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LOCAL SPECTRAL THEORY II

JONG-KWANG YOO

Abstract. In this paper we show that if $A \in L(X)$ and $B \in L(Y)$, X and Y complex Banach spaces, then $A \oplus B \in L(X \oplus Y)$ is subscalar if and only if both A and B are subscalar. We also prove that if $A, Q \in L(X)$ satisfies $AQ = QA$ and $Q^p = 0$ for some nonnegative integer p, then A has property (C) (resp. property (β)) if and only if so does $A + Q$ (resp. property (β)). Finally, we show that $A \in L(X, Y)$ and $B, C \in L(Y, X)$ satisfying operator equation $ABA = ACA$ and $BA \in L(X)$ is subscalar with property (δ) then both $Lat(BA)$ and $Lat(AC)$ are non-trivial.

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1. Introduction and preliminaries

Throughout this paper, X and Y denote complex Banach spaces and let $L(X, Y)$ denotes the Banach algebra of all bounded linear operators from X to Y. As usual, when $X = Y$, we simply write $L(X)$ for $L(X, X)$. For $A \in L(X)$, let $ker(A)$ and $A(X)$ stand for the kernel and range of A, respectively. The spectrum, the point spectrum and the resolvent set of A are denoted by $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$, respectively. Let $Lat(A)$ stand for the collection of all A-invariant closed linear subspaces of X. For a A-invariant closed linear subspace Y of X, let $A|Y$ denote the operator given by the restriction of A to Y.

The single-valued extension property was first introduced by Dunford and has received a systematic treatment in Dunford-Schwartz [3], [4] and [5]. The following localized version of SVEP was introduced by Finch [10]. SVEP has developed into one of the major tools in the local spectral theory and Fredholm theory for operators on Banach spaces.

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Definition 1.1. ([11]) An operator $A \in L(X)$ is said to have the *single-valued extension property* An operator $A \in L(X)$ is said to have the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for brevity), if for every open disc U centered at λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - A)f(\lambda) = 0$ for all $\lambda \in U$ is the constant function $f \equiv 0$. An operator $A \in L(X)$ is said to have the SVEP if A has the SVEP at every point $\lambda \in \mathbb{C}$.

It is clear that an operator $A \in L(X)$ has SVEP at a point $\lambda \in \mathbb{C}$ precisely when $\lambda I - A$ has SVEP at 0. Moreover, SVEP at a point is inherited by restrictions to closed invariant subspaces. Evidently, A has SVEP at every $\lambda \in \mathbb{C} \setminus int(\sigma_p(A))$. In particular, if $\sigma_p(A)$ has empty interior, for example, A is of finite rank, then A has SVEP.

For a bounded linear operator A defined on a complex Banach space X, the *local resolvent set* $\rho_A(x)$ of A at the point $x \in X$ defined as the union of all open subsets U of $\mathbb{C}$ such that there exists an analytic function $f : U \to X$ which satisfies

$$(\lambda I - A)f(\lambda) = x \quad \text{for all } \lambda \in U.$$

The *local spectrum* $\sigma_A(x)$ of A at x is the set defined by $\sigma_A(x) := \mathbb{C} \setminus \rho_A(x)$. Obviously, we have $\sigma_A(x)$ is a closed subset of $\sigma(A)$. The local ananlytic solutions occuring in the defintion of the local resolvent set will be unique for all $x \in X$ if and only if A has SVEP.

For every subset F subset of $\mathbb{C}$, the *local spectral subspace* of A associated with F is the set

$$X_A(F) := \{x \in X : \sigma_A(x) \subseteq F\}.$$

It is clear from the definition that $X_A(F)$ is a $A-$hyperinvariant linear subspace of X. In general, these linear subspaces $X_A(F)$ is not closed. Moreover, for every closed $F \subseteq \mathbb{C}$ we have

$$(\lambda I - A)X_A(F) = X_A(F) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus F,$$

see [12], Proposition 1.2.16.

