J. Appl. Math. & Informatics Vol. **39***(2021), No. 3 - 4, pp. 553 - 567*
https://doi.org/10.14317/jami.2021.553

MS-FUZZY IDEALS OF MS-ALGEBRAS

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Abstract. In this paper, we introduce concepts of MS-fuzzy ideals of MS-algebras. We reveal the connections between MS-fuzzy ideals and several kinds of fuzzy ideals as fuzzy prime ideals, kernel fuzzy ideals, e-fuzzy ideals and closure fuzzy ideals. We show that many of these classes are proper subclasses of the class of MS-fuzzy ideals. Finally some properties of the homomorphic images, inverse homomorphic images of MS-fuzzy ideals are studied.

AMS Mathematics Subject Classification : 06D99, 06D05, 06D30.
Key words and phrases : MS-algebra, Kernel fuzzy ideals, e-fuzzy ideals, MS-fuzzy ideals, Closure fuzzy ideals, Homomorphisms.

1. Introduction

Zadeh[20] introduced the concepts of a fuzzy set. Fuzzy sets have been found to be very useful in diversely applied areas of science and technology. Extensive applications of the fuzzy set theory have been found in various fields such as logic programming, medical diagnosis, decision making problems and microelectronic fault analysis. Also fuzzy set theory is conveniently and successfully applied in abstract Algebra. Rosenfeld [16] defined the notion of a fuzzy subgroup of a group. Then many algebraists took interest to introduce fuzzy theory in various algabraic structures by fuzzyfying the formal theory. [4, 5, 17, 18, 19] introduced fuzzy ideals in a distributive lattice. Recently, Alaba and Alemayehu [1] introduced the notion of clouser fuzzy ideals of MS-algebras. Alaba, Taye and Alemayehu [2, 3] introduced the concept of δ-fuzzy ideals in MS-algebras and fuzzy congruences on MS-algebras.

On the other hand, Blyth and Varlet [10, 11] introduced the notion of MS-algebras as a common abstraction of de Morgan algebras and Stone algebras. Badawy and Rao [6] introduced the notions of dominator ideals and closure ideals. They characterized closure ideals in terms of principal dominator ideals

Received September 9, 2020. Revised February 18, 2021. Accepted February 22, 2021.
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and studied some properties of closure ideals with respect to homomorphisms. Luo and Zheng [15] defined e-ideals, tail ideals of MS-algebras and characterized those MS-algebras whose e-ideals are kernel ideals. Badawy [8] introduced δ-ideals in MS-algebras.

More recently, Badawy, Seidy and Gaber [7] studied on MS-ideals of MS-algebras. These studies motivated us to study MS-fuzzy ideals of MS-algebras. In particular, we reveal the connections between MS-fuzzy ideals and several kinds of fuzzy ideals as fuzzy prime ideals, kernel fuzzy ideals, e-fuzzy ideals and closure fuzzy ideals. We show that many of these classes are proper sub-classes of the class of MS-fuzzy ideals. Finally, we studied some properties of the homomorphic images, inverse homomorphic images of MS-fuzzy ideals.

2. Preliminaries

In this section, we recall some definitions and results which will be used in this paper. For the fundamental crisp concepts of MS-ideals of MS-algebras we refer [7].

Definition 2.1 ([10, 11])**.** An MS-algebra is an algebra $(L, \vee, \wedge, ^\circ, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a \to a^\circ$ is a unary operation satisfies: $a \leq a^{\circ\circ}$, $(a \wedge b)^\circ = a^\circ \vee b^\circ$ and $1^\circ = 0$.

A Stone algebra $S = (S, \vee, \wedge, ^*, 0, 1)$ is also a bounded distributive lattice endowed with a unary operation $x \to x^*$ satisfying $(x \wedge y)^* = x^* \vee y^*,\ x \wedge x^* = 0$ and $0^* = 1$.

A de Morgan algebra is an algebra $(L, \vee, \wedge, ^-, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a \to \overline{a}$ is a unary operation satisfies: $\overline{\overline{a}} = a$, $\overline{(a \wedge b)} = \overline{a} \vee \overline{b}$ and $\overline{1} = 0$.

Lemma 2.2 ([11])**.** *For any two elements a, b of an MS-algebra L, we have the following:*

(1) $0^\circ = 1$,
(2) $a \leq b \Rightarrow b^\circ \leq a^\circ$,
(3) $a^{\circ\circ\circ} = a^\circ$,
(4) $(a \vee b)^\circ = a^\circ \wedge b^\circ$,
(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,
(6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

Definition 2.3 ([7])**.** An ideal I of an MS-algebra L is called an MS-ideal if $x^{\circ\circ} \in I$ for every $x \in I$.

In [7], For any ideal of an MS-algebra, define $I_{\circ\circ} = \{x \in L : x \leq a^{\circ\circ}$ for some $a \in I\}$ and $I^\circ = \{x \in L : i^\circ \leq x$ for some $i \in I\}$. It is known that $I_{\circ\circ}$ is an ideal of L and I° is a filter of L.

Definition 2.4 ([15])**.** Let I be an ideal of an MS-algebra L. Then I is called an e-ideal of L if $I = I_{\circ\circ}$ and $I \cap I^\circ = \emptyset$.

We recall that for any nonempty set L, the characteristic function of L,

$$\chi_L(x) = \begin{cases} 1 & if\ x \in L, \\ 0 & if\ \ x \notin L. \end{cases}$$

Definition 2.5 ([6]). An ideal I of an MS-algebra L is a closure ideal, if

$$\overleftarrow{\sigma}\sigma(I) = I$$

.

