt)\beta(ds) < \infty.$$

Using Leibniz's rule for differentiation under the integral sign we obtain that $D_h\left[(F\tilde{*}G)_q\right]$ exists and is given by (3.11). This completes the proof. □

Theorem 3.7. *Let F and G be functionals on Ω_0 of the form* (1.9). *Then the L_2 AFFTSB of $(F\tilde{*}G)_q$ exists and for SI-a.e. $y\circ\ell\in\Omega_0$ we have*

$$(3.12)\quad D_h\left[T_q(F\tilde{*}G)_q\right](y\circ\ell) = \int_0^T\int_0^T i\frac{h(\ell_t)+h(\ell_s)}{2}\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\Phi((t,s),\lambda)\cdot\alpha(dt)\beta(ds).$$

Proof. Let $h\in\mathbb{H}$ and $(t,s)\in[0,T]^2$ then

$$\left|i\frac{h(\ell_t)+h(\ell_s)}{2}\exp\left(i\frac{y(\ell_t)+y(\ell_s)}{2}\right)\Phi((t,s),-iq)\right| \le \|h'\|\frac{\ell_t+\ell_s}{2},$$

thus as in the proof of Theorem 3.6, it follows by Leibniz's rule for differentiation under the integral sign that $D_h\left[T_q\left(F\tilde{*}G\right)_q\right]$ exists and is given by (3.12), which completes the proof. $\square$

4. THE L_p-AFFTSB APPLIED TO FUNCTIONAL $F \in \mathcal{A}(p,n)$

In this section we show the existence of the L_p-AFFTSB and we give its expression for functionals F from $\mathcal{A}(2,n)$. The integrability with respect the subordinate Brownian motion is a difficult problem to deal with. We will consider a particular subordinator, the Lévy subordinator (see [1]) which can be defined as a first hitting time for one-dimensional standard Brownian motion $(B_t)_{t\geq 0}$. More precisely:

$$S_0 = 0, \qquad S_t = \inf\left\{s > 0;\ B_s = \frac{t}{\sqrt{2}}\right\}, \quad t > 0.$$

It is known by [1, Example 1.3.19, p.53] that S_t has a density given by the Lévy distribution

$$\phi_{S_t}(u) = \left(\frac{t}{2\sqrt{\pi}}\right) u^{-3/2} e^{-t^2/4u}, \quad u \geq 0. \tag{4.1}$$

For the subordinator S we consider the function $H_S(\vec{u},\lambda)$ defined by

$$H_S(\vec{u},\lambda) = \int_{\mathbb{S}} \prod_{k=1}^{n} \left(\sigma_{t_k} - \sigma_{t_{k-1}}\right)^{-1/2} \exp\left(-\frac{\lambda}{2}\sum_{k=1}^{n} \frac{(u_k - u_{k-1})^2}{\sigma_{t_k} - \sigma_{t_{k-1}}}\right) \mu(d\sigma), \tag{4.2}$$

for $\vec{u} \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}_+$ (if it exists). The following lemma will be helpful in the sequel.

Lemma 4.1. *Let $(S_t)_{t\in[0,T]}$ be a Lvy subordinator and H_S be defined by (4.2), then for all $\vec{u} \in \mathbb{R}^n$ and $\lambda \in \tilde{\mathbb{C}}_+$*

$$(i) \qquad H_S(\vec{u},\lambda) = \left(\frac{2}{\sqrt{\pi}}\right)^n \prod_{k=1}^{n} \frac{t_k - t_{k-1}}{2\lambda(u_k - u_{k-1})^2 + (t_k - t_{k-1})^2}, \tag{4.3}$$

$$(ii) \qquad \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}^+}} \|H_S(\cdot,\lambda) - H_S(\cdot,-iq)\|_{L^{p'}(\mathbb{R}^n)} = 0. \tag{4.4}$$

