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CO-FUZZY ANNIHILATOR FILTERS IN DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce the concept of relative co-fuzzy annihilator filters in distributive lattices. We give a set of equivalent conditions for a co-fuzzy annihilator to be a fuzzy filter and we characterize distributive lattices with the help of co-fuzzy annihilator filters. Furthermore, using the concept of relative co-fuzzy annihilators, we prove that the class of fuzzy filters of distributive lattices forms a Heyting algebra. We also study co-fuzzy annihilator filters. It is proved that the set of all co-fuzzy annihilator filters forms a complete Boolean algebra.

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1. Introduction

The theory of pseudo-complementation was introduced and extensively studied in semi-lattices and particularly in distributive lattices by O. Frink [11] and G. Birkhoff [10]. The pseudo-complement "a^*" of an element "a" is the greatest element disjoint from "a", if such an element exists. A lattice is said to be relatively pseudo-complemented if for every pair of elements "a" and "b" there exists a l.u.b. of $\{x : a \wedge x \leq b\}$ called the pseudo-complement of "a" relative to "b" and denoted by "a^*b". In 1970, M. Mandelker [13] introduced the concept of annihilators $\langle a, b\rangle$ of "a" relative to "b" as a natural generalization of relative pseudo-complement. That is, $\langle a, b\rangle = \{x : a \wedge x \leq b\}$. The greatest element of $\langle a, b\rangle$, if it exists, is the relative pseudo-complement a^*b. A lattice is relatively pseudo-complemented if and only if each annihilator has a greatest element, and hence is a principal ideal. He also characterized distributive lattices with the help of annihilators. Latter, properties of annihilators extensively studied by many authors, particularly T.P Speed in the papers [18] and [19]. In [15], M.S.

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Rao and A.E. Badawy studied the concept of co-annihilator filters in distributive lattices as a dual of the concept of annihilator ideals.

In 1965 Zadeh [22] mathematically formulated the fuzzy subset concept. He defined fuzzy subset of a non-empty set as a collection of objects with grade of membership in a continuum, with each object being assigned a value between 0 and 1 by a membership function. Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate or useful notions in describing the real life problems, because every object encountered in this real physical world carries some degree of fuzziness.

In 1971, A. Rosenfeld used the notion of a fuzzy subset of a set to introduce the concept of a fuzzy subgroup of a group [16]. His paper inspired the development of fuzzy abstract algebra. Since then, several authors have developed interesting results on fuzzy theory (see [2]-[8], [9, 12, 16, 17, 20, 21]).

In this paper, the concept of relative co-fuzzy annihilator filters is introduced in distributive lattices as a dual of the concept of fuzzy annihilator ideals studied by B.A. Alaba and W.Z. Norahun [1]. We give a set of equivalent conditions for a co-fuzzy annihilator to be a fuzzy filter and we characterize distributive lattice with the help of co-fuzzy annihilator filters. Basic properties of relative fuzzy annihilator filters also studied. We characterize relative co-fuzzy annihilators in terms of fuzzy points. Furthermore, using the concept of relative co-fuzzy annihilator, we prove that the class of fuzzy filters of distributive lattices forms a Heyting algebra. We also study co-fuzzy annihilator filters in distributive lattices. Basic properties of co-fuzzy annihilator filters also studied. It is proved that the set of all co-fuzzy annihilator filters forms a complete Boolean algebra.

2. Preliminaries

We refer to G. Birkhoff [10] for the elementary properties of lattices.

For nonempty subsets A and B of a lattice L the set

$$\langle A, B\rangle = \{x \in L : x \wedge a \in B \texttt{ for all } a \in A\}.$$

If $A = \{a\}$, we write $\langle a, B\rangle$ and if $B = \{0\}$ we write A^* instead of $\langle A, \{0\}\rangle$ and $(a]^*$ instead of $\{a\}^*$. That is,

$$(a]^* = \{x \in L : x \wedge a = 0\}.$$

$\langle a, b\rangle$ denotes $\langle \{a\}, \{b\}\rangle$. As observed by Mandelker [13], $\langle a, b\rangle = \{x \in L : x \wedge a \leq b\}$ is an ideal if and only if L is a distributive lattice. $\langle a, b\rangle$ is called relative annihilator of a and b (the annihilator of a relative to b). When B is an ideal it is also clear that $\langle A, B\rangle$ is an ideal. In general, let A be a nonempty subset of L and I be an ideal of L. Then the ideal $\langle A, I\rangle$ is called the annihilator of A relative to I. If $I = \{0\}$, then $\langle A, \{0\}\rangle = A^*$ is annihilator of A.

It can be easily defined for filters dually as follows:

For nonempty subsets A and B of a lattice L the set

$$(A : B)^+ = \{x \in L : x \vee a \in B \texttt{ for all } a \in A\}.$$

If $A = \{a\}$, we write $(a : B)^+$ and if $B = \{1\}$ we write A^+ instead of $(A : \{1\})^+$ and $(a)^+$ instead of $\{a\}^+$. That is,

$$(a)^+ = \{x \in L : x \vee a = 1\}.$$

For any nonempty subset A of L, A^+ is called the co-annihilator of A [15]. Clearly $L^+ = \{1\}$ and $(1)^+ = L$. For any subset A of a distributive lattice L, it is clear that A^+ is a filter in L.

For any $a, b \in L$ the co-annihilator of a relative to b defined as:

$$(a : b)^+ = \{x \in L : x \vee a \geq b\}.$$

It can be easily observed that, $(a : b)^+$ is a filter if and only if L is a distributive lattice.

Definition 2.1. [22] Let X be any nonempty set. A mapping $\mu : X \longrightarrow [0, 1]$ is called a fuzzy subset of X.

We often write $\wedge$ for minimum or infimum and $\vee$ for maximum or supremum. That is, for all $\alpha, \beta \in [0, 1]$ we have, $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$. The unit interval $[0, 1]$ together the operations min and max form a complete lattice satisfying the infinite meet distributive law; i.e.,

$$\alpha \wedge (\bigvee_{\beta \in M} \beta) = \bigvee_{\beta \in M} (\alpha \wedge \beta)$$

for all $\alpha \in [0, 1]$ and any $M \subseteq [0, 1]$.

