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EXPLICIT PROPERTIES OF q-SIGMOID POLYNOMIALS COMBINING q-COSINE FUNCTION†

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Abstract. In this paper, we introduce q-sigmoid polynomials combining q-cosine function. We find several properties and identities of these polynomials which are related to sigmoid function using deep learning.

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1. Introduction

In [6], Jackson who published influential papers on the subject introduce the q-number and its notation stems. We begin by introducing several definitions related to q-numbers used in this paper.

Thoughout this paper, the symbol, $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ denotes the set of natural numbers, the set of integers, the set of real numbers and the set of complex numbers, respectively.
Let $n, q \in \mathbb{R}$ with $q \neq 1$. the number

$$[n]_q = \frac{1-q^n}{1-q} \tag{1.1}$$

is called q-number. We note that $\lim_{q\to 1}[n]_q = n$. In particular, for $k \in \mathbb{Z}$, $[k]_q$ is called q-integer.

From q-number appearance, many mathematicians studied the this field such as q-differential equations, q-series, q-trigonometric function, and so on, see [1, 7, 9-11]. Of course, mathematicians constructed and researched about q-Gaussian binomial coefficients.

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Definition 1.1. The q-Gaussian binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \tag{1.2}$$

where m and r are non-negative integers. For $r = 0$ the value is 1 since the numerator and the denominator are both empty products. We note that $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$.

Definition 1.2. Let z be any complex numbers with $|z| < 1$. Two forms of q-exponential functions can be expressed as

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \qquad E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}. \tag{1.3}$$

We note that $\lim_{q\to 1} e_q(z) = e^z$. Indeed, the derivation of the power series expressions of the two q-exponential functions (1.3) are from Euler. After the limit formulas for the q-exponential functions, which given from Rawlings, several other interesting q-series expansions are presented in the classical book of Andrews. Moreover, Jackson extensively studied q-derivatives and q-integrals.

Theorem 1.3. *From Definition 1.2, we note that*

$$\begin{aligned} &\textit{(i)} \quad e_q(x)e_q(y) = e_q(x+y), \quad \textit{if} \quad yx = qxy. \\ &\textit{(ii)} \quad e_q(x)E_q(-x) = 1. \\ &\textit{(iii)} \quad e_{q^{-1}}(x) = E_q(x). \end{aligned} \tag{1.4}$$

Definition 1.4. The definition of the q-derivative operator of any function f follows that

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad x \neq 0, \tag{1.5}$$

and $D_q f(0) = f'(0)$. We can prove that f is differentiable at 0, and it is clear that $D_q x^n = [n]_q x^{n-1}$.

In [7], Victor and Pokman publised book about quantum calculus including q-derivative and q-analogue of $(x-a)^n$ and q-trigonometric functions and so on.

Definition 1.5. The q-analogues of $(x-a)^n$ and $(x+a)^n$ are defined by

$$\begin{aligned} &\text{(i)} \quad (x \ominus a)_q^n = \begin{cases} 1 & \text{if } n = 0 \\ (x-a)(x-qa)\cdots(x-q^{n-1}a) & \text{if } n \geq 1 \end{cases}, \\ &\text{(ii)} \quad (x \oplus a)_q^n = \begin{cases} 1 & \text{if } n = 0 \\ (x+a)(x+qa)\cdots(x+q^{n-1}a) & \text{if } n \geq 1 \end{cases}, \text{respectively.} \end{aligned} \tag{1.6}$$

Definition 1.6. The q-trigonometric functions are defined by

$$\begin{aligned} \sin_q(x) &= \frac{e_q(ix)-e_q(-ix)}{2i}, \qquad \mathrm{SIN}_q(x) = \frac{E_q(ix)-E_q(-ix)}{2i} \\ \cos_q(x) &= \frac{e_q(ix)+e_q(-ix)}{2}, \qquad \mathrm{COS}_q(x) = \frac{E_q(ix)+E_q(-ix)}{2}, \end{aligned} \tag{1.7}$$

where, $\mathrm{SIN}_q(x) = \sin_{q^{-1}}(x), \mathrm{COS}_q(x) = \cos_{q^{-1}}(x)$.