Proof. i) By independence and stationarity of the increments of S and using the probability density of S_t, we get

$$H_S(\vec{u},\lambda) = \int_{\mathbb{S}} \prod_{k=1}^{n} \left(\sigma_{t_k} - \sigma_{t_{k-1}}\right)^{-1/2} \exp\left(-\frac{\lambda}{2}\sum_{k=1}^{n} \frac{(u_k - u_{k-1})^2}{\sigma_{t_k} - \sigma_{t_{k-1}}}\right) \mu(d\sigma)$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{S}}\left[\prod_{k=1}^{n}\left(S_{t_k}-S_{t_{k-1}}\right)^{-1/2}\exp\left(-\frac{\lambda}{2}\sum_{k=1}^{n}\frac{(u_k-u_{k-1})^2}{S_{t_k}-S_{t_{k-1}}}\right)\right]\\
&= \prod_{k=1}^{n}\mathbb{E}_{\mathbb{S}}\left[\left(S_{t_k}-S_{t_{k-1}}\right)^{-1/2}\exp\left(-\frac{\lambda}{2}\sum_{k=1}^{n}\frac{(u_k-u_{k-1})^2}{S_{t_k}-S_{t_{k-1}}}\right)\right]\\
&= \prod_{k=1}^{n}\mathbb{E}_{\mathbb{S}}\left[\left(S_{t_k-t_{k-1}}\right)^{-1/2}\exp\left(-\frac{\lambda}{2}\sum_{k=1}^{n}\frac{(u_k-u_{k-1})^2}{S_{t_k-t_{k-1}}}\right)\right]\\
&= \prod_{k=1}^{n}\int_0^\infty s^{-1/2}\exp\left(-\frac{\lambda}{2}\frac{(u_k-u_{k-1})^2}{s}\right)\frac{t_k-t_{k-1}}{2\sqrt{\pi}}s^{-3/2}\\
&\qquad\qquad\qquad\qquad\qquad\cdot\exp\left(-\frac{(t_k-t_{k-1})^2}{4s}\right)ds\\
&= \left(\frac{2}{\sqrt{\pi}}\right)^n\prod_{k=1}^{n}\frac{t_k-t_{k-1}}{2\lambda(u_k-u_{k-1})^2+(t_k-t_{k-1})^2}.
\end{aligned}
$$

Thus (4.3) is proved. It is clear that $\lambda \mapsto H_S(\vec{u},\lambda)$ is continuous on $\mathbb{C}_+$, then we have for all $\vec{u} \in \mathbb{R}^n$

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}^+}} H_S(\vec{u},\lambda) = H_S(\vec{u},-iq).$$

Moreover, for $\lambda \in \mathbb{C}_+$ such that $|\lambda + iq| < |q|/2$, we have

$$|H_S(\vec{u},\lambda)| \leq \left(\frac{2}{\sqrt{\pi}}\right)^n\prod_{k=1}^{n}\frac{t_k-t_{k-1}}{\sqrt{|q|^2(u_k-u_{k-1})^4+(t_k-t_{k-1})^4}} \in L^{p'}(\mathbb{R}^n).$$

Therefore by the dominated convergence theorem we get (4.4) and the lemma follows. □

Remark 4.2. The expression (4.3) of H_S shows that for all $\vec{u}\in\mathbb{R}^n$, $\lambda \mapsto H_S(\vec{u},\lambda)$ is analytic on $\tilde{\mathbb{C}}_+$, and for all $\lambda \in \tilde{\mathbb{C}}_+$, $\vec{u} \mapsto H_S(\vec{u},\lambda)$ belongs to $L^{p'}(\mathbb{R}^n)$. Furthermore we have for all $\vec{u}\in\mathbb{R}^n$, $\lambda \in \tilde{\mathbb{C}}_+$

$$|H_S(\vec{u},\lambda)| \leq \gamma := \left(\frac{2}{\sqrt{\pi}}\right)^n\prod_{k=1}^{n}(t_k-t_{k-1})^{-1}. \tag{4.5}$$

Lemma 4.3. *Let $q \in \mathbb{R} - \{0\}$ and F a cylinder functionals of the form* (1.10)*. For SI-a.e. $y \circ \ell \in \Omega_0$, the analytic function space integral $T_\lambda F(y\circ\ell)$ exists and has the form*

$$T_\lambda F(y\circ\ell) = (2\pi)^{-n/2}\,(f * H_S(\cdot,\lambda))(\pi_{\vec{t}}(y\circ\ell) \tag{4.6}$$

for all $\lambda \in \mathbb{C}^+$, where $*$ denote the usual convolution product.