We often write $\wedge$ for minimum or infimum and $\vee$ for maximum or supremum. That is, for all $\alpha, \beta \in [0, 1]$ we have, $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$.

The characteristic function of any subset A of X is defined as:

$$\chi_A(x) = \begin{cases} 1 \text{ , if } x \in A \\ 0 \text{ , if } x \notin A. \end{cases}$$

Definition 2.2. [14] Let $Y \subseteq X$ and $\alpha \in [0, 1]$. Define $\alpha_Y \in [0, 1]^X$ as follows:

$$\alpha_Y(x) = \begin{cases} \alpha \text{ , if } x \in Y \\ 0 \text{ , if } x \in X - Y \end{cases}$$

In particular, if Y is a singleton, say $\{y\}$, then $\alpha_{\{y\}}$ is called a fuzzy point (or fuzzy singleton), and is sometimes denoted by y_α.

Definition 2.3. [16] Let μ and θ be fuzzy subsets of a set A. Define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of A as follows: for each $x \in A$,

$$(\mu \cup \theta)(x) = \mu(x) \vee \theta(x) \text{ and } (\mu \cap \theta)(x) = \mu(x) \wedge \theta(x).$$

Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ, respectively.

For any collection, $\{\mu_i : i \in I\}$ of fuzzy subsets of X, where I is a nonempty index set, the least upper bound $\bigcup_{i \in I} \mu_i$ and the greatest lower bound $\bigcap_{i \in I} \mu_i$ of the μ_i's are given by for each $x \in X$,

$$(\bigcup_{i \in I} \mu_i)(x) = \bigvee_{i \in I} \mu_i(x) \text{ and } (\bigcap_{i \in I} \mu_i)(x) = \bigwedge_{i \in I} \mu_i(x),$$

respectively.

For each $t \in [0,1]$, the set

$$\mu_t = \{x \in A : \mu(x) \geq t\}$$

is called the level subset of μ at t [22].

Definition 2.4. [20] A fuzzy subset μ of a bounded lattice L is said to be a fuzzy filter of L, if for all $x, y \in L$

(1) $\mu(1) = 1$
(2) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$
(3) $\mu(x \vee y) \geq \mu(x) \vee \mu(y)$

In [20], Swamy and Raju observed that, a fuzzy subset μ of a lattice L is a fuzzy filter of L if and only if

$$\mu(1) = 1 \text{ and } \mu(x \wedge y) = \mu(x) \wedge \mu(y) \text{ for all } x, y \in L.$$

A fuzzy filter μ of L is said to be a proper fuzzy filter if there exists $x \in L$ such that $\mu(x) \neq 1$.

Let μ be a fuzzy subset of a lattice L. The smallest fuzzy filter of L containing μ is called a fuzzy filter of L generated by μ and denoted by $[\mu)$ and

$$[\mu) = \bigcap\{\theta \in FF(L) : \mu \subseteq \theta\}.$$

Define binary operations "+" and "·" on the set of all fuzzy subsets of a distributive lattice L as:

$$(\mu + \theta)(x) = Sup\{\mu(y) \wedge \theta(z) : y, z \in L,\ y \vee z = x\} \text{ and}$$
$$(\mu \cdot \theta)(x) = Sup\{\mu(y) \wedge \theta(z) : y, z \in L,\ y \wedge z = x\}.$$

If μ and θ are fuzzy ideals of L, then $\mu \cdot \theta = \mu \wedge \theta = \mu \cap \theta$ and $\mu + \theta = \mu \vee \theta$.

If μ and θ are fuzzy filters of L, then $\mu + \theta = \mu \wedge \theta$ and $\mu \cdot \theta = \mu \vee \theta$.

The set of all fuzzy filters of L is denoted by $FF(L)$.

Note that a fuzzy subset μ of L is nonempty if there exists $x \in L$ such that $\mu(x) \neq 0$.

3. Relative co-fuzzy annihilators

In this section, we introduce the concept of relative co-fuzzy annihilator filters in distributive lattices. We give a set of equivalent conditions for a co-fuzzy annihilator to be a fuzzy filter and we characterize distributive lattice with the help of co-fuzzy annihilator filters. Basic properties of relative co-fuzzy annihilator filters also studied. We also characterize relative co-fuzzy annihilators in terms of fuzzy points. Finally, we prove that the set of all fuzzy filters of a distributive lattice forms a Heyting algebra.

Throughout the rest of this paper L stands for the distributive lattice with 1 unless otherwise mentioned.

Definition 3.1. Let μ be a nonempty fuzzy subset of L and θ be a fuzzy filter of L. The co-fuzzy annihilator of μ relative to θ is denoted by $(\mu : \theta)^+$ and defined as:

$$(\mu:\theta)^{+}=\bigcup\{\eta:\eta\in[0,1]^{L},\ \eta+\mu\subseteq\theta\}.$$

Lemma 3.2. *Let μ be a fuzzy subset of L. Then a fuzzy subset $\bar{\mu}$ of L defined as:*

$$\bar{\mu}(x)=Sup\{\alpha\in[0,1]:x\in[\mu_\alpha)\}\ \textit{for all}\ x\in L$$

is a fuzzy filter of L generated by μ.