From the above Definition, we note that

$$\text{(i)} \quad E_q(ix) = COS_q(x) + iSIN_q(x)$$
$$\text{(ii)} \quad E_q(-ix) = COS_q(x) - iSIN_q(x).$$

In a deep learning network, we pass the nonlinear function through the nonlinear function, rather than passing it directly to the next layer. The function used at this time is called the activation function. Among these activation functions, there is a sigmoid function. The definition of sigmoid function is as follows

Definition 1.7. Let $z \in \mathbb{C}$. Then sigmoid function is expressed as

$$s(z) = \frac{1}{1+e^{-z}}.$$

In order to find various applications, various studies were done by investigating the sigmoid function. For example, a variant sigmoid function with three parameters has been employed in order to explain hybrid sigmoidal networks and, also, sigmoid function, which is also called logistic function, has been defined using flexible sigmoidal mixed models based on logistic family curves for medical applications, see [2-5].

The definition of q-sigmoid polynomials of the third row are as follows, see [11].

Definition 1.8. The q-sigmoid polynomials can be expressed as

$$\sum_{n=0}^{\infty} \mathcal{S}_{n,q}(z)\frac{t^n}{[n]_q!} = \frac{1}{1+e_q(-t)}e_q(tz) \tag{1.8}$$

2. Some basic properties of q-cosine sigmoid polynomials

In this section, we define q-cosine sigmoid polynomials using q-trigonometric functions. From these polynomials, we derive some properties and identities.

Definition 2.1. Let $|q| < 1$ and $x, y \in \mathbb{R}$ with $i = \sqrt{-1}$. Then we define generating functions of the q-cosine sigmoid polynomials,

$$\sum_{n=0}^{\infty} {}_C\mathcal{S}_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{1}{1+e_q(-t)}e_q(tx)COS_q(ty). \tag{2.1}$$

From Definition 2.1, cosine sigmoid polynomials can be defined for $q \to 1$ such as $\sum_{n=0}^{\infty} {}_C\mathcal{S}_n(x,y)\frac{t^n}{n!} = \frac{1}{1+e^{-t}}e^{tx}cos(ty)$.

Theorem 2.2. *Let $x, y \in \mathbb{R}$ with $i = \sqrt{-1}$. Then we obtain*

$$\sum_{n=0}^{\infty} \left(\frac{\mathcal{S}_{n,q}((x \oplus iy)_q) + \mathcal{S}_{n,q}((x \ominus iy)_q)}{2} \right) \frac{t^n}{[n]_q!} = \frac{1}{1+e_q(-t)} e_q(tx) COS_q(ty). \tag{2.2}$$

Proof. To find the result, we consider $(x \oplus iy)_q$ instead of z in the generating function of q-sigmoid polynomials such as

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{S}_{n,q}((x \oplus iy)_q) \frac{t^n}{[n]_q!} &= \frac{1}{1+e_q(-t)} e_q(t(x \oplus iy)_q) \\ &= \frac{1}{1+e_q(-t)} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} x^k (iy)^{n-k} \right) \frac{t^n}{[n]_q!} \\ &= \frac{1}{1+e_q(-t)} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} q^{\binom{n}{2}} (iy)^n \frac{t^n}{[n]_q!} \\ &= \frac{1}{1+e_q(-t)} e_q(tx) E_q(ity). \end{aligned} \tag{2.3}$$

From a property of q-trigonometric functions, $E_q(ity) = COS_q(ty) + iSIN_q(ty)$, we have

$$\sum_{n=0}^{\infty} \mathcal{S}_{n,q}((x \oplus iy)_q) \frac{t^n}{[n]_q!} = \frac{1}{1+e_q(-t)} e_q(tx)(COS_q(ty) + iSIN_q(ty)). \tag{2.4}$$

Also, we find the following equation using the same similar method.