Proof. Let $\lambda \in \mathbb{C}^+$ and $y \circ \ell \in \Omega_0$, then we have

$$\begin{aligned}
&\mathbb{E}_{\Omega_0}[F(\lambda^{-1/2} \cdot + y \circ \ell)] \\
&\quad = \int \int_{\mathbb{S} \times \mathcal{C}_0} F(\lambda^{-1/2} x \circ \sigma + y \circ \ell) W(dx) \mu(d\sigma) \\
&\quad = \int \int_{\mathbb{S} \times \mathcal{C}_0} f(\lambda^{-1/2} \pi_{\vec{t}}(x \circ \sigma_t) + \pi_{\vec{t}}(y \circ \ell_t)) W(dx) \mu(d\sigma) \\
&\quad = \int_{\mathbb{S}} \int_{\mathcal{C}_0} f(\lambda^{-1/2} x(\sigma_{t_1}) + y(\ell_{t_1}), \ldots, \lambda^{-1/2} x(\sigma_{t_n}) + y(\ell_{t_n})) W(dx) \mu(d\sigma) \\
&\quad = \int_{\mathbb{S}} \int_{\mathbb{R}^n} f(\lambda^{-1/2} \vec{u} + \pi_{\vec{t}}(y \circ \ell_t)) \prod_{k=1}^{n} \left[2\pi(\sigma_{t_k} - \sigma_{t_{k-1}})\right]^{-1/2} \\
&\qquad\qquad\qquad \cdot \exp\left(-\frac{1}{2} \sum_{k=1}^{n} \frac{(u_k - u_{k-1})^2}{\sigma_{t_k} - \sigma_{t_{k-1}}} \right) d\vec{u}\, \mu(d\sigma) \\
&\quad = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\vec{u} + \pi_{\vec{t}}(y \circ \ell_t)) \int_{\mathbb{S}} \prod_{k=1}^{n} (\sigma_{t_k} - \sigma_{t_{k-1}})^{-1/2} \\
&\qquad\qquad\qquad \cdot \exp\left(-\frac{\lambda}{2} \sum_{k=1}^{n} \frac{(u_k - u_{k-1})^2}{\sigma_{t_k} - \sigma_{t_{k-1}}} \right) \mu(d\sigma) d\vec{u} \\
&\quad = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\vec{u} + \pi_{\vec{t}}(y \circ \ell_t)) H_S(\vec{u}, \lambda) d\vec{u} \\
&\quad = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\vec{u} + \pi_{\vec{t}}(y \circ \ell_t)) H_S(-\vec{u}, \lambda) d\vec{u} \\
&\quad = (2\pi)^{-n/2} (f * H_S(\cdot, \lambda))(\pi_{\vec{t}}(y \circ \ell)
\end{aligned}$$

Since $\vec{u} \mapsto H_S(\vec{u}, \lambda) \in L^{p'}(\mathbb{R}^n)$, and $f \in L^p(\mathbb{R}^n)$, then $f(\vec{u} + \pi_{\vec{t}}(y \circ \ell_t)) H_S(\vec{u}, \lambda) \in L^1(\mathbb{R}^n)$. Let (λ_j) be any sequence in $\mathbb{C}_+$ such that $\lambda_j \longrightarrow \lambda$, then there exists $j_0 \in \mathbb{N}$ such that $\mathcal{R}(\lambda_j) > \mathcal{R}(\lambda)/2$ for all $j \geq j_0$. Thus

$$|H_S(\vec{u}, \lambda_j)| \leq \left(\frac{2}{\sqrt{\pi}} \right)^n \prod_{k=1}^{n} \frac{t_k - t_{k-1}}{\mathcal{R}(\lambda)(u_k - u_{k-1})^2 + (t_k - t_{k-1})^2} \in L^{p'}(\mathbb{R}^n),$$