For a closed set $F \subseteq \mathbb{C}$, the *glocal spectral subspace* $\mathcal{X}_A(F)$ consists of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ such that

$$(\lambda I - A)f(\lambda) = x \quad \text{for each } \lambda \in \mathbb{C} \setminus F.$$

In general, $\mathcal{X}_A(F) \subseteq X_A(F)$ for every closed $F \subseteq \mathbb{C}$, and neither the local spectral subspace nor the glocal spectral subspaces have to be closed. But the two concepts of local spectral subspace and glocal spectral subspace coincide if A has SVEP, see [12], Proposition 3.3.2.

An operatpr $A \in L(X)$ is said to have *Dunford's property (C)* (abbreviated *property (C)*) if the local spectral subspace $X_A(F)$ is closed for every closed subset F of $\mathbb{C}$.

Theorem 1.2. *([11]) Let $A \in L(X)$. Then $\mathcal{X}_A(\phi) = \{0\}$, $\mathcal{X}_A(F) = \mathcal{X}_A(F \cap \sigma(A))$ and $\mathcal{X}_A(F) \subseteq X_A(F)$ for all closed $F \subseteq \mathbb{C}$. Moreover, the following assertions are equivalent.*
(a) A has SVEP.
(b) $\mathcal{X}_A(F) = X_A(F)$ for all closed $F \subseteq \mathbb{C}$.
(c) $X_A(\phi)$ is closed.
(d) $X_A(\phi) = \{0\}$.

Proof. Proposition 1.1 of [11]. □

Let $\lambda \in \rho_A(x)$ and U denote an open neighborhood of λ. If $f : U \to X$ is analytic function satisfies the equation $(\lambda I - A)f(\lambda) = x$ for all $\lambda \in U$, then $\sigma_A(f(\lambda)) = \sigma_A(x)$ for all $\lambda \in U$, see [12], Lemma 1.2.14. It is clear that

$$x \in X_{A+\lambda I}(F) \Leftrightarrow \sigma_{A+\lambda I}(x) \subseteq F \Leftrightarrow \sigma_A(x) \subseteq F - \lambda \Leftrightarrow x \in X_A(F - \lambda),$$

where $F - \lambda := \{\mu - \lambda : \mu \in F\}$. This implies that $\sigma_{A+\lambda I}(x) = \sigma_A(x) + \lambda$ for all $\lambda \in \mathbb{C}$ and we conclude that $X_{A+\lambda I}(F) = X_A(F - \lambda)$ for every $F \subseteq \mathbb{C}$.

Let $O(U, X)$ denote the Fréchet algebra of all $X-$valued analytic functions on the open subset $U \subseteq \mathbb{C}$ endowed with uniform convergence on compact subsets of U. An operator $A \in L(X)$ is said to have *Bishop's property* (β) if for every open subset U of $\mathbb{C}$ and for every sequence $f_n : U \to X$ of analytic functions such that $(\lambda I - A)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of U, $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of U. An operator $A \in L(X)$ is said to have the *decomposition property* (δ) if $X = \mathcal{X}_A(\overline{U}) + \mathcal{X}_A(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. It is well known that $A \in L(X)$ has property (δ) if and only if its dual $A^* \in L(X^*)$ has property (β), see [1] and [12].

An operator $A \in L(X)$ is called *decomposable* provided that for each open cover $\{U, V\}$ of the complex plane $\mathbb{C}$, there exist $Y, Z \in Lat(A)$ for which $Y + Z = X$, $\sigma(A|Y) \subseteq U$ and $\sigma(A|Z) \subseteq V$. This class contains all compact operators, normal operators, spectral operators and generalized scalar operators. In particular, a simple application of the Riesz functional calculus shows that all operators with totally disconnected spectrum are decomposable. In particular, all algebraic operators are decomposable. It has been observed in [1] that an operator $A \in L(X)$ is decomposable if and only if it has both properties (β) and (δ). We refer the reader to [1], [2] and [12] for more details and further defintions.