$$\sum_{n=0}^{\infty} \mathcal{S}_{n,q}((x \ominus iy)_q) \frac{t^n}{[n]_q!} = \frac{1}{1+e_q(-t)} e_q(tx)(COS_q(ty) - iSIN_q(ty)). \tag{2.5}$$

From (2.4) and (2.5), we obtain

$$\sum_{n=0}^{\infty} \left(\mathcal{S}_{n,q}((x \oplus iy)_q) + \mathcal{S}_{n,q}((x \ominus iy)_q) \right) \frac{t^n}{[n]_q!} = \frac{2}{1+e_q(-t)} e_q(tx) COS_q(ty). \tag{2.6}$$

From the above equation, we obtain the required result. □

Remark 2.1. From Definition 2.1 and Theorem2.2, the following holds

$$\mathcal{S}_{n,q}((x \oplus iy)_q) + \mathcal{S}_{n,q}((x \ominus iy)_q) = 2 {}_C\mathcal{S}_{n,q}(x, y).$$

In [8], authors define $C_n(x, y)$ which is related to the Bernoulli and Euler polynomials. In addition, by combining q-numbers in $C_n(x, y)$, [9] is introduced as follows.

$$\sum_{n=0}^{\infty} C_{n,q}(x, y) \frac{t^n}{[n]_q!} = e_q(tx) COS_q(ty). \tag{2.7}$$

Theorem 2.3. *Let $|q| < 1$. Then we derive*

$$
{}_C\mathcal{S}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{S}_{k,q} C_{n-k,q}(x,y), \tag{2.8}
$$

where $\mathcal{S}_{n,q}$ is the q-sigmoid numbers.

Proof. From the generating function of q-cosine sigmoid polynomials and $C_{n,q}(x,y)$, there exists a relation between q-cosine sigmoid polynomials and q-sigmoid numbers as follows.

$$
\begin{aligned}
\sum_{n=0}^{\infty} {}_C\mathcal{S}_{n,q}(x,y) \frac{t^n}{[n]_q!} &= \frac{1}{1+e_q(-t)} e_q(tx) COS_q(ty) \\
&= \sum_{n=0}^{\infty} \mathcal{S}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} C_{n,q}(x,y) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{S}_{k,q} C_{n-k,q}(x,y) \right) \frac{t^n}{[n]_q!}.
\end{aligned} \tag{2.9}
$$

From (2.9), we find the following result. □

Corollary 2.4. *Setting $q \to 1$, one holds*

$$
{}_C\mathcal{S}_{n}(x,y) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{S}_{k} C_{n-k}(x,y),
$$

where $C_n(x,y) = e^{tx} cos(ty)$, see [8].

Theorem 2.5. *Let $e_q(-t) \neq -1$ with $|q| < 1$. Then we have*

$$
C_{n,q}(x,y) = {}_C\mathcal{S}_{n,q}(x,y) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} {}_C\mathcal{S}_{k,q}(x,y). \tag{2.10}
$$

Proof. Consider that $e_q(-t) \neq -1$. Then, we have

$$
\sum_{n=0}^{\infty} {}_C\mathcal{S}_{n,q}(x,y) \frac{t^n}{[n]_q!} (1 + e_q(-t)) = e_q(tx) COS_q(ty). \tag{2.11}
$$

The left-hand side of (2.11) can be transformed as

$$
\begin{aligned}
&\sum_{n=0}^{\infty} {}_C\mathcal{S}_{n,q}(x,y) \frac{t^n}{[n]_q!} (1 + e_q(-t)) \\
&= \sum_{n=0}^{\infty} \left({}_C\mathcal{S}_{n,q}(x,y) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} {}_C\mathcal{S}_{k,q}(x,y) \right) \frac{t^n}{[n]_q!}
\end{aligned} \tag{2.12}
$$

and the right-hand side of (2.11) can be changed such as

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left({}_{C}\mathcal{S}_{n,q}(x,y)+\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q(-1)^{n-k}{}_{C}\mathcal{S}_{k,q}(x,y)\right)\frac{t^n}{[n]_q!} \\
&=e_q(tx)COS_q(ty)=\sum_{n=0}^{\infty}C_{n,q}(x,y)\frac{t^n}{[n]_q!}.
\end{aligned} \tag{2.13}$$