Hence, using the dominated convergence theorem, it follows that $\mathbb{E}_{\Omega_0}[F(\lambda^{-1/2} \cdot + y \circ \ell)]$ is continuous on $\mathbb{C}_+$. It is clear that $H_S(\vec{u}, \lambda)$ is analytic in λ on $\mathbb{C}_+$, then by the Fubini theorem, the Cauchy theorem, and the Morera theorem we obtain as in Lemma 2.3 that $\lambda \mapsto \mathbb{E}_{\Omega_0}[F(\lambda^{-1/2} \cdot + y \circ \ell)]$ admits an analytic extension on $\mathbb{C}_+$. This completes the proof. □

Theorem 4.4. *Let $q \in \mathbb{R} - \{0\}$ and F a cylinder functionals of the form* (1.10)*, then the L_p-AFFTSB of F exists and has the form*

$$T_qF(y\circ\ell) = (f * H_S(\cdot, -iq))(\pi_{\vec{t}}(y\circ\ell)) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(\vec{u}+\pi_{\vec{t}}(y\circ\ell))H_S(\vec{u}, -iq)d\vec{u}, \tag{4.7}$$

for SI-a.e. $y\circ\ell \in \Omega_0$.

Proof. Lemma 4.3 shows that $\mathbb{E}_{\Omega_0}[F(\lambda^{-1/2}\cdot +y\circ\ell)]$ admits an analytic extension for SI-e.a. $y\circ\ell \in \Omega_0$ given by (4.7), hence to obtain (4.7) it remains to prove that for all $\varrho > 0$

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \mathbb{E}_{\Omega_0}\left[|T_\lambda F(\varrho y\circ\ell) - T_{-iq}F(\varrho y\circ\ell)|^{p'}\right] = 0.$$

Let $\varrho > 0$ and using the Hölder inequality we obtain that

$$\begin{aligned}
&\mathbb{E}_{\Omega_0}\left[|T_\lambda F(\varrho y\circ\ell) - T_{-iq}F(\varrho y\circ\ell)|^{p'}\right]\\
&\quad= \int\int_{\mathbb{S}\times\mathcal{C}_0} |T_\lambda F(\varrho y\circ\ell) - T_{-iq}F(\varrho y\circ\ell)|^{p'}\, W(dy)\mu(d\ell)\\
&\quad= \int\int_{\mathbb{S}\times\mathcal{C}_0} \left|(2\pi)^{-n/2}\int_{\mathbb{R}^n} f(\vec{u}+\varrho\pi_{\vec{t}}(y\circ\ell_t)[H_S(\vec{u},\lambda) - H_S(\vec{u},-iq)]d\vec{u}\right|^{p'} W(dy)\mu(d\ell)\\
&\quad\le \int\int_{\mathbb{S}\times\mathcal{C}_0} (2\pi)^{-p'n/2}\|f(\varrho\pi_{\vec{t}}(y\circ\ell)+\vec{\cdot})\|^{p'}_{L^p(\mathbb{R}^n)} W(dy)\mu(d\ell)\\
&\qquad\qquad\cdot \|H_S(\vec{\cdot},\lambda) - H_S(\vec{\cdot},-iq)\|^{p'}_{L^{p'}(\mathbb{R}^n)}\\
&\quad= (2\pi)^{-p'n/2}\,\|f\|^{p'}_{L^p(\mathbb{R}^n)}\,\|H_S(\vec{\cdot},\lambda) - H_S(\vec{\cdot},-iq)\|^{p'}_{L^{p'}(\mathbb{R}^n)}.
\end{aligned}$$

Using Lemma 4.1 we finish the proof. □

5. Operator Gradient and Convolution Product applied to Functional $F \in \mathcal{A}(n,2)$

In this section we will concentrate on the the fucntions of

Proposition 5.1. *Let f and g be measurable functions of $L^2(\mathbb{R}^n)$, and let $f\otimes g$ be defined by*

$$(f\otimes g)(\vec{u},\lambda) = \int_{\mathbb{R}^n} f(\vec{u}+\vec{v})g(\vec{u}-\vec{v})H_S(\vec{v},\lambda)\,d\vec{v}, \quad \vec{u}\in\mathbb{R}^n,\ \lambda\in\mathbb{C}_+. \tag{5.1}$$