Proof. Let μ be any fuzzy subset of L. First, we need to show $\bar{\mu}$ is a fuzzy filter of L. Since $[\mu_t)$ is a filter of L for all $t\in[0,1]$, we have $1\in[\mu_t)$ and $\bar{\mu}(1)=1$. For any $x,y\in L$,

$$\begin{aligned}\bar{\mu}(x)\wedge\bar{\mu}(y) &= Sup\{\alpha\in[0,1]:x\in[\mu_\alpha)\}\wedge Sup\{\beta\in[0,1]:y\in[\mu_\beta)\}\\ &= Sup\{\alpha\wedge\beta\in[0,1]:x\in[\mu_\alpha),\ y\in[\mu_\beta)\}\end{aligned}$$

Put $t=\alpha\wedge\beta$. Then $t\le\alpha,\ t\le\beta$ and $[\mu_\alpha)\subseteq[\mu_t),\ [\mu_\beta)\subseteq[\mu_t)$. Since $x\in[\mu_\alpha)$ and $y\in[\mu_\beta)$, we have that $x,y\in[\mu_t)$ and $x\wedge y\in[\mu_t)$. Using this fact we have,

$$\begin{aligned}\bar{\mu}(x)\wedge\bar{\mu}(y) &\le Sup\{t\in[0,1]:x\wedge y\in[\mu_t)\}\\ &= \bar{\mu}(x\wedge y)\end{aligned}$$

On the other hand,

$$\begin{aligned}\bar{\mu}(x)=Sup\{\alpha\in[0,1]:x\in[\mu_\alpha)\} &\le Sup\{\alpha\in[0,1]:x\vee y\in[\mu_\alpha)\}\\ &=\bar{\mu}(x\vee y).\end{aligned}$$

Similarly, $\bar{\mu}(y)\le\bar{\mu}(x\vee y)$. This shows that $\bar{\mu}(x\vee y)\ge\bar{\mu}(x)\vee\bar{\mu}(y)$. Thus $\bar{\mu}$ is a fuzzy filter of L.

Now we proceed to show that $\bar{\mu}$ is the smallest fuzzy filter containing μ. Clearly $\mu\subseteq\bar{\mu}$. Let θ be any fuzzy filter containing μ. Then $\mu_t\subseteq\theta_t$ for all $t\in[0,1]$. For any $x\in L$,

$$\bar{\mu}(x)=Sup\{\alpha\in[0,1]:x\in[\mu_\alpha)\}\le Sup\{\alpha\in[0,1]:x\in\theta_\alpha\}=\theta(x).$$

This shows that $\bar{\mu}(x)\le\theta(x)$ for all $x\in L$. Thus $\bar{\mu}$ is the smallest fuzzy filter containing μ. So $\bar{\mu}$ is a fuzzy filter of L generated by μ. □

Lemma 3.3. *For any two fuzzy subsets μ and θ of a distributive lattice L, we have*

$$[\mu+\theta)=[\mu)\wedge[\theta).$$

Proof. Let μ and θ be fuzzy subsets of L. Clearly we have that $\mu+\theta\subseteq[\mu)\wedge[\theta)$. Since $[\mu+\theta)$ is the smallest fuzzy filter containing $\mu+\theta$, we get that $[\mu+\theta)\subseteq[\mu)\wedge[\theta)$.

Now we proceed to show the other inclusion. In a distributive lattice L, for any subsets A and B of L, we have $[A\vee B)=[A)\wedge[B)$, where $A\vee B=\{a\vee b: a\in A,\ b\in B\}$.

Since $[\mu)$ and $[\theta)$ are fuzzy filters, we have $[\mu)\wedge[\theta)=[\mu)\cap[\theta)$. Now,

$$([\mu)\wedge[\theta))(x) = [\mu)(x)\wedge[\theta)(x)$$

$$\begin{aligned}
&= Sup\{t_1 \in [0,1] : x \in [\mu_{t_1})\} \wedge Sup\{t_2 \in [0,1] : x \in [\theta_{t_2})\} \\
&= Sup\{t_1 \wedge t_2 : x \in [\mu_{t_1}),\ x \in [\theta_{t_2})\} \\
&\leq Sup\{t : x \in [\mu_t) \wedge [\theta_t)\} \\
&\leq Sup\{t : x \in [(\mu+\theta)_t)\} \quad (\texttt{Since } \mu_t \vee \theta_t \subseteq (\mu+\theta)_t) \\
&= [\mu+\theta)(x)
\end{aligned}$$

Thus $[\mu) \wedge [\theta) \subseteq [\mu+\theta)$. So $[\mu) \wedge [\theta) = [\mu+\theta)$. □

Now we will have the following result.

Lemma 3.4. *Let μ be a nonempty fuzzy subset of L and θ be a fuzzy filter of L. Then*

$$(\mu : \theta)^+ = \bigcup\{\eta : \eta \in FF(L),\ \eta + \mu \subseteq \theta\}.$$

Proof. Clearly $\bigcup\{\eta : \eta \in FF(L),\ \eta + \mu \subseteq \theta\} \subseteq \bigcup\{\delta : \delta \in [0,1]^L,\ \delta + \mu \subseteq \theta\}$.
On the other hand,

$$\begin{aligned}
(\mu : \theta)^+(x) &= Sup\{\eta(x) : \eta \in [0,1]^L,\ \eta + \mu \subseteq \theta\} \\
&\leq Sup\{[\eta)(x) : (\eta] \in FF(L),\ [\eta) + \mu \subseteq \theta\},
\end{aligned}$$

since $[\eta + \mu) = [\eta) \wedge [\mu)$.

Thus $(\mu : \theta)^+ \subseteq \bigcup\{\eta : \eta \in FF(L),\ \eta + \mu \subseteq \theta\}$. So

$$(\mu : \theta)^+ = \bigcup\{\eta : \eta \in FF(L),\ \eta + \mu \subseteq \theta\}.$$

□

Theorem 3.5. *Let μ be a nonempty fuzzy subset of L and θ be a fuzzy filter of L. Then $(\mu : \theta)^+$ is a fuzzy filter of L.*

Proof. Let μ be a nonempty fuzzy subset of L and θ be a fuzzy filter of L. Since $\theta + \mu \subseteq \theta$ and $\theta(1) = 1$, we get that $(\mu : \theta)^+(1) = 1$.
Again for any $x, y \in L$,

$$\begin{aligned}
& (\mu : \theta)^+(x) \wedge (\mu : \theta)^+(y) \\
={}& Sup\{\eta(x) : \eta \in FF(L),\ \eta + \mu \subseteq \theta\} \\
& \wedge Sup\{\sigma(y) : \sigma \in FF(L),\ \sigma + \mu \subseteq \theta\} \\
={}& Sup\{\eta(x) \wedge \sigma(y) : \eta, \sigma \in FF(L),\ \eta + \mu \subseteq \theta,\ \sigma + \mu \subseteq \theta\} \\
\leq{}& Sup\{(\eta \vee \sigma)(x) \wedge (\eta \vee \sigma)(y) : \eta + \mu \subseteq \theta,\ \sigma + \mu \subseteq \theta\}
\end{aligned}$$