Therefore, we find the required result by the comparison of coefficients. □

Corollary 2.6. *In Theorem 2.5, the following holds*

$$C_n(x,y)={}_{C}\mathcal{S}_n(x,y)+\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}{}_{C}\mathcal{S}_k(x,y).$$

Corollary 2.7. *From Theorem 2.5, one holds*

$$2{}_{C}\mathcal{S}_{n,q}(x,y)-{}_{C}\mathcal{E}_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q({}_{C}\mathcal{E}_{k,q}(x,y)-2(-1)^{n-k}{}_{C}\mathcal{S}_{k,q}(x,y)),$$

where ${}_{C}\mathcal{E}_{n,q}(x,y)$ is the q-cosine Euler polynomials.

Theorem 2.8. *For $|q|<1$, we find*

$$\frac{\partial}{\partial x}{}_{C}\mathcal{S}_{n,q}(x,y)=[n]_q{}_{C}\mathcal{S}_{n-1,q}(x,y). \tag{2.14}$$

Proof. Using partial q-derivative for q-cosine sigmoid polynomials, we have

$$\begin{aligned}
\frac{\partial}{\partial x}\sum_{n=0}^{\infty}{}_{C}\mathcal{S}_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\frac{1}{1+e_q(-t)}COS_q(ty)\frac{\partial}{\partial x}e_q(tx) \\
&=\frac{t}{1+e_q(-t)}e_q(tx)COS_q(ty) \\
&=\sum_{n=0}^{\infty}[n]_q{}_{C}\mathcal{S}_{n-1,q}(x,y)\frac{t^n}{[n]_q!}.
\end{aligned} \tag{2.15}$$

Thus, we have the required result. □

Theorem 2.9. *For $|q|<1$ and $k(\geq 0)\in\mathbb{Z}$, we investigate*

$$ {}_{C}\mathcal{S}_{n,q}(x,y)=\sum_{k=0}^{[\frac{n}{2}]}\begin{bmatrix} n \\ 2k \end{bmatrix}_q(-1)^k q^{(2k-1)k}y^{2k}\mathcal{S}_{n-k,q}(x), \tag{2.16}$$

where $[x]$ is the greatest integer not exceeding x.

Proof. From the generating function of q-cosine sigmoid polynomials, we can find a relation as

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{C}\mathcal{S}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{1+e_q(-t)}e_q(tx)COS_q(ty)\\ &= \sum_{n=0}^{\infty}\mathcal{S}_{n,q}(x)\frac{t^n}{[n]_q!}COS_q(ty),\end{aligned} \tag{2.17}$$

where $\mathcal{S}_{n,q}(x)$ is the q-sigmoid polynomials.
Applying $COS_q(x) = \sum_{n=0}^{\infty}(-1)^n q^{(2n-1)n}\frac{x^{2n}}{[2n]_q!}$ in (2.17), we rewrite (2.18) such as

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_{C}\mathcal{S}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\ &= \sum_{n=0}^{\infty}\mathcal{S}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(-1)^n q^{(2n-1)n}y^{2n}\frac{t^{2n}}{[2n]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k q^{(2k-1)k}\mathcal{S}_{n-k,q}(x)y^{2k}\right)\frac{t^{n+k}}{[n-k]_q![2k]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}\begin{bmatrix} n\\ 2k\end{bmatrix}_q(-1)^k q^{(2k-1)k}y^{2k}\mathcal{S}_{n-k,q}(x)\right)\frac{t^n}{[n]_q!}.\end{aligned} \tag{2.18}$$