Definition 2.6 ([5]). Let μ be a fuzzy subset of $(L, \wedge, \vee, 0, 1)$. For any $\alpha \in [0, 1]$, we shall denote the level subset $\mu^{-1}([\alpha, 1])$ by simply μ_α, i.e.,

$$\mu_\alpha = \{x \in L : \alpha \leq \mu(x)\}.$$

A fuzzy subset μ of L is proper, if it is a non constant function. A fuzzy subset μ is improper if it is constant function.

Definition 2.7 ([16]). Let μ and θ be fuzzy subsets of a set L. Define fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of L as follows: for each $x \in L$,

$$(\mu \cup \theta)(x) = \mu(x) \vee \theta(x) \text{ and } (\mu \cap \theta)(x) = \mu(x) \wedge \theta(x).$$

Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ respectively.

We define the binary operations " $\vee$ " and " $\wedge$ " on the set of all fuzzy subsets of L as:

$$(\mu \vee \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \vee b = x\}$$

and

$$(\mu \wedge \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \wedge b = x\}.$$

If μ and θ are fuzzy ideals of L, then $\mu \wedge \theta = \mu \cap \theta$ and $\mu \vee \theta$ is a fuzzy ideal generated by $\mu \cup \theta$.

Definition 2.8 ([17]). A fuzzy subset μ of a lattice L is called a fuzzy ideal of L if, for all $x, y \in L$ the following condition satisfies:

(1) $\mu(0) = 1$,
(2) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,
(3) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

Definition 2.9 ([17]). A fuzzy subset μ of a lattice L is called a fuzzy filter of L if, for all $x, y \in L$ the following condition satisfies:

(1) $\mu(1) = 1$,
(2) $\mu(x \vee y) \geq \mu(x) \vee \mu(y)$,
(3) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.

In [17], Swamy and Raju observed that,

(1) A fuzzy subset μ of a lattice L is a fuzzy ideal of L if and only if

$$\mu(0) = 1 \text{ and } \mu(x \vee y) = \mu(x) \wedge \mu(y) \text{ for all } x, y \in L.$$

(2) A fuzzy subset μ of a lattice L is a fuzzy filter of L if and only if

$$\mu(1) = 1 \text{ and } \mu(x \wedge y) = \mu(x) \wedge \mu(y) \text{ for all } x, y \in L.$$

Lemma 2.10 ([5])**.** *Let μ be a fuzzy ideal of L. If $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y \in L$.*

Definition 2.11.
(1) A proper fuzzy ideal μ of L is called a prime fuzzy ideal if for any two fuzzy ideals η, ν of L, $\eta \cap \nu \subset \mu$ implies $\eta \subset \mu$ or $\nu \subset \mu$.
(2) A proper fuzzy filter μ of L is called a prime fuzzy filter if for any two fuzzy filters η, ν of L, $\eta \cap \nu \subset \mu$ implies $\eta \subset \mu$ or $\nu \subset \mu$.

Theorem 2.12 ([20])**.** *Let μ be a fuzzy subset of L. Then μ is a fuzzy ideal of L if and only if for any $\alpha \in [0, 1]$, μ_α is an ideal of L.*

Theorem 2.13 ([18])**.** *For any $\alpha \in [0, 1)$, the fuzzy subset P^1_α of L given by*

$$P^1_\alpha(x) = \begin{cases} 1 & if\ x \in P, \\ \alpha & if\ \ x \notin P \end{cases}$$

for each $x \in L$ is prime fuzzy ideal of L if and only if P is prime ideal of L.

Theorem 2.14 ([19])**.** *Let $f : L \to L^{'}$ be an onto homomorphism. Then $f(\mu)$ is a fuzzy ideal of $L^{'}$ if μ is a fuzzy ideal of L.*

Theorem 2.15 ([4])**.** *Let $f : L \to L^{'}$ be homomorphism. Then $f^{-1}(\mu)$ is a fuzzy ideal of L if μ is a fuzzy ideal of $L^{'}$.*

Definition 2.16 ([1])**.** A fuzzy relation ϕ on an MS-algebra L is called fuzzy congruence relation on L if the following are satisfied:

(1) $\phi(x \wedge z, y \wedge w) \wedge \phi(x \vee z, y \vee w) \geq \phi(x, y) \wedge \phi(z, w)$ for all $x, y, z, w \in L$,
(2) $\phi(x^\circ, y^\circ) \geq \phi(x, y)$ for all $x, y \in L$.

Definition 2.17 ([1])**.** A kernel fuzzy ideal μ of an MS-algebra L is a fuzzy ideal μ of L for which there exists a fuzzy congruence ϕ of L such that $\mu = ker\phi$.

Definition 2.18 ([2])**.** For any non-empty fuzzy subset μ of an MS-algebra L, define the dominator of μ as follows: $\mu_{\circ\circ}(x) = \sup\{\mu(a) : x \leq a^{\circ\circ}, a \in L\}$, for all $x \in L$.

Definition 2.19 ([2])**.** A fuzzy ideal μ of an MS-algebra L is called a closure fuzzy ideal, if $\overleftarrow{\sigma}\sigma(\mu) = \mu$, where for any fuzzy ideal μ of L,

$$\sigma(\mu)((a]_{\circ\circ}) = \sup\{\mu(b) : (a]_{\circ\circ} = (b]_{\circ\circ},\ b \in L\}.$$

and for any fuzzy ideal θ of the set of principal dominator ideal of L,

$$\overleftarrow{\sigma}(\theta)(a) = \theta((a]_{\circ\circ}),$$

for all $a \in L$.

Definition 2.20 ([3])**.** Let L be MS-algebra. Then for any fuzzy filter μ of L, denote the fuzzy subset $\sigma(\mu)$ as follows:

$$\sigma(\mu)(x) = \mu(x^\circ)$$

for all $x \in L$.