Let $\mathcal{E}(U, X)$ denote the Fréchet algebra of all $X-$valued infinitely continuously differentiable functions on the open subset $U \subseteq \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of U of such functions and their derivatives. We say that an $A \in L(X)$ is said to have the *property* (β_ϵ) at $\lambda \in \mathbb{C}$ if there exists a neighborhood V of λ such that for each open subset $U \subseteq V$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{E}(U, X)$,

$$\lim_{n\to\infty} (\mu I - A)f_n(\mu) = 0$$

in $\mathcal{E}(U,X)$ implies $\lim_{n\to\infty} f_n(\mu)=0$ in $\mathcal{E}(U,X)$. Let $\sigma_{\beta_\epsilon}(A)$ be the set where A fails to satisfy (β_ϵ). We say that an operator $A\in L(X)$ has property (β_ϵ) if $\sigma_{\beta_\epsilon}(A)=\phi$. It is clear that $A\in L(X)$ has property (β_ϵ) precisely when A is subscalar. The class of subscalar operators contains all hyponormal operators, $p-$hyponormal operators, $k-$quasihyponormal operators on a complex Hilbert space, see [9] and [14]. Note that every generalized scalar operator is decomposable, and hence a generalized scalar operator has property (β). Since the restriction of an operator with property (β) to a closed invariant subspace certainly inherits this property. We conclude that every subscalar operator has property (β). It is well known from [8] and [12] that

$$\text{subscalar} \Rightarrow \text{Bishop's property}(\beta) \Rightarrow \text{Dunford's property}(C) \Rightarrow \text{SVEP}.$$

In general, the converse implications do not hold, see [2], [8] and [12]. As an immediate application of Theorem 1.2, we obtain

Proposition 1.3. *If $A\in L(X)$ has SVEP (resp. property (C), property (β)) then so does $A+\lambda I$ (resp. property (C), property (β)) for all $\lambda\in\mathbb{C}$.*

2. Main results

Theorem 2.1. *Let $A\in L(X)$ and $B\in L(Y)$. Then $\sigma_{\beta_\epsilon}(A\oplus B)=\sigma_{\beta_\epsilon}(A)\cup\sigma_{\beta_\epsilon}(B)$. Moreover, $A\oplus B\in L(X\oplus Y)$ is subscalar if and only if both A and B are subscalar.*

Proof. If $\lambda\notin\sigma_{\beta_\epsilon}(A\oplus B)$ then $A\oplus B$ has property (β_ϵ) at $\lambda\in\mathbb{C}$. Suppose that $(g_n)_n$ is a sequence in $\mathcal{E}(U,X)$ such that $\lim_{n\to\infty}(\lambda I-A)g_n(\mu)=0$ in $\mathcal{E}(U,X)$ and $(h_n)_n$ is a sequence in $\mathcal{E}(U,Y)$ such that $\lim_{n\to\infty}(\lambda I-B)h_n(\mu)=0$ in $\mathcal{E}(U,Y)$. We define $f_n:U\to X\oplus Y$ by

$$f_n(\mu):=g_n(\mu)+h_n(\mu)\quad\text{for all } \mu\in U.$$

Then clearly $(f_n)_n\subseteq\mathcal{E}(U,X\oplus Y)$ and

$$\lim_{n\to\infty}(\lambda I-(A\oplus B))f_n(\mu)=\lim_{n\to\infty}(\lambda I-A)g_n(\mu)+\lim_{n\to\infty}(\lambda I-B)h_n(\mu)=0$$

in $\mathcal{E}(U,X\oplus Y)$. Since $A\oplus B$ has property (β_ϵ) at $\lambda\in\mathbb{C}$,