For each $\eta, \sigma \in FF(L)$ such that $\eta + \mu \subseteq \theta$ and $\sigma + \mu \subseteq \theta$, $\eta \vee \sigma \in FF(L)$ and $(\eta \vee \sigma) + \mu \subseteq \theta$. Then

$$\begin{aligned}
(\mu : \theta)^+(x) \wedge (\mu : \theta)^+(y) &\leq Sup\{\lambda(x) \wedge \lambda(y) : \lambda \in FF(L),\ \lambda + \mu \subseteq \theta\} \\
&= Sup\{\lambda(x \wedge y) : \lambda \in FF(L),\ \lambda + \mu \subseteq \theta\} \\
&= (\mu : \theta)^+(x \wedge y).
\end{aligned}$$

Thus, $(\mu : \theta)^+(x \wedge y) \geq (\mu : \theta)^+(x) \wedge (\mu : \theta)^+(y)$.

Now we show that $(\mu:\theta)^+(x) \leq (\mu:\theta)^+(x\vee y)$ and $(\mu:\theta)^+(y) \leq (\mu:\theta)^+(x\vee y)$.

$$\begin{aligned}(\mu:\theta)^+(x) &= Sup\{\eta(x):\eta\in FF(L),\ \eta+\mu\subseteq\theta\}\\ &\leq Sup\{\eta(x\vee y):\eta\in FF(L),\ \eta+\mu\subseteq\theta\}\\ &= (\mu:\theta)^+(x\vee y)\end{aligned}$$

Similarly, $(\mu:\theta)^+(y)\leq(\mu:\theta)^+(x\vee y)$.

So $(\mu:\theta)^+(x\vee y)\geq(\mu:\theta)^+(x)\vee(\mu:\theta)^+(y)$. Hence $(\mu:\theta)^+$ is a fuzzy filter of L. □

In the above theorem we have shown a relative co-fuzzy annihilator is a fuzzy filter whenever L is a distributive lattice. In the following theorem, we characterize a distributive lattice with the help of relative co-fuzzy annihilator filters.

Theorem 3.6. *For any lattice L, the following are equivalent:*
(1) L is distributive,
(2) $(\mu:\theta)^+$ is a fuzzy filter for all $\mu,\theta\in FF(L)$,
(3) $FF(L)$ is a distributive lattice.

Proof. $1\Rightarrow 2$: It is obvious.

$2\Rightarrow 3$: Assume that condition (2) holds. Let $\mu,\theta,\gamma\in FF(L)$. Now we need to show $\mu\wedge(\theta\vee\gamma)\subseteq(\mu\wedge\theta)\vee(\mu\wedge\gamma)$. Put $\lambda=(\mu\wedge\theta)\vee(\mu\wedge\gamma)$. Then by our assumption, $(\mu:\lambda)^+$ is a fuzzy filter and $(\mu:\lambda)^+=Sup\{\eta:\eta\in[0,1]^L,\ \eta+\mu\subseteq\lambda\}$.

Since $\mu\wedge\theta\subseteq\lambda$ and $\mu\wedge\gamma\subseteq\lambda$, we get that $\theta\subseteq(\mu:\lambda)^+$ and $\gamma\subseteq(\mu:\lambda)^+$. It follows that $\theta\vee\gamma\subseteq(\mu:\lambda)^+$. This shows that $\mu\wedge(\theta\vee\gamma)\subseteq\lambda$. Thus

$$\mu\wedge(\theta\vee\gamma)\subseteq(\mu\wedge\theta)\vee(\mu\wedge\gamma)\text{ for all }\mu,\theta,\gamma\in FF(L).$$

So $FF(L)$ is a distributive lattice.

$3\Rightarrow 1$: Suppose $FF(L)$ is a distributive lattice. Let $x,y,z\in L$. Then $\chi_{[x)},\chi_{[y)},\chi_{[z)}$ are fuzzy filters of L. Since $FF(L)$ is a distributive lattice, we get that

$$\chi_{[x)}\wedge(\chi_{[y)}\vee\chi_{[z)})=(\chi_{[x)}\wedge\chi_{[y)})\vee(\chi_{[x)}\wedge\chi_{[z)})$$

Which implies that $\chi_{[x\wedge(y\vee z))}=\chi_{[(x\wedge y)\vee(x\wedge z))}$. This implies that

$$[x\wedge(y\vee z))=[(x\wedge y)\vee(x\wedge z)).$$

Thus

$$x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)\text{ for all }x,y,z\in L.$$

So L is distributive. □

In the following theorem, we characterize relative co-fuzzy annihilators in terms of fuzzy points.

Theorem 3.7. *Let μ be a nonempty fuzzy subset of L and θ be a fuzzy filter of L. Then for each $x \in L$,*

$$(\mu : \theta)^+(x) = Sup\{\alpha \in [0,1] : x_\alpha + \mu \subseteq \theta\}.$$

Proof. For each $x \in L$, let us define two sets A_x and B_x as follows:

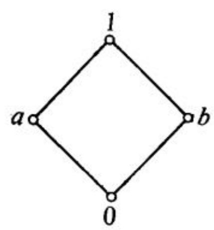
$$A_x = \{\eta(x) : \eta \in [0,1]^L,\ \eta + \mu \subseteq \theta\} \text{ and } B_x = \{\alpha \in [0,1] : x_\alpha + \mu \subseteq \theta\}.$$

Since $\theta + \mu \subseteq \theta$, then both A_x and B_x are nonempty subsets of $[0,1]$.