Definition 2.21 ([3])**.** Let L be an MS-algebra. A fuzzy ideal μ of L is called δ-fuzzy ideal if $\mu = \delta(\eta)$ for some fuzzy filter η of L.

3. MS-fuzzy ideals of MS-algebras

In this section, we introduced the concepts of MS-fuzzy ideals of MS-algebras. We study relation between MS-fuzzy ideals and several kinds of fuzzy ideals as fuzzy prime ideals, kernel fuzzy ideals, e-fuzzy ideals, closure fuzzy ideals ande δ-fuzzy ideeals. Also, we show that many of these classes are proper subclasses of the class of MS-fuzzy ideals.

Definition 3.1. A fuzzy ideal μ of an MS-algebra L is called an MS- fuzzy ideal if $\mu(x^{\circ\circ}) \geq \mu(x)$.

Theorem 3.2. *μ is an MS-fuzzy ideal of an MS-algebra L if and only if, $\forall\, \alpha \in [0,1]$, μ_α is an MS-ideal of L.*

Proof. Suppose that μ is an MS-fuzzy ideal of an MS-algebra L. Let $x \in \mu_\alpha$, $\alpha \in [0,1]$, then $\mu(x^{\circ\circ}) \geq \mu(x) \geq \alpha$. This implies $x^{\circ\circ} \in \mu_\alpha$. Hence μ_α is an MS-ideal of L.
Conversely, let $\mu(x) = \alpha$, then $x \in \mu_\alpha$. Since μ_α is an MS-ideal of L, $x^{\circ\circ} \in \mu_\alpha$. This implies $\mu(x^{\circ\circ}) \geq \alpha = \mu(x)$. Hence μ is an MS-fuzzy ideal of L. □

Corollary 3.3. *I is an MS-ideal of L if and only if χ_I is an MS-fuzzy ideal of L.*

Lemma 3.4. *Let μ be a prime fuzzy filter of an MS-algebra L, and $\mu(1) = 0$. Then a fuzzy subset $\ell(\mu)$ of L defined as $\ell(\mu)(x) = \mu^c(x^\circ)$ for all $x \in L$ is an MS-fuzzy ideal of L.*

Proof. First we show that $\ell(\mu)$ is a fuzzy ideal of L.
$\ell(\mu)(0) = \mu^c(0^\circ) = 1 - \mu(1) = 1$
For any $x, y \in L$,

$$\begin{aligned}
\ell(\mu)(x \wedge y) &= \mu^c((x \wedge y)^\circ) \\
&= 1 - \mu((x \wedge y)^\circ) \\
&= 1 - \mu(x^\circ \vee y^\circ) \\
&\geq 1 - (\mu(x^\circ) \wedge \mu(y^\circ)) \\
&= (1 - \mu(x^\circ)) \vee (1 - \mu(y^\circ)) \\
&= \ell(\mu)(x) \vee \ell(\mu)(y)
\end{aligned}$$

And also

$$\begin{aligned}\ell(\mu)(x\vee y) &= \mu^c((x\vee y)^\circ)\\ &= 1-\mu((x\vee y)^\circ)\\ &= 1-\mu(x^\circ\wedge y^\circ)\\ &\geq 1-(\mu(x^\circ)\vee\mu(y^\circ))\\ &= (1-\mu(x^\circ))\wedge(1-\mu(y^\circ))\\ &= \ell(\mu)(x)\vee\ell(\mu)(y)\end{aligned}$$

Hence $\ell(\mu)$ is fuzzy ideal of L.
Also $\ell(\mu)(x)=\mu^c(x^\circ)=\mu^c(x^{\circ\circ\circ})=\ell(\mu)(x^{\circ\circ})$. Thus $\ell(\mu)$ is an MS-fuzzy ideal of L. □

Definition 3.5. For any fuzzy ideal μ of an MS-algebra L. The perpendicular of μ is defined to be the fuzzy set $\mu^\perp(x)=\sup\{\mu(a): x^{\circ\circ}\wedge a=0,\ a\in L\}$ for all $x\in L$.

Lemma 3.6. *Let L be an MS-algebra. Then $\mu^\perp$ is an MS-fuzzy ideal for any fuzzy ideal μ of L.*

Proof. First prove that μ is a fuzzy ideal of an MS-algebra. $\mu^\perp(0)=\sup\{\mu(a): 0^{\circ\circ}\wedge a=0,\ a\in L\}\geq\mu(0)=1$. Hence $\mu^\perp(0)=1$. For any $x,y\in L$,

$$\begin{aligned}\mu^\perp(x)\wedge\mu^\perp(y) &= \sup\{\mu(a): x^{\circ\circ}\wedge a=0,\ a\in L\}\wedge\sup\{\mu(b): y^{\circ\circ}\wedge b=0,\ b\in L\}\\ &= \sup\{\mu(a)\wedge\mu(b): x^{\circ\circ}\wedge a=0,\ y^{\circ\circ}\wedge b=0\ a,\ b\in L\}\\ &\leq \sup\{\mu(a\vee b): (x\vee y)^{\circ\circ}\wedge(a\vee b)=0,\ a,\ b\in L\}\\ &= \mu^\perp(x\vee y).\end{aligned}$$

Finally,

$$\begin{aligned}\mu^\perp(x)\vee\mu^\perp(y) &= \sup\{\mu(a): x^{\circ\circ}\wedge a=0,\ a\in L\}\vee\sup\{\mu(b): y^{\circ\circ}\wedge b=0,\ b\in L\}\\ &= \sup\{\mu(a)\vee\mu(b): x^{\circ\circ}\wedge a=0,\ y^{\circ\circ}\wedge b=0\ a,\ b\in L\}\\ &\leq \sup\{\mu(a\wedge b): (x\wedge y)^{\circ\circ}\wedge(a\wedge b)=0,\ a,\ b\in L\}\\ &= \mu^\perp(x\wedge y).\end{aligned}$$