From the above equation, we obtain Theorem 2.9. □

Corollary 2.10. *Putting $y = 1$ in Theorem 2.9, the following holds*

$${}_{C}\mathcal{S}_{n,q}(x,1) = \sum_{k=0}^{[\frac{n}{2}]}\begin{bmatrix} n\\ 2k\end{bmatrix}_q(-1)^k q^{(2k-1)k}\mathcal{S}_{n-k,q}(x).$$

Corollary 2.11. *Considering $q \to 1$ in Theorem 2.9, one holds*

$${}_{C}\mathcal{S}_{n}(x,y) = \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}(-1)^k y^{2k}\mathcal{S}_{n-k}(x).$$

3. Some special properties of q-cosine signoid polynomials

In this section, we derive some special identities of q-cosine sigmoid polynomials. We can find various properties by applying some formulae of q-exponential functions and q-trigonometric functions.

Lemma 3.1. *Let $|q| < 1$ and $a, x \in \mathbb{R}$. Then, we have*

$$\begin{aligned}&(i)\quad {}_{C}\mathcal{S}_{n,q}((x\oplus a)_q,y) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{n-k}{2}}\,{}_{C}\mathcal{S}_{k,q}(x,y)a^{n-k},\\ &(ii)\quad {}_{C}\mathcal{S}_{n,q}((x\ominus a)_q,y) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q(-1)^{n-k}q^{\binom{n-k}{2}}\,{}_{C}\mathcal{S}_{k,q}(x,y)a^{n-k}.\end{aligned} \tag{3.1}$$

Proof. (i) If we substitute $(x \oplus a)_q$ instead of x, then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_C\mathcal{S}_{n,q}((x \oplus a)_q, y)\frac{t^n}{[n]_q!} \\
&= \frac{1}{1+e_q(-t)} e_q((x \oplus a)_q t) COS_q(ty) \\
&= \frac{1}{1+e_q(-t)} e_q(tx)E_q(ta)COS_q(ty) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} {}_C\mathcal{S}_{k,q}(x,y)a^{n-k}\right)\frac{t^n}{[n]_q!}.
\end{aligned} \tag{3.2}$$

Comparing the coefficients of the both sides in the (3.2), we find the required result.
(ii) Similarly, consider $(x \ominus a)_q$ in the q-cosine sigmoid polynomials. Then we have

$$\sum_{n=0}^{\infty} {}_C\mathcal{S}_{n,q}((x \ominus a)_q, y)\frac{t^n}{[n]_q!} = \frac{1}{1+e_q(-t)} e_q(tx)E_q(-ta)COS_q(ty). \tag{3.3}$$

From (3.3), we obtain the alternative finite summation which is the required result. ☐

Theorem 3.2. *Let $|q| < 1$ and $a, x \in \mathbb{R}$. Then we obtain*

$$\begin{aligned}
&{}_C\mathcal{S}_{n,q}((x \oplus a)_q, y) + {}_C\mathcal{S}_{n,q}((x \ominus a)_q, y) \\
&= \begin{cases} 2\sum_{k=0}^{n} \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_q q^{\binom{n-(2k+1)}{2}} {}_C\mathcal{S}_{2k+1,q}(x,y)a^{n-(2k+1)}, & \textit{if} \quad n : \textit{odd} \\ 2\sum_{k=0}^{n} \begin{bmatrix} n \\ 2k \end{bmatrix}_q q^{\binom{n-2k}{2}} {}_C\mathcal{S}_{2k,q}(x,y)a^{n-2k}, & \textit{if} \quad n : \textit{even} \end{cases}.
\end{aligned} \tag{3.4}$$

Proof. Using Lemma 3.1 (i) and (ii), we derive

$$\begin{aligned}
&{}_C\mathcal{S}_{n,q}((x \oplus a)_q, y) + {}_C\mathcal{S}_{n,q}((x \ominus a)_q, y) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1+(-1)^{n-k}) q^{\binom{n-k}{2}} {}_C\mathcal{S}_{k,q}(x,y)a^{n-k}.
\end{aligned} \tag{3.5}$$