Now we proceed to show that $\bigvee A_x = \bigvee B_x$. Let $\alpha \in A_x$. Then $\alpha = \eta(x)$ for some fuzzy subset η of L satisfying $\eta + \mu \subseteq \theta$. If $\alpha = 0$, then we can find $\beta \in B_x$ such that $\alpha \leq \beta$. On the other hand, suppose that $\alpha \neq 0$. Then x_α is a fuzzy point of L such that $x_\alpha \subseteq \eta$. Which implies $x_\alpha + \mu \subseteq \eta + \mu$ and $\alpha \in B_x$. Thus $A_x \subseteq B_x$. So $\bigvee A_x \leq \bigvee B_x$.

To show $\bigvee B_x \leq \bigvee A_x$, let $\beta \in B_x$. Then x_β is a fuzzy point of L such that $x_\beta + \mu \subseteq \theta$. This shows that $\beta \in A_x$. Thus $B_x \subseteq A_x$. So $\bigvee B_x \leq \bigvee A_x$. Hence $\bigvee B_x = \bigvee A_x$. □

Example 3.8. Consider the distributive lattice $L = \{0, a, b, 1\}$ whose Hasse diagram is given below.



A fuzzy subset θ of L defined by $\theta(1) = 1,\ \theta(b) = \frac{1}{4},\ \theta(a) = \frac{1}{2},\ \theta(0) = \frac{1}{4}$ is a fuzzy filter. Let μ be a fuzzy subset of L defined as: $\mu(0) = \frac{1}{3},\ \mu(a) = \frac{1}{5},\ \mu(b) = \frac{1}{5},\ \mu(1) = \frac{1}{3}$. Now we can easily find the value of $(\mu : \theta)^+(x)$ for each $x \in L$. For any $\eta \in [0,1]^L$ with $\eta + \mu \subseteq \theta$, we can determine the value of $\eta(x)$. Then we have $(\mu : \theta)^+(1) = (\mu : \theta)^+(a) = 1,\ (\mu : \theta)^+(0) = (\mu : \theta)^+(b) = \frac{1}{4}$. Thus $(\mu : \theta)^+$ is a fuzzy filter of L.

In the following lemma, some basic properties of relative co-fuzzy annihilators can be observed.

Lemma 3.9. *Let η and δ be fuzzy subsets and μ, θ and λ fuzzy filters of L. Then*

(1) $(\eta : \mu)^+ = \chi_L \Leftrightarrow \eta \subseteq \mu$,
(2) $\theta \subseteq (\eta : \theta)^+$,
(3) $\eta \subseteq \delta \Rightarrow (\delta : \mu)^+ \subseteq (\eta : \mu)^+$,
(4) $\mu \subseteq \theta \Rightarrow (\delta : \mu)^+ \subseteq (\delta : \theta)^+$,
(5) $(\eta : \mu \cap \theta)^+ = (\eta : \mu)^+ \cap (\eta : \theta)^+$,
(6) $([\eta) : \mu)^+ = (\eta : \mu)^+$,
(7) $(\eta \cup \delta : \mu)^+ = (\eta : \mu)^+ \cap (\delta : \mu)^+$,

(8) $(\mu \vee \theta : \lambda)^+ = (\mu : \lambda)^+ \cap (\theta : \lambda)^+$,
(9) $(\mu : \theta)^+ = (\mu \vee \theta : \theta)^+ = (\mu : \mu \wedge \theta)^+$,
(10) $[\eta) \cap \theta \subseteq \mu \Leftrightarrow \theta \subseteq (\eta : \mu)^+$.

Proof. Let η and δ be fuzzy subsets and μ, θ and λ fuzzy filters of L.

(1) Let $(\eta : \mu)^+ = \chi_L$. We need to show $\eta \subseteq \mu$. Suppose not. Then there is $x \in L$ such that $\eta(x) > \mu(x)$. This implies that $\gamma(x) \leq \mu(x)$, for each γ such that $\gamma + \eta \subseteq \mu$. Thus $\mu(x)$ is an upper bound of $\{\gamma(x) : \gamma + \eta \subseteq \mu\}$. Which implies

$$1 = Sup\{\gamma(x) : \gamma + \eta \subseteq \mu\} \leq \mu(x).$$

So $\mu(x) \geq \eta(x)$ which is a contradiction. Hence $\eta \subseteq \mu$.

Conversely, assume that $\eta \subseteq \mu$. Then $\chi_L + \eta \subseteq \eta$. This implies that $\chi_L + \eta \subseteq \mu$. Thus $(\eta : \mu)^+ = \chi_L$.

(2) $(\eta : \theta)^+ = Sup\{\gamma : \gamma \in FF(L),\ \gamma + \eta \subseteq \theta\}$. Since $\theta \wedge [\eta) \subseteq \theta$, we have $\theta + \eta \subseteq \theta$. Thus $\theta \subseteq (\eta : \theta)^+$.

(3) $(\delta : \mu)^+ = Sup\{\gamma : \gamma \in FF(L),\ \gamma + \delta \subseteq \mu\}$. Since $\eta \subseteq \delta$, then $\lambda + \eta \subseteq \lambda + \delta$ for fuzzy filter λ of L. Thus $(\delta : \mu)^+ \subseteq (\eta : \mu)^+$.

(4) It is clear.