Thus $\mu^\perp$ is a fuzzy ideal of L. Now,

$$\begin{aligned}\mu^\perp(x) &= \sup\{\mu(a): x^{\circ\circ}\wedge a=0,\ a\in L\}\\ &= \sup\{\mu(a): x(^{\circ\circ})^{\circ\circ}\wedge a=0,\ a\in L\}\\ &= \mu^\perp(x^{\circ\circ}).\end{aligned}$$

Hence $\mu^\perp$ is an MS-fuzzy ideal for any fuzzy ideal μ of L. □

Lemma 3.7. *For any fuzzy ideal μ of an MS-algebra L, $\mu_{\circ\circ}$ is an MS-fuzzy ideal of L.*

Proof.

$$\begin{aligned}\mu_{\circ\circ}(x) &= \sup\{\mu(a) : x \le a^{\circ\circ}\}\\ &\le \sup\{\mu(a) : x^{\circ\circ} \le a^{\circ\circ}\}\\ &= \mu_{\circ\circ}(x^{\circ\circ})\end{aligned}$$

Hence $\mu_{\circ\circ}$ is an MS-fuzzy ideal of L. □

Lemma 3.8. *If μ is a kernel fuzzy ideal of an MS-algebra L, then $\mu = \mu_{\circ\circ}$ and $(\mu \cap \mu^\circ)(x) \le \alpha$ for all $x \in L$ and for all $\alpha \in Im(\mu)$.*

Proof. Easily proved that $\mu = \mu_{\circ\circ}$

$$\begin{aligned}(\mu \cap \mu^\circ)(x) &= \mu(x) \wedge \mu^\circ(x)\\ &= \mu_{\circ\circ}(x) \wedge \mu^\circ(x)\\ &= \sup\{\mu(a) : x \le a^{\circ\circ}\} \wedge \sup\{\mu(b) : b^\circ \le x\}\\ &= \sup\{ker\phi(a) : x \le a^{\circ\circ}\} \wedge \sup\{ker\phi(b) : b^\circ \le x\}\\ &\quad \text{for some fuzzy congruence } \phi \text{ such that } ker\phi = \mu\\ &= \sup\{\phi(a,0) : x \le a^{\circ\circ}\} \wedge \sup\{\phi(b,0) : b^\circ \le x\}\\ &\le \sup\{\phi(a^{\circ\circ},0) : x \le a^{\circ\circ}\} \wedge \sup\{\phi(b^{\circ\circ},0) : x^\circ \le b^{\circ\circ}\}\\ &\le \mu(x) \wedge \mu(x^{\circ\circ})\\ &\le \mu(x \vee x^{\circ\circ})\\ &\le \mu(x),\ \text{put } \mu(x) = \alpha \in Im\mu\end{aligned}$$

Hence the result hold. □

Definition 3.9. Let μ be a fuzzy ideal of an MS-algebra L. Then, μ is called an e-fuzzy ideal of L if $\mu = \mu_{\circ\circ}$ and $(\mu \cap \mu^\circ)(x) \le \alpha$ for all $x \in L$ and for all $\alpha \in Im(\mu)$.

The following corollaries follow directly from the previous lemma and the definitions of kernel fuzzy ideal and e-fuzzy ideal of an MS-algebra.

Corollary 3.10. *Any kernel fuzzy ideal of an MS-algebra is an MS-fuzzy ideal.*

Corollary 3.11. *Any e-fuzzy ideal of an MS-algebra is an MS-fuzzy ideal.*

Lemma 3.12. *Any δ-fuzzy ideal μ of an MS-algebra L is an MS-fuzzy ideal.*

Proof. Suppose that μ is a δ-fuzzy ideal, there exists a fuzzy filter η of L such that $\mu = \delta(\eta)$. Now for any $a \in L$,

$$\mu(x) = \delta(\eta)(x) = \eta(x^\circ) \le \eta(x^{\circ\circ\circ}) = \delta(\eta)(x^{\circ\circ}) = \mu(x^{\circ\circ})$$

Hence any δ-fuzzy ideal of L is MS-fuzzy ideal. □

Lemma 3.13. *Let μ be a proper MS-fuzzy ideal of an MS-algebra L. Then $\mu(x) \le \alpha$ for all x in the set of dense element D and $\alpha \in Im(\mu)$.*

Proof. Suppose that μ is a proper MS-fuzzy ideal of an MS-algebra L and $x \in D$. Then $\mu(x) \leq \mu(x^{\circ\circ}) = \mu(1)$. Hence $\mu(x) \leq \alpha$ for all x in the set of dense element D and $\alpha \in Im(\mu)$, since $\mu(1) \leq \mu(y)$ for all $y \in L$. □

Let L be an MS-algebra. For any non empty fuzzy subset μ and any element a of L, define $\mu_a(x) = \mu(x \wedge a)$. It is easy to see that μ_a is not a fuzzy ideal in general but it is a fuzzy ideal if μ is a fuzzy ideal. In the following we give some properties of these fuzzy subsets.

Lemma 3.14. *Let μ be a fuzzy ideal of an MS-algebra L. Then*

(1) *for any element $a \in L$, $\mu \subseteq \mu_a$,*
(2) *$\mu_1 = \mu$,*
(3) *for any element $a \in L$, $\mu_a(x) = 1$ if and only if $\mu(a) = 1$,*
(4) *let $a, b \in L$. If $a \leq b$, then $\mu_b \subseteq \mu_a$,*
(5) *for any element $a \in L$, $\mu_{a^{\circ\circ}} \subseteq \mu_a$.*

Proof. (1) For any $x, y \in L$, $\mu_a(x) = \mu(x \wedge a) \geq \mu(x)$, Since μ is a fuzzy ideal. Hence $\mu \subseteq \mu_a$.