If we consider two cases when n is odd or even, then we obtain the required result in the q-cosine sigmoid polynomials. ☐

Corollary 3.3. *Setting $a=1$ in Theorem 3.2, the following holds*

$$
\begin{aligned}
&{}_C\mathcal{S}_{n,q}((x\oplus 1)_q,y)+{}_C\mathcal{S}_{n,q}((x\ominus 1)_q,y)\\
&=\begin{cases}
2\sum_{k=0}^{n}\begin{bmatrix} n\\ 2k+1\end{bmatrix}_q q^{\binom{n-(2k+1)}{2}}{}_C\mathcal{S}_{2k+1,q}(x,y), & \text{if} \quad n\ :\ odd\\[2ex]
2\sum_{k=0}^{n}\begin{bmatrix} n\\ 2k\end{bmatrix}_q q^{\binom{n-2k}{2}}{}_C\mathcal{S}_{2k,q}(x,y), & \text{if} \quad n\ :\ even
\end{cases}.
\end{aligned}
$$

Theorem 3.4. *Let $a\in\mathbb{R}$ with $|q|<1$. Then we find*

$$
{}_C\mathcal{S}_{n,q}(-1,y)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q(-1)^{n-k}q^{\binom{n-k}{2}}(C_{k,q}(a,y)-{}_C\mathcal{S}_{n,q}(a,y))a^{n-k}. \tag{3.6}
$$

Proof. Setting $x=-1$ and using a property of q-exponential function which is $e_q(x)E_q(-x)=1$, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}{}_C\mathcal{S}_{n,q}(-1,y)\frac{t^n}{[n]_q!}\\
&=\frac{1}{1+e_q(-t)}(1+e_q(-t)-1)COS_q(ty)\\
&=COS_q(ty)-\frac{1}{1+e_q(-t)}COS_q(ty)\\
&=\left(e_q(at)COS_q(ty)-\frac{1}{1+e_q(-t)}e_q(at)COS_q(ty)\right)E_q(-at).
\end{aligned}
\tag{3.7}
$$

Using $[n]_{q^{-1}}!=q^{-\binom{n}{2}}[n]_q!$ and $E_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]_{q^{-1}}!}$ in the (3.7), we find

$$
\begin{aligned}
&\sum_{n=0}^{\infty}{}_C\mathcal{S}_{n,q}(-1,y)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}(C_{n,q}(a,y)-{}_C\mathcal{S}_{n,q}(a,y))\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}a^n\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q(-1)^{n-k}q^{\binom{n-k}{2}}(C_{k,q}(a,y)-{}_C\mathcal{S}_{n,q}(a,y))a^{n-k}\right)\frac{t^n}{[n]_q!},
\end{aligned}
\tag{3.8}
$$

which is the required result. □

Corollary 3.5. *Putting $a=1$ in Theorem 3.4, the following holds*

$$
{}_C\mathcal{S}_{n,q}(-1,y)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q(-1)^{n-k}q^{\binom{n-k}{2}}(C_{k,q}(1,y)-{}_C\mathcal{S}_{n,q}(1,y)).
$$

Corollary 3.6. *Setting $q\to 1$ in Theorem 3.4, one holds*

$$
{}_C\mathcal{S}_{n}(-1,y)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(C_{k}(a,y)-{}_C\mathcal{S}_{n}(a,y))a^{n-k}.
$$

4. Some identities which are related to q-cosine sigmoid polynomials

In this section, we find some symmetric relations for q-cosine sigmoid polynomials using various ways.