(5) By (4), we have $(\eta : \mu \cap \theta)^+ \subseteq (\eta : \mu)^+ \cap (\eta : \theta)^+$. On the other hand,

$$\begin{aligned} & (\eta : \mu)^+ \cap (\eta : \theta)^+ \\ = \ & Sup\{\gamma_1 : \gamma_1 \in FF(L),\ \gamma_1 + \eta \subseteq \mu\} \\ & \wedge Sup\{\gamma_2 : \gamma_2 \in FF(L),\ \gamma_2 + \eta \subseteq \theta\} \\ = \ & Sup\{\gamma_1 \wedge \gamma_2 : \gamma_1,\ \gamma_2 \in FF(L),\ \gamma_1 + \eta \subseteq \mu,\ \gamma_2 + \eta \subseteq \theta\} \end{aligned}$$

Since $\gamma_1 + \eta \subseteq \mu$ and $\gamma_2 + \eta \subseteq \theta$, we can find a fuzzy filter γ of L contained in γ_1 and γ_2 such that $\gamma + \eta \subseteq \mu$ and $\gamma + \eta \subseteq \theta$. Based on this fact we have,

$$\begin{aligned} (\eta : \mu)^+ \cap (\eta : \theta)^+ & \leq Sup\{\gamma : \gamma \in FF(L),\ \gamma + \eta \subseteq \mu \cap \theta\} \\ & = (\eta : \mu \cap \theta)^+ \end{aligned}$$

Then $(\eta : \mu)^+ \cap (\eta : \theta)^+ = (\eta : \mu \cap \theta)^+$.

(6) Since $[\gamma + \eta) = [\gamma) \wedge [\eta) = \gamma \wedge [\eta)$, for every $\gamma \in FF(L)$, we have

$$\begin{aligned} (\eta : \mu)^+ & = Sup\{\gamma : \gamma \in FF(L),\ \gamma + \eta \subseteq \mu\} \\ & = Sup\{\gamma : \gamma \wedge [\eta) \subseteq \mu\} \\ & = ([\eta) : \mu)^+ \end{aligned}$$

Then $(\eta : \mu)^+ = ([\eta) : \mu)^+$.

(7) By property (3), we have that $(\eta \cup \delta : \mu)^+ \subseteq (\eta : \mu)^+ \cap (\delta : \mu)^+$. On the other hand,

$$\begin{aligned} (\eta : \mu)^+ \cap (\delta : \mu)^+ & = ([\eta) : \mu)^+ \cap ([\delta), \mu)^+ \\ & = Sup\{\gamma_1 : \gamma_1 \in FF(L),\ \gamma_1 \wedge [\eta) \subseteq \mu\} \\ & \quad \wedge Sup\{\gamma_2 : \gamma_2 \in FF(L),\ \gamma_2 \wedge [\delta) \subseteq \mu\} \end{aligned}$$

$$= \quad Sup\{\gamma_1 \wedge \gamma_2 : \gamma_1 \wedge [\eta) \subseteq \mu,\ \gamma_2 \wedge [\delta) \subseteq \mu\}.$$

Since $\gamma_1 \wedge [\eta) \subseteq \mu$ and $\gamma_2 \wedge [\delta) \subseteq \mu$, we can find a fuzzy filter γ of L contained in γ_1 and γ_2 such that $\gamma \wedge [\eta) \subseteq \mu$ and $\gamma \wedge [\delta) \subseteq \mu$. This implies that $(\gamma \wedge ([\eta) \vee [\delta))) \subseteq \mu$. This shows that

$$\begin{aligned}(\eta : \mu)^+ \cap (\delta : \mu)^+ &\leq Sup\{\gamma : \gamma \wedge ([\eta) \vee [\delta)) \subseteq \mu\}\\ &\leq Sup\{\gamma : \gamma + (\eta \cup \delta) \subseteq \mu\}\\ &= (\eta \cup \delta : \mu)^+.\end{aligned}$$

(8) Since $\mu \vee \theta = [\mu \cup \theta)$, by (6) we get $(\mu \vee \theta : \lambda)^+ = (\mu \cup \theta : \lambda)^+$. Thus

$$(\mu \vee \theta : \lambda)^+ = (\mu : \lambda)^+ \cap (\theta : \lambda)^+.$$

(9) Since $(\theta : \theta)^+ = \chi_L$, by (8) we get $(\mu \vee \theta : \theta)^+ = (\mu : \theta)^+$. On the other hand, let $\gamma \wedge \mu \subseteq \theta$ for some fuzzy filter γ of L. Since $\gamma \wedge \mu \subseteq \mu$, we get that $\gamma \wedge \mu \subseteq \mu \wedge \theta$. Thus $(\mu : \theta)^+ \subseteq (\mu : \mu \wedge \theta)^+$. Since $\mu \wedge \theta \subseteq \theta$, by (4) we have $(\mu : \mu \wedge \theta)^+ \subseteq (\mu : \theta)^+$. Hence $(\mu : \theta)^+ = (\mu : \mu \wedge \theta)^+$. So

$$(\mu : \theta)^+ = (\mu \vee \theta : \theta)^+ = (\mu : \mu \wedge \theta)^+.$$

(10) If $[\eta) \cap \theta \subseteq \mu$, then by (6) and by the definition of relative co-fuzzy annihilator $\theta \subseteq (\eta : \mu)^+$.

Conversely, suppose $\theta \subseteq (\eta : \mu)^+$. Since θ is a fuzzy filter, we can express θ as follows:

$$\theta = \bigvee_{x_\alpha \subseteq \theta} [x_\alpha).$$

Let x_α be a fuzzy point of L such that $x_\alpha \subseteq \theta$. Since $\theta \subseteq (\eta : \mu)^+$, we get $x_\alpha \subseteq (\eta : \mu)^+$. Thus $x_\alpha + \eta \subseteq \mu$. Now,

$$\begin{aligned}[\eta) \cap \theta &= (\bigvee_{x_\alpha \subseteq \theta} [x_\alpha)) \cap [\eta)\\ &= \bigvee_{x_\alpha \subseteq \theta} [x_\alpha + \eta) \qquad \textit{by Lemma 3.3}\\ &\subseteq \mu\end{aligned}$$

Thus $[\eta) \cap \theta \subseteq \mu$. ☐

Theorem 3.10. *Let θ be a fuzzy filter of L. If $\{\mu_\alpha\}_{\alpha \in \Delta}$ is a class of fuzzy filters of L, then*

$$(\bigcup_{\alpha \in \Delta} \mu_\alpha : \theta)^+ = \bigcap_{\alpha \in \Delta} (\mu_\alpha : \theta)^+.$$