The proof of (2) is straightforward.

(3) Suppose that for any element $a \in L$, $\mu_a(x) = 1$. This implies $1 = \mu_a(x) = \mu(a \wedge x) = \mu(a)$, where taking $x = 1$.

Conversely $1 = \mu(a) \leq \mu(x \wedge a) = \mu_a(x)$. This implies $\mu_a(x) = 1$

(4) If $a \leq b$, then $x \wedge a \leq x \wedge b$ for any $x \in L$. This implies $\mu(x \wedge b) \leq \mu(x \wedge a)$. Thus $\mu_b(x) \leq \mu_a(x)$. Hence $\mu_b \subseteq \mu_a$.

(5) Since $a \leq a^{\circ\circ}$, by (4) $\mu_{a^{\circ\circ}} \subseteq \mu_a$. □

Lemma 3.15. *Let L be an MS-algebra. If μ is an MS-fuzzy ideal of L, then μ_a is an MS-fuzzy ideal of L for any $a \in L$.*

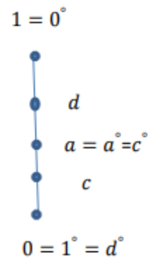
Proof. Let $x \in L$, $\mu_a(x) = \mu(a \wedge x) \leq \mu((a \wedge x)^{\circ\circ}) = \mu(a^{\circ\circ} \wedge x^{\circ\circ}) = \mu_{a^{\circ\circ}}(x^{\circ\circ}) \leq \mu_a(x^{\circ\circ})$. Hence μ_a is an MS-fuzzy ideal of L for any $a \in L$. □

Lemma 3.16. *Let I be an ideal of L. Then for any $a \in L$, $(\chi_I)_a = \chi_{I_a}$.*

Lemma 3.17. *Let μ be an MS-fuzzy ideal of an MS-algebra L. Then $\mu_a = \mu_{a^{\circ\circ}}$, for all $a \in L$.*

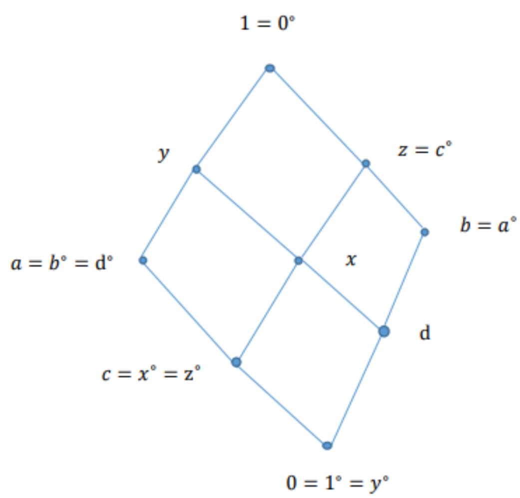
The following examples, show that the converses of Corollaries 3.10, 3.11, and Lemma 3.12 are not true.

Example 3.18. Consider the following MS-algebra:



Define the fuzzy subset μ on L as $\mu(0) = \mu(c) = \mu(a) = 1,\ \mu(d) = 0.7$ and $\mu(1) = 0.5$. Easily we can verify that μ is an MS-fuzzy ideal of L. We see that $(\mu \cap \mu^{\circ})(a) = 1$. It follows that μ is not an e-fuzzy ideal of L. Consequently, μ is not a kernel fuzzy ideal of L. This indicates the converse of Corollaries 3.10 and 3.11 are not true.

Example 3.19. Consider the following MS-algebra:



We have $I = \{0, b, c, d, x, z\}$ is an MS-ideal while it is not a δ-ideal of L. This implies χ_I is an MS-fuzzy ideal but not a δ-fuzzy ideal of L. This implies the converse of Lemma 3.12 is not true.

Lemma 3.20. *Let μ be an MS-fuzzy ideal of an MS-algebra L. Then, $\mu_a = \mu$ for any $a \in D$.*

Proof. For any $a \in D$ and $x \in L$, $\mu_a(x) = \mu(x \wedge a) \leq \mu((x \wedge a)^{\circ\circ}) = \mu(x^{\circ\circ} \wedge a^{\circ\circ}) = \mu(x^{\circ\circ} \wedge 1) = \mu_1(x^{\circ\circ}) = \mu(x^{\circ\circ}) \leq \mu(x)$. Hence $\mu_a = \mu$ for any $a \in D$. □

Lemma 3.21. *Let μ be a fuzzy ideal of an MS-fuzzy algebra L. If μ is fuzzy prime ideal then μ_a is fuzzy prime ideal.*

Lemma 3.22. *Let μ and λ be two fuzzy ideals of an MS-algebra L. Then, for every $a \in L$, we have*

(1) $(\mu \cap \lambda)_a = \mu_a \cap \lambda_a$,
(2) $\mu \subseteq \lambda$ *implies* $\mu_a \subseteq \lambda_a$.

Proof. For any $x \in L$,

(1) $(\mu \cap \lambda)_a(x) = (\mu \cap \lambda)(x \wedge a) = \mu(x \wedge a) \wedge \lambda(x \wedge a) = \mu_a(x) \wedge \lambda_a(x)$. Hence $(\mu \cap \lambda)_a = \mu_a \cap \lambda_a$.