$$\lim_{n\to\infty} f_n(\mu)=0$$

in $\mathcal{E}(U,X\oplus Y)$. It follows that $\lim_{n\to\infty} g_n(\mu)=0$ in $\mathcal{E}(U,X)$ and $\lim_{n\to\infty} h_n(\mu)=0$ in $\mathcal{E}(U,Y)$. Hence A and B have the property (β_ϵ) at $\lambda\in\mathbb{C}$, which implies $\lambda\notin\sigma_{\beta_\epsilon}(A)\cup\sigma_{\beta_\epsilon}(B)$. Conversely, assume that $\lambda\notin\sigma_{\beta_\epsilon}(A)\cup\sigma_{\beta_\epsilon}(B)$. Then there exist neighborhoods V, W of λ such that $V\cap\sigma_{\beta_\epsilon}(A)=\phi$ and $W\cap\sigma_{\beta_\epsilon}(B)=\phi$. Let $U:=V\cap W$. Then clearly U is a neighborhood of λ such that $U\cap(\sigma_{\beta_\epsilon}(A)\cup\sigma_{\beta_\epsilon}(B))=\phi$. Suppose that $(f_n)_n$ is a sequence in $\mathcal{E}(U,X\oplus Y)$ such that

$$\lim_{n\to\infty}(\lambda I-(A\oplus B))f_n(\mu)=0$$

in $\mathcal{E}(U, X\oplus Y)$. Clearly, $P_1 f_n \in \mathcal{E}(U, X)$ and $P_2 f_n \in \mathcal{E}(U, Y)$ where $P_1 : X\oplus Y \to X$ and $P_2 : X \oplus Y \to Y$ are projections. Then clearly,

$$P_1 f_n(\mu) + P_2 f_n(\mu) = f_n(\mu)$$

for all $\mu \in U$. It is clear that

$$\lim_{n\to\infty} (\lambda I - A)P_1 f_n(\mu) = 0$$

in $\mathcal{E}(U, X)$ and

$$\lim_{n\to\infty} (\lambda I - B)P_2 f_n(\mu) = 0$$

in $\mathcal{E}(U, Y)$. Since A and B have the property (β_ϵ) at λ, we have

$$\lim_{n\to\infty} P_1 f_n(\mu) = 0$$

in $\mathcal{E}(U, X)$ and

$$\lim_{n\to\infty} P_2 f_n(\mu) = 0$$

in $\mathcal{E}(U, Y)$. It follows that $\lim_{n\to\infty} f_n(\mu) = 0$ in $\mathcal{E}(U, X \oplus Y)$. This shows that $\lambda \notin \sigma_{\beta_\epsilon}(A \oplus B)$. □

For given operators $A \in L(X)$ and $B \in L(Y)$, we consider the corresponding commutator $C(B, A) : L(X, Y) \to L(X, Y)$ defined by $C(B, A)(T) := BT - TA$ for all $T \in L(X, Y)$. It is clear that

$$C(B, A)^n(T) := C(B, A)^{n-1}(BT - TA) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k B^{n-k} T A^k$$

for all $n \in \mathbb{N}$ and for all $T \in L(X, Y)$. In particular, if $A, B \in L(X)$ and there exists an integer $n \in \mathbb{N}$ for which $C(A, B)^n(I) = 0 = C(B, A)^n(I)$, then the operators A and B are said to be *nilpotent equivalent.* For $A, B \in L(X)$ with $AB = BA$, it is easily seen that $C(B, A)^n(I) = (A - B)^n = C(A, B)^n(I)$ for all $n \in \mathbb{N}$. In this case, A and B are nilpotent precisely when $A - B$ is nilpotent. see [2], [11] and [12].