Proof. We know that $\mu_\alpha \subseteq \bigcup_{\alpha \in \Delta} \mu_\alpha$ for each $\alpha \in \Delta$. Thus by Lemma 3.9(3), we get $(\bigcup_{\alpha \in \Delta} \mu_\alpha : \theta)^+ \subseteq (\mu_\alpha : \theta)^+$ for each $\alpha \in \Delta$. Thus

$$(\bigcup_{\alpha \in \Delta} \mu_\alpha : \theta)^+ \subseteq \bigcap_{\alpha \in \Delta} (\mu_\alpha : \theta)^+.$$

On the other hand, put $\eta = \bigcap_{\alpha\in\Delta}(\mu_\alpha : \theta)^+$. Then $\eta \subseteq (\mu_\alpha : \theta)^+$ for each $\alpha \in \Delta$. By Lemma 3.9(10), we have $\mu_\alpha \cap \eta \subseteq \theta$ for each $\alpha \in \Delta$. This implies

$$(\bigvee_{\alpha\in\Delta} \mu_\alpha) \cap \eta = \bigvee_{\alpha\in\Delta} (\mu_\alpha \cap \eta) \subseteq \theta.$$

So by Lemma 3.9(10), we have $\eta \subseteq (\bigvee_{\alpha\in\Delta} \mu_\alpha : \theta)^+$. Thus

$$\bigcap_{\alpha\in\Delta} (\mu_\alpha : \theta)^+ \subseteq (\bigcup_{\alpha\in\Delta} \mu_\alpha : \theta)^+.$$

So

$$(\bigcup_{\alpha\in\Delta} \mu_\alpha : \theta)^+ = \bigcap_{\alpha\in\Delta} (\mu_\alpha : \theta)^+.$$

□

In the following theorem we prove that, $(\mu : \theta)^+$ is a relative pseudo-complement of μ and θ in the class of $FF(L)$.

Theorem 3.11. *Let η be fuzzy subset and μ and θ be fuzzy filters of L. Then*

(1) *$(\eta : \mu)^+$ is the largest fuzzy filter such that $[\eta) \cap (\eta : \mu)^+ \subseteq \mu$,*
(2) *$(\mu : \theta)^+$ is the largest fuzzy filter such that $\mu \cap (\mu : \theta)^+ \subseteq \theta$.*

Proof. First we have to show $[\eta) \cap (\eta : \mu)^+ \subseteq \mu$. For any $x \in L$ we have,

$$\begin{aligned}
([\eta) \cap (\eta : \mu)^+)(x) &= [\eta)(x) \wedge ([\eta) : \mu)^+(x) \\
&= [\eta)(x) \wedge Sup\{\lambda(x) : \lambda \in FF(L),\ \lambda \wedge [\eta) \subseteq \mu\} \\
&= Sup\{[\eta)(x) \wedge \lambda(x) : \lambda \in FF(L),\ \lambda \wedge [\eta) \subseteq \mu\} \\
&= Sup\{([\eta) \cap \lambda)(x) : \lambda \in FI(L),\ \lambda \cap [\eta) \subseteq \mu\} \\
&\leq \mu(x)
\end{aligned}$$

This implies $([\eta) \cap (\eta : \mu)^+(x) \leq \mu(x)$ for each $x \in L$. Thus $[\eta) \cap (\eta : \mu)^+) \subseteq \mu$.

Now we show that $(\eta : \mu)^+$ is the largest fuzzy filter such that $[\eta) \cap (\eta : \mu)^+ \subseteq \mu$. Suppose not. Then there exists a fuzzy filter λ containing $(\eta : \mu)^+$ such that $[\eta) \cap \lambda \subseteq \mu$. Then by lemma 3.9 (10) we get that $\lambda \subseteq (\eta : \mu)^+$. Which is a contradiction. Therefore, $(\eta : \mu)^+$ is the largest fuzzy filter such that $[\eta) \cap (\eta : \mu)^+ \subseteq \mu$. □

The concept of fuzzy filters of a lattice have been studied by different scholars, but they observed that the class of all fuzzy filters of a lattice can be made a complete distributive lattice. Here by the presence of co-fuzzy annihilator filters of a distributive lattice we observe that, the class of all fuzzy filters of a distributive lattice forms a Heyting algebra.

Theorem 3.12. *The set $FF(L)$ of all fuzzy filters of L is a Heyting algebra.*

Proof. We know that the set $(FF(L), \vee, \cap, \chi_{\{1\}}, \chi_L)$ of all fuzzy filters of L is a complete distributive lattice. For any fuzzy filters μ and θ of L, by Theorem 3.11, $(\mu : \theta)^+$ is the largest fuzzy filter of $\{\lambda \in FF(L) : \lambda \cap \mu \subseteq \theta\}$. Thus

$$\mu \longrightarrow \theta = (\mu : \theta)^+.$$

So $(FF(L), \vee, \cap, \longrightarrow, \chi_{\{1\}}, \chi_L)$ is a Heyting algebra. □

4. Co-fuzzy annihilator

In this section, we study co-fuzzy annihilator filters in distributive lattices. Some basic properties of co-fuzzy annihilator filters also studied. It is proved that the set of all co-fuzzy annihilator filters forms a complete Boolean algebra.

Definition 4.1. For any nonempty fuzzy subset μ of L. The fuzzy filter $(\mu : \chi_{\{1\}})^+$ is denoted by μ^+ and μ^+ is called co-fuzzy annihilator of μ.

Lemma 4.2. *Let μ be a nonempty fuzzy subset of L. Then*

(1) $\chi_{\{1\}} \subseteq \mu^+$,
(2) $\mu + \mu^+ \subseteq \chi_{\{1\}}$,
(3) $\mu + \mu^+ = \chi_{\{1\}}$, *whenever* $\mu(1) = 1$,
(4) $\mu^+ \cap \mu^{++} = \chi_{\{1\}}$.