(2) Suppose that $\mu \subseteq \lambda$. Then $\mu_a(x) = \mu(x \wedge a) \leq \lambda(x \wedge a) = \lambda_a(x)$. Hence $\mu_a \subseteq \lambda_a$. □

Theorem 3.23. *Let μ be a fuzzy ideal of an MS-algebra L. Let $\mu_L = \{\mu_a : a \in L\}$ ordered by set inclusion. Then*

(1) *μ_L is a bounded distributive lattice,*
(2) *If μ is an MS-fuzzy ideal of L, then μ_L is a de Morgan algebra with $\overline{\mu_a} = \mu_{a^\circ}$.*

Proof. The greatest and least elements of μ_L are μ_0 and μ_1, respectively. Next, we prove that $\mu_a \cap \mu_b = \mu_{a \vee b}$ and $\mu_a \vee \mu_b = \mu_{a \wedge b}$ for any $a, b \in L$. Clearly $\mu_{a \wedge b} \subseteq \mu_a$ and $\mu_{a \wedge b} \subseteq \mu_b$ and so $\mu_{a \wedge b} \subseteq \mu_a \cap \mu_b$.
Conversely,

$$\begin{aligned}
(\mu_a \cap \mu_b)(x) &= \mu_a(x) \wedge \mu_b(x) \\
&= \mu(x \wedge a) \wedge \mu(x \wedge b) \\
&\leq \mu((x \wedge a) \vee (x \wedge b)) \\
&= \mu((x \wedge (a \vee b)) \\
&= \mu_{a \vee b}(x).
\end{aligned}$$

Hence $\mu_{a \vee b} = \mu_a \cap \mu_b$. Also clearly $\mu_a, \mu_b \subseteq \mu_{a \wedge b}$. This implies $\mu_{a \wedge b}$ is the upper bound of $\{\mu_a, \mu_b\}$. Let $\mu_a, \mu_b \subseteq \mu_z$, for some $z \in L$.
Now,

$$\begin{aligned}
\mu_{a \wedge b}(x) &= \mu(x \wedge a \wedge b) \\
&= \mu_a(x \wedge b) \\
&\leq \mu_z(x \wedge b).
\end{aligned}$$

Smilarly $\mu_{a \wedge b}(x) \leq \mu_z(x \wedge a)$. This implies

$$\begin{aligned}
\mu_{a \wedge b}(x) &\leq \mu_z(x \wedge a) \wedge \mu_z(x \wedge b) \\
&= \mu(z \wedge x \wedge a) \wedge \mu(z \wedge x \wedge b) \\
&= \mu(z \wedge x \wedge (a \vee b)) \\
&= \mu_{a \vee b}(z \wedge x) = (\mu_a \cap \mu_b)(z \wedge x) \\
&= \mu_a(z \wedge x) \wedge \mu_b(z \wedge x) \\
&\leq \mu_z(z \wedge x) \wedge \mu_z(z \wedge x) \\
&= \mu_z(x).
\end{aligned}$$

Hence $\mu_{a\wedge b} = \mu_a \vee \mu_b$. Finally, we prove distributivity. For any $a, b, c \in L$, we have

$$\begin{aligned}(\mu_a \vee \mu_b) \wedge \mu_c &= \mu_{a\wedge b} \wedge \mu_c \\ &= \mu_{(a\wedge b)\vee c} \\ &= \mu_{(a\vee c)\wedge(b\vee c)} \\ &= \mu_{a\vee c} \vee \mu_{b\vee c} \\ &= (\mu_a \wedge \mu_c) \vee (\mu_b \wedge \mu_c).\end{aligned}$$

Hence, μ_L is a bounded distributive lattice.

(2) Clearly $\overline{\overline{\mu_a}} = \mu_{a^{\circ\circ}} = \mu_a$ for all $a \in L$. Now,

$$\begin{aligned}\overline{\mu_a \vee \mu_b} &= \overline{\mu_{a\wedge b}} = \mu_{(a\wedge b)^\circ} \\ &= \mu_{a^\circ \vee b^\circ} \\ &= \mu_{a^\circ} \cap \mu_{b^\circ} \\ &= \overline{\mu_a} \cap \overline{\mu_b}.\end{aligned}$$

Finally, $\overline{\mu_0} = \mu_1$. Hence μ_L is a de Morgan algebra. □

Theorem 3.24. *Let μ be a fuzzy ideal of L. μ is closure fuzzy ideal of an MS-algebra L if and only if it is an MS-fuzzy ideal.*

4. MS-fuzzy ideals and Homomorphisms

In this section, we present some results on the homomorphic images and inverse homomorphic images of MS-fuzzy ideals. We show that any isomorphism between MS-algebras L and M induces an isomorphism between μ_L and μ_M for any fuzzy ideal μ of L.

Theorem 4.1. *Let $f : L \longrightarrow M$ be a homomorphism of MS-algebras. Then,*

(1) *The image of an MS-fuzzy ideal of L is an MS-fuzzy ideal of M,*
(2) *The inverse image of an MS-fuzzy ideal of M is an MS-fuzzy ideal of L.*

Proof. Suppose that μ is an MS-fuzzy ideal of L.
Case-1: If $f^{-1}(y) = \emptyset$ for some $y \in L$. Clearly, $f(\mu)(y) = 0 \leq f(\mu)(y^{\circ\circ})$.
Case-2: If $f^{-1}(y) \neq \emptyset$ for some $y \in M$.

$$\begin{aligned}f(\mu)(y) &= \sup\{\mu(x) : f(x) = y\} \\ &\leq \sup\{\mu(x^{\circ\circ}) : f(x^{\circ\circ}) = y^{\circ\circ}\} \\ &= f(\mu)(y^{\circ\circ})\end{aligned}$$

Hence $f(\mu)$ is an MS-fuzzy ideal of M.