Proposition 2.2. *Let $A, Q \in L(X)$ satisfies $AQ = QA$ and $Q^p = 0$ for some nonnegative integer p. Then $X_{A+Q}(F) = X_A(F)$ for every $F \subseteq \mathbb{C}$. Moerover, A has Dunford's property (C) if and only if so does $A + Q$.*

Proof. Clearly, $C(A + Q, A)^k(I) = Q^k$ and $C(A, A + Q)^k(I) = (-1)^k Q^k$ for all $k \in \mathbb{N}$. Thus $C(A + Q, A)^p(I) = 0 = C(A, A + Q)^p(I)$, which implies $A + Q$ and A are nilpotent equivalent. It follows from Corollary 3.4.5 [12] that $X_{A+Q}(F) = X_A(F)$ for every $F \subseteq \mathbb{C}$. □

Theorem 2.3. *Let $A, Q \in L(X)$ satisfies $AQ = QA$ and $Q^p = 0$ for some nonnegative integer p. Then A has Bishop's property (β) if and only if so does $A + Q$.*

Proof. Suppose that A has Bishop's property (β). Let $f_n : U \to X$ be any sequence of analytic functions on an arbitrary open set U such that

$$(\lambda I - (A + Q))f_n(\lambda) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U. Then

$$(\lambda I - A)f_n(\lambda) - Qf_n(\lambda) \to 0 \quad \text{as } n \to \infty \tag{1}$$

uniformly on all compact subsets of U. Since $Q^p = 0$ and $AQ = QA$, we have

$$(\lambda I - A)Q^{p-1}f_n(\lambda) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U. Since A has Bishop's property (β),

$$Q^{p-1}f_n(\lambda) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U. From (1), we have

$$(\lambda I - A)Q^{p-2}f_n(\lambda) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U and A has Bishop's property (β), we have

$$Q^{p-2}f_n(\lambda) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U. By induction, we can show that

$$f_n(\lambda) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U which implies $A+Q$ has Bishop's property (β). Conversely, suppose that $A+Q$ has Bishop's property (β). Let $f_n : U \to X$ be any sequence of analytic functions on an arbitrary open set U such that

$$(\lambda I - A)f_n(\lambda) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U. Then we have

$$(\lambda I - (A + Q))f_n(\lambda) + Qf_n(\lambda) = (\lambda I - A)f_n(\lambda) \to 0 \quad \text{as} \quad n \to \infty$$

uniformly on all compact subsets of U. Thus we have

$$Q^{p-1}((\lambda I - (A+Q))f_n(\lambda) + Qf_n(\lambda)) = (\lambda I - (A+Q))Q^{p-1}f_n(\lambda) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U. Since $A + Q$ has Bishop's property (β),

$$Q^{p-1}f_n(\lambda) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U. Clearly, we have

$$(\lambda I - (A+Q))Q^{p-2}f_n(\lambda) = Q^{p-2}((\lambda I - (A+Q))f_n(\lambda) + Qf_n(\lambda)) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U. Since $A + Q$ has Bishop's property (β),

$$Q^{p-2}f_n(\lambda) \to 0 \quad \text{as } n \to \infty$$

uniformly on all compact subsets of U. By induction, $f_n(\lambda) \to 0$ as as $n \to \infty$ uniformly on all compact subsets of U. Hence A has Bishop's property (β). □

Given an operator $A \in L(X)$ on a complex Banach space X and a linear subspace M of X, we say that M is an *invariant subspace* of A if $A(M) \subseteq M$. Obviously $\{0\}$ and X are invariant subspaces and M invariant implies $\overline{M}$ invariant. The invariant subspace problem is the question whether every bounded linear operator $A \in L(X)$ has a non-trivial invariant subspave. Read and Enflo proved that the invariant subspace problem has a negative answer on Banach spaces, see more details [6], [15] and [16]. Eshmeier and Prunaru [7] also proved that if $A \in L(X)$ has either property (β) or property (δ), then $Lat(A)$ is non-trivial provided that $\sigma(A)$ is thick, and that $Lat(A)$ is rich in the sense that it contains the lattice of all closed subspaces of some infinite-dimensional Banach space provided that the essential spectrum $\sigma_e(A)$ is thick, see [12], Proposition 2.6.2.

Read [16] proved that there exist quasi-nilpotent operators on a complex Banach spaces without non-trivial closed invariant subspaces.