(if it exists). Then

(1) For all $\vec{u}\in\mathbb{R}^n$, $\lambda\in\tilde{\mathbb{C}}_+$ $(f\otimes g)(\vec{u},\lambda)$ exists, and for any $\vec{u}\in\mathbb{R}^n$, $\lambda\mapsto (f\otimes g)(\vec{u},\lambda)$ is continuous on $\tilde{\mathbb{C}}_+$ and analytic on $\mathbb{C}_+$.

(2) For all $\lambda \in \tilde{\mathbb{C}}_+$, $(f \otimes g)(\cdot,\lambda) \in L^2(\mathbb{R}^n)$ *and satisfy*

$$|(f \otimes g)(\vec{u},\lambda)| \leq \gamma \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}, \tag{5.2}$$

$$\|(f \otimes g)(\cdot,\lambda)\|_{L^2(\mathbb{R}^n)} \leq \|H_S(\cdot,\lambda)\|_{L^2(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} \tag{5.3}$$

Proof. Since $\lambda \mapsto H_S(\vec{u},\lambda)$ is analytic on $\tilde{\mathbb{C}}_+$, bounded by inequality (4.5), and $f,\ g \in L^2(\mathbb{R}^n)$ then $(f \otimes g)(\vec{u},\lambda)$ exists for all $\vec{u} \in \mathbb{R}^n,\ \lambda \in \tilde{\mathbb{C}}_+$ and continuous on $\tilde{\mathbb{C}}_+$. Using the Morera theorem, the Cauchy theorem, and the Fubini theorem we get that $\lambda \mapsto (f \otimes g)(\vec{u},\lambda)$ is analytic on $\mathbb{C}_+$ and continuous on $\tilde{\mathbb{C}}_+$ for all $\vec{u} \in \mathbb{R}^n$.

By the Cauchy-Schwartz inequality and taking account of (4.5) we get (5.2).

Since $f, g \in L^2(\mathbb{R}^n)$ and H_S is bounded, then by the Cauchy-Schwartz inequality and the Fubini theorem we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} |(f \otimes g)(\vec{u},\lambda)|^2\, d\vec{u} &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\vec{u}+\vec{v})^2\, d\vec{v} \int_{\mathbb{R}^n} g(\vec{u}-\vec{v})^2 H_S(\vec{v},\lambda)^2\, d\vec{v}\, d\vec{u} \\ &= \|f\|^2_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(\vec{u}-\vec{v})^2 d\vec{u}\ H_S(\vec{v},\lambda)^2\, d\vec{v} \\ &= \|f\|^2_{L^2(\mathbb{R}^n)} \|g\|^2_{L^2(\mathbb{R}^n)} \|H_S(\vec{\cdot},\lambda)\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

This completes the proof. □

Theorem 5.2. *Let* $q \in \mathbb{R} - \{0\}$, *let* F *and* G *be functionals from* $\mathcal{A}(n,2)$. *Then* $(F \tilde{*} G)_q$ *exists, belongs to* $L^2(\Omega_0)$, *and has the form*

$$(F \tilde{*} G)_q(y \circ \ell) = (f \otimes g)(\pi_{\vec{t}}(y \circ \ell), -iq), \tag{5.4}$$

for all SI-a.e. $y \circ \ell \in \Omega_0$, *where* Θ *is given by* (5.1). *Moreover we have*

$$\|(F \tilde{*} G)_q\|_{L^2(\Omega_0)} \leq \gamma \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} \tag{5.5}$$