(2) Suppose that λ is an MS-fuzzy ideal of M. Now for any $x \in L$

$$\begin{aligned}f^{-1}(\lambda)(x) &= \lambda(f(x)) \\ &\leq \lambda((f(x))^{\circ\circ}) \\ &= \lambda(f(x^{\circ\circ}))\end{aligned}$$

$$= \quad f^{-1}(\lambda)(x^{\circ\circ})$$

Hence $f^{-1}(\mu)$ is an MS-fuzzy ideal of L. □

Theorem 4.2. *Let $f : L \longrightarrow M$ be a homomorphism of MS-algebras. Let μ be a fuzzy ideal of L and λ be a fuzzy ideal of M. Then, for every $a \in L$:*

(1) $f(\mu_a) \subseteq (f(\mu))_{f(a)}$,
(2) $f^{-1}(\mu_{f(a)}) = (f^{-1}(\mu))_a$,
(3) *If f is an isomorphism, then* $f(\mu_a) = (f(\mu))_{f(a)}$.

Proof. (1) Suppose that μ is a fuzzy ideal of L and λ is a fuzzy ideal of M. For any $y \in M$.
Case-1: If $f^{-1}(y) = \emptyset$ for some $y \in L$. Then $f(\mu_a)(y) = 0 \leq (f(\mu))_{f(a)}(y)$
Case-2: If $f^{-1}(y) \neq \emptyset$ for some $y \in L$. Then

$$\begin{aligned} f(\mu_a)(y) &= \sup\{\mu_a(x) : f(x) = y\} \\ &= \sup\{\mu(a \wedge x) : f(x) = y\} \\ &\leq \sup\{\mu(a \wedge x) : f(x) \wedge f(a) = y \wedge f(a)\} \\ &= \sup\{\mu(a \wedge x) : f(x \wedge a) = y \wedge f(a)\} \\ &= f(\mu)(y \wedge f(a)) \\ &= f(\mu)_{f(a)}(y). \end{aligned}$$

Hence for every $a \in L$, $f(\mu_a) \subseteq (f(\mu))_{f(a)}$.

(2) For any $x \in L$, $f^{-1}(\mu_{f(a)})(x) = \mu_{f(a)}(f(x)) = \mu(f(a) \wedge f(x)) = \mu(f(a \wedge x)) = f^{-1}(\mu)(a \wedge x) = (f^{-1}(\mu))_a(x)$.

(3) Suppose that f is an isomorphism. Then by (1) $f(\mu_a) \subseteq (f(\mu))_{f(a)}$. Conversely,

$$\begin{aligned} (f(\mu))_{f(a)}(y) &= f(\mu)(f(a) \wedge y) \\ &\leq \mu(f^{-1}(f(a) \wedge y)) \\ &= \mu(a \wedge f^{-1}(y)) \\ &= \mu_a(f^{-1}(y)) \\ &= f(\mu_a)(y)) \end{aligned}$$

This implies $(f(\mu))_{f(a)} \subseteq f(\mu_a)$. Hemce $(f(\mu))_{f(a)} = f(\mu_a)$. □

Theorem 4.3. *Let $f : L \longrightarrow M$ be an isomorphism of MS-algebras. Then:*

(1) *μ_L is isomorphic to $f(\mu)_M$, for any fuzzy ideal μ of L,*
(2) *μ_M is isomorphic to $f^{-1}(\mu)_L$, for any fuzzy ideal μ of M.*

Proof. Define $\varphi : \mu_L \to f(\mu)_M$ by $\varphi(\mu_a) = f(\mu_a)$, for any $a \in L$. Then, for any $a, b \in L$, we have

$$\begin{aligned} \varphi(\mu_a \cap \mu_b) &= \varphi(\mu_{a\vee b}) \\ &= f(\mu_{a\vee b}) \\ &= f(\mu)_{f(a\vee b)} \end{aligned}$$

$$\begin{aligned}
&= f(\mu)_{f(a)\vee f(b)} \\
&= f(\mu)_{f(a)} \cap f(\mu)_{f(b)} \\
&= f(\mu_a) \cap f(\mu_b) \\
&= \varphi(\mu_a) \cap \varphi(\mu_b) \\
\varphi(\mu_a \vee \mu_b) &= \varphi(\mu_{a\wedge b}) \\
&= f(\mu_{a\wedge b}) \\
&= f(\mu)_{f(a\wedge b)} \\
&= f(\mu)_{f(a)\wedge f(b)} \\
&= f(\mu)_{f(a)} \vee f(\mu)_{f(b)} \\
&= f(\mu_a) \vee f(\mu_b) \\
&= \varphi(\mu_a) \vee \varphi(\mu_b)
\end{aligned}$$

Let $\mu_a, \mu_b \in \mu_{L_1}$ such that $\varphi(\mu_a) = \varphi(\mu_b)$. This implies $f(\mu_a) = f(\mu_b)$ and $f(\mu)_{f(a)}) = f(\mu)_{f(b)})$. Now, for any $x \in L$,

$$\begin{aligned}
\mu_a(x) &= f^{-1}f(\mu_a)(x) \\
&= f(\mu_a)(f(x)) \\
&= f(\mu_b)(f(x)) \\
&= f^{-1}f(\mu_b)(x) \\
&= \mu_b(x).
\end{aligned}$$

Hence φ is one-to-one. Let $f(\mu)_b \in f(\mu)_M,\ b \in M$. Since f is on to, then there exists $a \in L$ such that $f(a) = b$. Therefore $f(\mu)_b = f(\mu)_{f(a)} = f(\mu_a)$ as $\mu_a \in \mu_L$. Hence φ is on to. Thus φ is an isomorphism between μ_L and $f(\mu)_M$.