Q. Zeng and H. Zhong [17] proved that if $A \in L(X,Y)$ and $B, C \in L(Y,X)$ satisfying operator equation $ABA = ACA$ then AC and BA share the local specral properties such as Bishop's property (β), property (δ), decomposability, subscalarity and Dunford's property (C).

Theorem 2.4. *(*[17]*) An operator $A \in L(X)$ is said to have the property (K) at a point $\lambda_0 \in \mathbb{C}$ Let $A \in L(X,Y)$ and $B, C \in L(Y,X)$ satisfying operator equation $ABA = ACA$. Then*
(1) AC is subscalar if and only if BA is subscalar;
(2) AC satisfies decomposition property (δ) if and only if so does BA.

Proof. Theorem 2.1 and Corollary 2.2 of [17]. □

An operator $A \in L(X)$ is said to be *algebraic* if $p(A) = 0$ for some non-zero complex polynomial p.

Lemma 2.5. *Let $A \in L(X,Y)$ and $B \in L(Y,X)$. Suppose that there exists $n \in \mathbb{N}$ such that $\bigcap_{\mu\in\mathbb{C}}(\mu I - BA)^n X = \{0\}$. Then BA is algebraic if and only if $\sigma(BA)$ is finite.*

Proof. If $BA \in L(X)$ is algebraic then by the spectral mapping theorem, $\sigma(BA)$ is finite. Conversely, assume that $\sigma(BA) = \{\mu_1, \mu_2, \cdots, \mu_k\}$ is finite. It is clear that

$$X_{BA}(\phi) = (\mu I - BA)^n X_{BA}(\phi) \subseteq (\mu I - BA)^n(X)$$

for all $\mu \in \mathbb{C}$, and which implies $X_{BA}(\phi) = \{0\}$. By Theorem 1.2, BA has SVEP. Since $\sigma(BA) = \{\mu_1, \mu_2, \cdots, \mu_k\}$, it follows from Proposition 1.2.16 [12] that

$$X = X_{BA}(\sigma(BA)) = X_{BA}(\{\mu_1\}) \oplus X_{BA}(\{\mu_2\}) \oplus \cdots \oplus X_{BA}(\{\mu_k\}),$$

holds as an algebraic direct sum. We prove that $X_{BA}(\{\mu\}) = ker(\mu I - BA)^n$ for all $\mu \in \mathbb{C}$. It follows from Propositiion 1.2.16 [12] that $ker(\mu I - BA)^n \subseteq X_{BA}(\{\mu\})$ for all $\mu \in \mathbb{C}$ and $n \in \mathbb{N}$. We prove that $(\mu I - BA)^n(X_{BA}(\{\mu\})) = \{0\}$.

It suffices to show that

$$(\mu I - BA)^n X_{BA}(\{\mu\}) \subseteq (\lambda I - BA)^n(X) \text{ for all } \lambda \in \mathbb{C}.$$

It is clear that if $\mu = \lambda$ then $(\mu I - BA)^n X_{BA}(\{\mu\}) \subseteq (\mu I - BA)^n(X)$, because of $X_{BA}(\{\mu\}) \subseteq X$. If $\mu \neq \lambda$ then by Proposition 1.2.16 (b) [12],

$$(\lambda I - BA)^n X_{BA}(\{\lambda\}) = X_{BA}(\{\mu\}).$$

From this, it then follows that

$$(\mu I - BA)(X_{BA}(\{\mu\})) \subseteq X_{BA}(\{\mu\}) = (\lambda I - BA)^n X_{BA}(\{\mu\}) \subseteq (\lambda I - BA)^n(X),$$

for all $n \in \mathbb{N}$. Hence $X_{BA}(\{\mu\}) = ker(\mu I - BA)^n$ for all $\mu \in \mathbb{C}$. It follows that

$$X = ker(\mu_1 I - BA)^n \oplus ker(\mu_2 I - BA)^n \oplus \cdots \oplus ker(\mu_k I - BA)^n.$$