Proof. Let $\lambda > 0$ then

$$\begin{aligned}&\mathbb{E}_{\Omega_0}\left[F\left(\frac{y \circ \ell + \lambda^{-1/2}\cdot}{\sqrt{2}}\right) G\left(\frac{y \circ \ell - \lambda^{-1/2}\cdot}{\sqrt{2}}\right)\right] \\ &\quad = \iint_{\mathbb{S} \times \mathcal{C}_0} F\left(\frac{y \circ \ell + \lambda^{-1/2} x \circ \sigma}{\sqrt{2}}\right) G\left(\frac{y \circ \ell - \lambda^{-1/2} x \circ \sigma}{\sqrt{2}}\right) W^{\mu}(dx \circ \sigma)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{S}}\int_{C_0} f\left(\frac{y\circ\ell+\lambda^{-1/2}\pi_{\vec{t}}(x\circ\sigma)}{\sqrt{2}}\right) g\left(\frac{y\circ\ell-\lambda^{-1/2}\pi_{\vec{t}}(x\circ\sigma)}{\sqrt{2}}\right) W(dx)\mu(d\sigma)\\
&= \int_{\mathbb{S}}\int_{\mathbb{R}^n} f\left(\frac{y\circ\ell+\vec{u}}{\sqrt{2}}\right) g\left(\frac{y\circ\ell-\vec{u}}{\sqrt{2}}\right) \prod_{k=1}^{n}\left[2\pi(\sigma_{t_k}-\sigma_{t_{k-1}})\right]^{-1/2}\\
&\qquad\qquad \cdot \exp\left(-\frac{\lambda}{2}\sum_{k=1}^{n}\frac{(u_k-u_{k-1})^2}{\sigma_{t_k}-\sigma_{t_{k-1}}}\right) d\vec{u}\,\mu(d\sigma)\\
&= \int_{\mathbb{R}^n} f\left(\frac{y\circ\ell+\vec{u}}{\sqrt{2}}\right) g\left(\frac{y\circ\ell-\vec{u}}{\sqrt{2}}\right) \int_{\mathbb{S}} \prod_{k=1}^{n}\left[2\pi(\sigma_{t_k}-\sigma_{t_{k-1}})\right]^{-1/2}\\
&\qquad\qquad \cdot \exp\left(-\frac{\lambda}{2}\sum_{k=1}^{n}\frac{(u_k-u_{k-1})^2}{\sigma_{t_k}-\sigma_{t_{k-1}}}\right) \mu(d\sigma)d\vec{u}\\
&= \int_{\mathbb{R}^n} f\left(\frac{y\circ\ell+\vec{u}}{\sqrt{2}}\right) g\left(\frac{y\circ\ell-\vec{u}}{\sqrt{2}}\right) H_S(\vec{u},\lambda)d\vec{u}\\
&= (f\otimes g)(\pi_{\vec{t}}(y\circ\ell),\lambda).
\end{aligned}
$$

By Lemma 5.1, we obtain the existence of $(F\tilde{*}G)_\lambda(y\circ\ell)$, and

$$
\begin{aligned}
(F\tilde{*}G)_q(y\circ\ell) &= \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \mathbb{E}^{anw_\lambda}_{\Omega_0}\left[F\left(\frac{y\circ\ell+\cdot}{\sqrt{2}}\right)G\left(\frac{y\circ\ell-\cdot}{\sqrt{2}}\right)\right]\\
&= \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} (f\otimes g)(\pi_{\vec{t}}(y\circ\ell),\lambda)\\
&= (f\otimes g)(\pi_{\vec{t}}(y\circ\ell),-iq).
\end{aligned}
$$

Using the inequality (5.2) we have

$$
\begin{aligned}
\mathbb{E}_{\Omega_0}\left[(F\tilde{*}G)_q^2\right] &= \int\!\!\int_{\mathbb{S}\times C_0} \left|(f\otimes g)(\pi_{\vec{t}}(y\circ\ell),-iq)\right|^2 W^\mu(dy\circ\ell)\\
&\le \gamma^2 \int\!\!\int_{\mathbb{S}\times C_0} \|f\|^2_{L^2(\mathbb{R}^n)}\|g\|^2_{L^2(\mathbb{R}^n)} W^\mu(dy\circ\ell)\\
&= \gamma^2\|f\|^2_{L^2(\mathbb{R}^n)}\|g\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

thus we get (5.5) and the theorem follows. □

Theorem 5.3. *Suppose that $q\in\mathbb{R}-\{0\}$. Let F and G be measurable functionals of $\mathcal{A}(n,2)$, and $h\in\mathbb{H}$. Then the L_p-AFFTSB of $(F\tilde{*}G)_q$ exists and is given by*

$$T_q\left[\left(F\tilde{*}G\right)_q\right](y\circ\ell)=((f\otimes g)(\cdot,-iq)*H_S(\cdot,-iq))(\pi_{\vec{t}}(y\circ\ell),-iq). \tag{5.6}$$

for SI-a.e. $y\circ\ell\in\Omega_0$.