Theorem 4.1. *Let $a, b(\neq 0) \in \mathbb{R}$. Then we have*

$$\begin{aligned}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} b^k {}_C\mathcal{S}_{n-k,q}\left(\frac{x}{a}, \frac{y}{a}\right) {}_C\mathcal{S}_{k,q}\left(\frac{x}{b}, \frac{y}{b}\right) \\ = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^{n-k} a^k {}_C\mathcal{S}_{n-k,q}\left(\frac{x}{b}, \frac{y}{b}\right) {}_C\mathcal{S}_{k,q}\left(\frac{x}{a}, \frac{y}{a}\right).\end{aligned} \tag{4.1}$$

Proof. Suppose that

$$A := \frac{\left(e_q(tx)COS_q(ty)\right)^2}{1+(e_q(-at))(1+e_q(-bt))}. \tag{4.2}$$

From form A, we find

$$\begin{aligned} A &= \frac{1}{1+e_q(-at)} e_q(tx)COS_q(ty) \frac{1}{1+e_q(-bt)} e_q(tx)COS_q(ty) \\ &= \sum_{n=0}^{\infty} {}_C\mathcal{S}_{n,q}\left(\frac{x}{a}, \frac{y}{a}\right) \frac{(at)^n}{[n]_q!} \sum_{n=0}^{\infty} {}_C\mathcal{S}_{n,q}\left(\frac{x}{b}, \frac{y}{b}\right) \frac{(bt)^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} b^k {}_C\mathcal{S}_{n-k,q}\left(\frac{x}{a}, \frac{y}{a}\right) {}_C\mathcal{S}_{k,q}\left(\frac{x}{b}, \frac{y}{b}\right) \right) \frac{t^n}{[n]_q!}. \end{aligned} \tag{4.3}$$

Also, we can transform the form A such as follows.

$$\begin{aligned} A &= \frac{1}{1+e_q(-bt)} e_q(tx)COS_q(ty) \frac{1}{1+e_q(-at)} e_q(tx)COS_q(ty) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^{n-k} a^k {}_C\mathcal{S}_{n-k,q}\left(\frac{x}{b}, \frac{y}{b}\right) {}_C\mathcal{S}_{k,q}\left(\frac{x}{a}, \frac{y}{a}\right) \right) \frac{t^n}{[n]_q!}. \end{aligned} \tag{4.4}$$

Comparing the coefficient in (4.3) and (4.4), we can find the desired result. □

Corollary 4.2. *If $q \to 1$ in Theorem 4.1, we have*

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k {}_C\mathcal{S}_{n-k}\left(\frac{x}{a}, \frac{y}{a}\right) {}_C\mathcal{S}_{k}\left(\frac{x}{b}, \frac{y}{b}\right) \\ = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k {}_C\mathcal{S}_{n-k}\left(\frac{x}{b}, \frac{y}{b}\right) {}_C\mathcal{S}_{k}\left(\frac{x}{a}, \frac{y}{a}\right),\end{aligned}$$

where ${}_C\mathcal{S}_n(x, y)$ is the cosine sigmoid polynomials.

Corollary 4.3. *Set $a = 1$ in Theorem 4.1. Then one holds*

$$\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q b^k{}_C\mathcal{S}_{n-k,q}\left(x,y\right){}_C\mathcal{S}_{k,q}\left(\frac{x}{b},\frac{y}{b}\right)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q b^{n-k}{}_C\mathcal{S}_{n-k,q}\left(\frac{x}{b},\frac{y}{b}\right){}_C\mathcal{S}_{k,q}\left(x,y\right).\end{aligned}$$

In [9], we can find the Definition of q-cosine Bernoulli polynomials. From the Definitions of these polynomials, we can obtain some relations which are related to q-cosine sigmoid polynomials.

Theorem 4.4. *Let $e_q(-t) \neq -1$ and $t \neq 0$. Then, we derive*

$$\begin{aligned}&[n]_{q C}\mathcal{S}_{n-1,q}(x,y)+{}_CB_{n,q}(x,y)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left({}_CB_{k,q}(x,y)+(-1)^{n-k-1}[k]_{q C}\mathcal{S}_{k-1,q}(x,y)\right),\end{aligned}\tag{4.5}$$

where ${}_CB_{n,q}(x,y)$ is the q-cosine Bernoulli polynomials.