We conclude that $(\mu_1 I - BA)^n(\mu_2 I - BA)^n \cdots (\mu_k I - BA)^n = 0$ on X. Hence BA is algebraic. □

Theorem 2.6. *Let $A \in L(X, Y)$ and $B, C \in L(Y, X)$ satisfying operator equation $ABA = ACA$. Suppose that $BA \in (X)$ is subscalar with property (δ). Then both $Lat(BA)$ and $Lat(AC)$ are non-trivial.*

Proof. Suppose that $BA \in L(X)$ is subscalar with property (δ). Since every subscalar operator has property (β), BA is decomposable. By Theorem 2.4, $AC \in L(Y)$ is subscalar with property (δ) and hence AC is decomposable. At first, we prove that if $\sigma(BA)$ contains at least two points then $Lat(BA)$ is non-trivial. Let $\mu \in \sigma(BA)$. It follows from Proposition 1.2.20 [12] that $X_{BA}(\{\mu\})$ is a closed invariant subspace of BA and $\sigma(BA|X_{BA}(\{\mu\})) \subseteq \{\mu\}$. It suffices to prove that $X_{BA}(\{\mu\})$ is non-trivial. Let W be an open neighborhood of μ. Then there exists an open set V of $\mathbb{C}$ such that $\{W, V\}$ is an open covering of $\sigma(BA)$ and $\mu \in \mathbb{C} \setminus V$. It follows from the definition of decomposable that $X_{BA}(\{\mu\}) + X_{BA}(V) = X$,

$$\sigma(BA|X_{BA}(\{\mu\})) \subseteq W \text{ and } \sigma(BA|X_{BA}(V)) \subseteq V.$$

Suppose that $X_{BA}(\{\mu\}) = \{0\}$. Then $\sigma(BA) = \sigma(BA|X_{BA}(V)) \subseteq V$, this contradicts $\mu \notin V$ and $\mu \in \sigma(BA)$. Hence $X_{BA}(\{\mu\}) \neq \{0\}$. Suppose that $X_{BA}(\{\mu\}) = X$. Then $\sigma(BA) = \sigma(BA|X_{BA}(\{\mu\})) \subseteq \{\mu\}$. This contradicts that $\sigma(BA)$ contains at least two points. This contradiction shows that $X_{BA}(\{\mu\}) \neq X$. It follows that $X_{BA}(\{\mu\}) \in Lat(BA)$ for all $\mu \in \sigma(BA)$. Hence $Lat(BA)$ is non-trivial. It remains to consisder the case of subscalar operator BA such that X is at least two-dimensional and $\sigma(BA)$ is a singleton. Since BA is subscalar, BA has SVEP and $X_{BA}(\phi) = \{0\}$, by Theorem 1.2. By Theorem 4 [13], there exists an integer $n \in \mathbb{N}$ such that

$$X_{BA}(F) = \bigcap_{\mu \in \mathbb{C} \setminus F} (\mu I - BA)^n(X)$$

for all closed subset F of $\mathbb{C}$, which implies that

$$\bigcap_{\mu\in\mathbb{C}}(\mu I - BA)^n(X) = X_{BA}(\phi) = \{0\}.$$

It follows from Lemma 2.5 that $BA = Q + \mu I$ for some nilpotent operator $Q \in L(X)$ and for some $\mu \in \mathbb{C}$. Let $m \in \mathbb{N}$ be the smallest integer for which $Q^m = 0$ and choose an $x \in X$ for which $Q^{m-1}x$ is not zero. The linear subspace generated by $Q^{m-1}x$ is a one-dimensional $BA-$invariant linear subspace of X. Hence $Lat(BA)$ is non-trivial. The same argument above proves the second assertion. This completes the proof. □

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Jong-Kwang Yoo received M.Sc. from Chonnam National University and Ph.D. at Sogang University. Since 1994 he has been at Chodang University. His research interests include operator theory and functional analysis.

Department of Flight Operation, Chodang University, Chonnam 534-701, Korea.
e-mail: jkyoo@cdu.ac.kr