(2) Define $\varphi : \lambda_{L_2} \mapsto f^{-1}(\lambda)_{L_1}$ by $\varphi(\lambda_{f(a)}) = f^{-1}(\lambda f(a))$, for any $a \in L$. Then, for any $a, b \in L$, we have

$$\begin{aligned}
\varphi(\lambda_{f(a)} \cap \lambda_{f(b)}) &= f^{-1}(\lambda_{f(a)} \cap \lambda_{f(b)}) \\
&= f^{-1}(\lambda_{f(a\vee b)}) \\
&= (f^{-1}(\lambda))_{a\vee b} \\
&= (f^{-1}(\lambda)_a \cap (f^{-1}(\lambda))_b \\
&= f^{-1}(\lambda_{f(a)}) \cap f^{-1}(\lambda_{f(b)}) \\
&= \varphi(\lambda_{f(a)}) \cap \varphi(\lambda_{f(b)}).
\end{aligned}$$

Also,

$$\begin{aligned}
\varphi(\lambda_{f(a)} \vee \lambda_{f(b)}) &= f^{-1}(\lambda_{f(a)} \vee \lambda_{f(b)}) \\
&= f^{-1}(\lambda_{f(a\wedge b)}) \\
&= (f^{-1}(\lambda))_{a\wedge b} \\
&= (f^{-1}(\lambda)_a \vee (f^{-1}(\lambda))_b
\end{aligned}$$

$$= f^{-1}(\lambda_{f(a)}) \vee f^{-1}(\lambda_{f(b)})$$
$$= \varphi(\lambda_{f(a)}) \vee \varphi(\lambda_{f(b)}).$$

Let $\lambda_{f(a)}$, $\lambda_{f(b)} \in \lambda_M$ with $\varphi(\lambda_{f(a)}) = \varphi(\lambda_{f(b)})$. That is, $f^{-1}(\lambda_{f(a)}) = f^{-1}(\lambda_{f(b)})$. We want to prove that $\lambda_{f(a)} = \lambda_{f(b)}$. Suppose that $\lambda_{f(a)} \neq \lambda_{f(b)}$. This implies there exist $x \in L$ such that $\lambda_{f(a)}(x) < \lambda_{f(b)}(x)$ or $\lambda_{f(a)}(x) > \lambda_{f(b)}(x)$. With out loss of generality $\lambda_{f(a)}(x) < \lambda_{f(b)}(x)$. This implies $f^{-1}(\lambda_{f(a)})(f^{-1}(x)) < f^{-1}(\lambda_{f(b)})(f^{-1}(x))$. Hence $f^{-1}(\lambda_{f(a)}) \neq f^{-1}(\lambda_{f(b)})$. Which is a contradictions. Hence φ is one to one. Let for any $(f^{-1}(\lambda))_a \in f^{-1}(\lambda)_{L_1}$. Now $f^{-1}(\lambda_{f(a)}) = f^{-1}(\lambda)_a$. Hence φ is on to. □

Lemma 4.4. *Let θ be a fuzzy congruence on an MS-algebra L. Then $Ker(\theta)$ is an MS-fuzzy ideal.*

Theorem 4.5. *Let μ be an MS-fuzzy ideal of an MS-algebra L. Define $\theta(\mu)$ on L by $\theta(\mu)(x, y) = 1 \Leftrightarrow \mu_x = \mu_y$. Then*

(1) *$\theta(\mu)$ is a lattice fuzzy congruence,*
(2) *$\theta(\mu)$ is not a fuzzy congruence on L in general.*

Proof. (1) Clearly $\mu_x = \mu_x$. This implies $\theta(\mu)(x, x) = 1$. Suppose that $\theta(\mu)(x, y) = 1$. This implies $\mu_x = \mu_y$ and $\mu_y = \mu_x$. Thus $\theta(\mu)(y, x) = 1$. Hence $\theta(\mu)(x, y) = \theta(\mu)(y, x)$.

Suppose that $\theta(\mu)(x, y) = 1$, $\theta(\mu)(y, z) = 1$. This implies $\mu_x = \mu_y$, $\mu_y = \mu_z$. Thus $\mu_x = \mu_z$ and so $\theta(\mu)(x, z) = 1$. Hence $\theta(\mu)(x, y) \wedge \theta(\mu)(y, z) \leq \theta(\mu)(x, z)$. Thus implies $\theta(\mu)$ is equivalence relations. Let $\theta(\mu)(a, b) = 1$. Then $\mu_a = \mu_b$. For any $c \in L$,

$$\mu_{a\wedge c}(x) = \mu(a \wedge c \wedge x) = \mu_a(c \wedge x) = \mu_b(c \wedge x) = \mu(b \wedge c \wedge x) = \mu_{b\wedge c}(x).$$

Hence $\theta(\mu)(a \wedge c, b \wedge c) = 1 \geq \theta(\mu)(a, b)$. Therefore $\theta(\mu)$ is a lattice fuzzy congruence.

(2) Here we give an example. Consider example 3.19 . Let $\mu(0) = \mu(d) = \mu(b) = 1$, $\mu(c) = \mu(x) = \mu(z) = \mu(z) = \mu(a) = \mu(y) = \mu(1) = 0.7$. Then $\mu_a = \mu = \mu_x$ and so $\theta(\mu)(a, x) = 1$. On the other hand we have $\mu_{a^\circ}(c) = \mu_b(c) = \mu(b \wedge c) = \mu(0) = 1$, $\mu_{x^\circ}(c) = \mu_c(c) = \mu(c \wedge c) = \mu(c) = 0.7$. This implies $\mu_{a^\circ} \neq \mu_{x^\circ}$. Thus implies $\theta(\mu)(a^\circ, x^\circ) \neq 1$. Hence, $\theta(\mu)$ is not a fuzzy congruence on L. □

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