Proof. By Theorem 5.2, $\left(F\tilde{*}G\right)_q$ exists for SI-a.e. $y\circ\ell\in\Omega_0$ and is equal to $(f\otimes g)(\cdot,-iq)$, but since $(f\otimes g)(\cdot,-iq)\in L^2(\mathbb{R}^n)$, then by Theorem 4.4, L_p-AFFTSB of $\left(F\tilde{*}G\right)_q$ exists for SI-a.e. $y\circ\ell\in\Omega_0$ and is given by (5.6). □

Theorem 5.4. *Suppose that* $q\in\mathbb{R}-\{0\}$. *Let* F *and* G *be measurable functionals of* $\mathcal{A}(n,2)$. *Then the convolution product of* T_qF *and* T_qG *exists and is given by*

$$\left(T_qF\tilde{*}T_qG\right)_q(y\circ\ell)=((f*H_S(\cdot,-iq))\otimes(g*H_S(\cdot,-iq)))\left(\pi_{\vec{t}}(y\circ\ell),-iq\right), \tag{5.7}$$

for SI-a.e. $y\circ\ell\in\Omega_0$.

Proof. According to Theorem 4.4, T_qF and T_qG exist and are given by $f*H_S(\cdot,-iq)$ and $g*H_S(\cdot,-iq)$ respectively. Since $f,g\in L^2(\mathbb{R}^n)$ and $H_S\in L^1(\mathbb{R}^n)$, then $f*H_S(\cdot,-iq),g*H_S(\cdot,-iq)\in L^2(\mathbb{R}^n)$. Thus by Theorem 5.2 the convolution product of T_qF and T_qG exists and is given by (5.7). □

Theorem 5.5. *Let* F *and* G *be measurable functionals of* $\mathcal{A}(n,2)\cap\mathfrak{F}_b^{\infty}$, *and* $h\in\mathbb{H}$. *Then* $D_h(F\tilde{*}G)_q$ *exists and is given by*

$$D_h\left[(F\tilde{*}G)_q\right](y\circ\ell)=\sum_{k=1}^{n}y(\ell_{t_k})\left(f_k\otimes g+f\otimes g_k\right)\left(\pi_{\vec{t}}(y\circ\ell,-iq\right) \tag{5.8}$$

for SI-a.e. $y\circ\ell\in\Omega_0$.

Proof. By Theorem 5.2, $(F\tilde{*}G)_q$ exists as an element of $L^2(\Omega_0)$ and is given by (5.4). Hence by (1.7), $D_h\left[(F\tilde{*}G)_q\right](y\circ\ell)$ exists and is given by

$$\begin{aligned}D_h\left[(F\tilde{*}G)_q\right](y\circ\ell)&=D_h\left[(f\otimes g)(\pi_{\vec{t}}(y\circ\ell),-iq)\right]\\&=\sum_{k=1}^{n}y(\ell_{t_k})(\partial_k(f\otimes g)(\cdot,-iq))(\pi_{\vec{t}}(y\circ\ell),-iq).\end{aligned}$$

Since f_k,g_k,f,g are bounded and belong to $L^2(\mathbb{R}^n)$, and H_S is bounded, then by Leibniz's rule for differentiation under the integral sign we obtain (5.8), which completes the proof. □

Theorem 5.6. *Let* F *be measurable functional of* $\mathcal{A}(n,2)\cap\mathfrak{F}_b^{\infty}$, *and* $h\in\mathbb{H}$. *Then* $D_h\left[T_qF\right]$ *exists and is given by*

$$D_h\left[T_qF\right](y\circ\ell)=\sum_{k=1}^{n}(f_k*H_S(\cdot,-iq))(\pi_{\vec{t}}(y\circ\ell)) \tag{5.9}$$