Proof. To find a relation between q-cosine sigmoid polynomials and q-cosine Bernoulli polynomials, we transform q-cosine sigmoid polynomials such as

$$\sum_{n=0}^{\infty}{}_C\mathcal{S}_{n,q}(x,y)\frac{t^n}{[n]_q!}=\frac{e_q(t)-1}{t(1+e_q(-t))}\sum_{n=0}^{\infty}{}_CB_{n,q}(x,y)\frac{t^n}{[n]_q!}.\tag{4.6}$$

From the above equation, we find

$$\begin{aligned}&\sum_{n=0}^{\infty}{}_C\mathcal{S}_{n,q}(x,y)\frac{t^{n+1}}{[n]_q!}\left(\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{[n]_q!}+1\right)\\&=\sum_{n=0}^{\infty}{}_CB_{n,q}(x,y)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}-1\right).\end{aligned}\tag{4.7}$$

The left-hand side of (4.7) can be transformed as

$$\begin{aligned}&\sum_{n=0}^{\infty}{}_C\mathcal{S}_{n,q}(x,y)\frac{t^{n+1}}{[n]_q!}\left(\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{[n]_q!}+1\right)\\&=\sum_{n=0}^{\infty}[n]_{q C}\mathcal{S}_{n-1,q}(x,y)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{[n]_q!}+1\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{n-k}[k]_{q C}\mathcal{S}_{k-1,q}(x,y)+[n]_{q C}\mathcal{S}_{n-1,q}(x,y)\right)\frac{t^n}{[n]_q!}.\end{aligned}\tag{4.8}$$

The right-hand side of (4.7) can be changed as

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_C B_{n,q}(x,y)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}-1\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q {}_C Bk,q(x,y)-{}_C B_{n,q}(x,y)\right)\frac{t^n}{[n]_q!}.\end{aligned}\tag{4.9}$$

Comparing coefficients of (4.8) and (4.9), we find the required result. □

Corollary 4.5. *If $q \to 1$ in Theorem 4.4, one holds*

$$n_C\mathcal{S}_{n-1}(x,y)+{}_C B_n(x,y)=\sum_{k=0}^{n}\binom{n}{k}\left({}_C B_k(x,y)+(-1)^{n-k-1}k_C\mathcal{S}_{k-1}(x,y)\right),$$

where ${}_C B_n(x,y)$ is the cosine Bernoulli polynomials.

References

1. H. Exton, *q-Hypergeometric Functions and Applications*, New York: Halstead Press, Chichester: Ellis Horwood, 1983.
2. J. Han, C. Moraga, *The influence of the sigmoid function parameters on the speed of backpropagation learning*, International Workshop on Artificial Neural Networks doi: https://doi.org/10.1007/3 − 540 − 59497 − 3175.
3. J. Han, R.S. Wilson, S.E. Leurgans, *Sigmoidal mixed models for longitudinal data*, Statistical Methods in Medical Research doi: https://doi.org/10.1177/0962280216645632.
4. Y. Ito, *Representation of functions by superpositions of a step or sigmoid function and their applications to neural network theory*, Neural Networks **4** (1991), 385-394.
5. H.K. Kwan, *Simple sigmoid-like activation function suitable for digital hardware implementation*, Electronics Letters **28** (1992), doi: 10.1049/el:19920877
6. H.F. Jackson, *q-Difference equations*, Am. J. Math. **32** (1910), 305-314.
7. V. Kac, P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, 2002.
8. T. Kim, C.S. Ryoo, *Some identities for Euler and Bernoulli polynomials and their zeros*, Axioms **7** (2018), 1-19.
9. J.Y. Kang, C.S. Ryoo, *Various structures of the roots and explicit properties of q-cosine Bernoulli Polynomials and q-sine Bernoulli Polynomials*, Mathematics **8** (2020), 1-18.
10. J.Y. Kang, *Some relationships between sigmoid polynomials and other polynomials*, J. Appl. & Pure Math. **1** (2019), 57-67.
11. J.Y. Kang, *Some Properties and Distribution of the Zeros of the q-Sigmoid Polynomials*, Discrete Dynamics in Nature and Society **2020** (2020), 1-10.

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