Proof. By Theorem 4.4, the L_2-AFFTSB of F exists for SI-a.e. $y\circ\ell\in\Omega_0$ and is given by (4.7). Furthermore we have f_k are bounded and belongs to $L^2(\mathbb{R}^n)$ and $H_S(\cdot,-iq)\in L^1(\mathbb{R}^n)$. Then by (1.7) we get

$$\begin{aligned}D_h\left[T_qF\right](y\circ\ell)&=D_h\left[f*H_S(\cdot,-iq)((\pi_{\vec{t}}(y\circ\ell)))\right]\\&=\sum_{k=1}^{n}(f_k*H_S(\cdot,-iq))(\pi_{\vec{t}}(y\circ\ell)).\end{aligned}$$

This completes the proof. □

Theorem 5.7. *Let F and G be measurable functionals of $\mathcal{A}(n,2)\cap\mathfrak{F}_b^{\infty}$. Then*

$$D_h\left(T_q\left[(F\tilde{*}G)_q\right]\right)(y\circ\ell)=\sum_{k=1}^{n}y(\ell_{t_k})((f_k\otimes g+f\otimes g_k)(\cdot,-iq)*H_S(\cdot,-iq))(\pi_{\vec{t}}(y\circ\ell),-iq) \tag{5.10}$$

Proof. Since the proof here is basically the same as Theorem 5.5 we omit it. □

References

1. D. Applebaum: Lévy processes and stochastic calculus. Cambridge Studies in Advanced Mathematics, 93, Cambridge University Press, Cambridge, 2004.
2. J. Bertoin: Subordinators: examples and applications. Lectures on probability theory and statistics (Saint-Flour, 1997), 1-91, Lecture Notes in Math., 1717, Springer, Berlin.
3. M. D. Brue: A Functional Transform for Feynman Integrals Similar to the Fourier Transform. Ph.D. Thesis, University of Minnesota, Minneapolis, 1972.
4. R.H. Cameron & D.A. Storvick: An L_2 analytic Fourier-Feynman transform. Michigan Math. J. **23** (1976), 1-30.
5. S.J. Chang, J.G. Choi & D. Skoug: Generalized Fourier-Feynman transforms, convolution products, and first variations on function space. Rocky Mountain J. Math. **40** (2010), 761-788.
6. S.J. Chang, J.G. Choi & D. Skoug: Integration by parts formulas involving generalized Fourier-Feynman transforms on function space. Trans. Amer. Math. Soc. **355** (2003), 2925-2948.
7. S.J. Chang & J.G. Choi: Rotation of Gaussian paths on Wiener space with applications. Banach J. Math. Anal. **12** (2018), 651-672.

8. S.J. Chang, H.S. Chung & I.Y. Lee: A new approach method to obtain the L_1 generalized analytic Fourier-Feynman transform. Integral Transforms Spec. Funct. **29** (2018), 745-760.
9. C.S. Deng & R.L. Schilling: On a Cameron-Martin type quasi-invariance theorem and applications to subordinate Brownian motion. Stoch. Anal. Appl. **33** (2015), 975-993.
10. G.W. Johnson & D.L. Skoug: An L_p analytic Fourier-Feynman transform. Michigan Math. J. **26** (1979), 103-127.
11. J.G. Kim & et al. :Relationships among transforms, convolutions, and first variations. Int. J. Math. Math. Sci. **22** (1999), 191-204.
12. D. Skoug & D. Storvick: A survey of results involving transforms and convolutions in function space. Rocky Mountain J. Math. **34** (2004), 1147-1175.
13. R.L. Schilling, R. Song & Z. Vondraček: Bernstein functions. second edition, De Gruyter Studies in Mathematics, 37, Walter de Gruyter & Co., Berlin, 2012.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE SEMLALIA, CADI AYYAD UNIVERSITY, MARRAKECH 40000, MOROCCO.
Email address: `medelkoufi@gmail.com`