Proof. Here it is enough to prove property (3). Let μ be a nonempty fuzzy subset of L. For any $x \in L$,

$$\begin{aligned} & (\mu + \mu^+)(x) \\ = \;& Sup\{\mu(a) \wedge \mu^+(b) : x = a \vee b\} \\ = \;& Sup\{\mu(a) \wedge Sup\{\eta(b) : \eta \in FF(L),\ \eta + \mu \subseteq \chi_{\{1\}}\} :\ x = a \vee b\} \\ = \;& Sup\{Sup\{\mu(a) \wedge \eta(b) : \eta \in FF(L),\ \eta + \mu \subseteq \chi_{\{1\}}\} :\ x = a \vee b\} \\ = \;& Sup\{Sup\{\mu(a) \wedge \eta(b) : x = a \vee b\} :\ \eta \in FF(L),\ \eta + \mu \subseteq \chi_{\{1\}}\} \\ = \;& Sup\{(\mu + \eta)(x) : \eta + \mu \subseteq \chi_{\{1\}}\} \\ \leq \;& \chi_{\{1\}}(x) \end{aligned}$$

This shows that $\mu + \mu^+ \subseteq \chi_{\{1\}}$. If $\mu(1) = 1$, then $(\mu + \mu^+)(1) = 1$ and $\chi_{\{1\}} = \mu + \mu^+$. □

The proof of the following lemmas are quite routine and will be omitted.

Lemma 4.3. *Let μ and θ be nonempty fuzzy subsets of L. Then*

(1) $\mu \subseteq \theta \Rightarrow \theta^+ \subseteq \mu^+$,
(2) $\theta + \mu \subseteq \chi_{\{1\}} \Leftrightarrow \theta \subseteq \mu^+$,
(3) $\theta + \mu = \chi_{\{1\}} \Leftrightarrow \theta \subseteq \mu^+$, *whenever* $\mu(1) = 1 = \theta(1)$,
(4) $\mu \subseteq \mu^{++}$,
(5) $\mu^+ = \mu^{+++}$.

Lemma 4.4. *Let μ and θ be fuzzy filters of L. Then*

(1) $(\chi_{\{1\}})^+ = \chi_L$,
(2) $(\chi_L)^+ = \chi_{\{1\}}$,
(3) $(\mu \vee \theta)^+ = \mu^+ \cap \theta^+$,
(4) $(\mu \vee \mu^+)^+ = \chi_{\{1\}}$,

(5) $\mu^+ = \chi_L \Leftrightarrow \mu = \chi_L$.

Theorem 4.5. *The set $FF(L)$ of all fuzzy filters of L is a pseudo-complemented lattice.*

Proof. Let μ be a fuzzy filter of L. Then it is clear that μ^+ is a fuzzy filter of L and that $\mu \cap \mu^+ = \chi_{\{1\}}$. Suppose now $\theta \in FF(L)$ such that $\mu \cap \theta = \chi_{\{1\}}$. Then by Lemma 4.3(2), $\theta \subseteq \mu^+$ and consequently μ^+ is the pseudo-complement of μ. □

Lemma 4.6. *If $\mu_i \in [0,1]^L$ for every $i \in I$, then*

$$(\bigcup_{i\in I} \mu_i)^+ = \bigcap_{i\in I} \mu_i^+.$$

Proof. Let $\{\mu_i : i \in I\}$ be family of fuzzy subsets of L. Since $\mu_i \subseteq (\bigcup_{i\in I} \mu_i)$ for each $i \in I$, by Lemma 4.3(1), we have $(\bigcup_{i\in I} \mu_i)^+ \subseteq \mu_i^+$. Thus

$$(\bigcup_{i\in I} \mu_i)^+ \subseteq \bigcap_{i\in I} \mu_i^+.$$

To prove $\bigcap_{i\in I} \mu_i^+ \subseteq (\bigcup_{i\in I} \mu_i)^+$ it is enough to show that $(\bigcap_{i\in I} \mu_i^+) + (\bigcup_{j\in I} \mu_j) \subseteq \chi_{\{1\}}$. For any $x \in L$,

$$\begin{aligned}
((\bigcap_{i\in I} \mu_i^+) + (\bigcup_{j\in I} \mu_j))(x) &= Sup\{(\bigcap_{i\in I} \mu_i^+)(a) \wedge (\bigcup_{j\in I} \mu_j)(b) : a \vee b = x\} \\
&= Sup\{(\bigwedge_{i\in I} \mu_i^+(a)) \wedge (\bigvee_{j\in I} \mu_j(b)) : a \vee b = x\} \\
&= Sup\{\bigvee_{j\in I} ((\bigwedge_{i\in I} \mu_i^+(a)) \wedge \mu_j(b))) : a \vee b = x\} \\
&\leq Sup\{\bigvee_{j\in I} ((\mu_j^+(a) \wedge \mu_j(b))) : a \vee b = x\} \\
&\leq Sup\{\bigvee_{j\in I} ((\mu_j^+ + \mu_j)(x)) : a \vee b = x\} \\
&\leq \chi_{\{1\}}(x)
\end{aligned}$$

Thus by Lemma 4.3(2), we get that $(\bigcap_{i\in I} \mu_i^+) \subseteq (\bigcup_{i\in I} \mu_i)^+$. So

$$(\bigcap_{i\in I} \mu_i^+) = (\bigcup_{i\in I} \mu_i)^+.$$

□

Definition 4.7. A fuzzy filter μ of L is called a direct factor of L if there exists a proper fuzzy filter θ such that $\mu \cap \theta = \chi_{\{1\}}$ and $\mu \vee \theta = \chi_L$.

Now we give the definition of co-fuzzy annihilator filter.

Definition 4.8. A fuzzy filter μ of L is called a co-fuzzy annihilator filter, if $\mu = \theta^+$, for some nonempty fuzzy subset θ of L, or equivalently, if $\mu = \mu^{++}$.

We denote the class of all co-fuzzy annihilator filters of L by $FF^+(L)$.

Lemma 4.9. *Let $\mu, \theta \in FF^+(L)$. Then*

(1) $\mu \cap \theta = (\mu^+ \vee \theta^+)^+$,
(2) $\mu \cap \theta = (\mu \cap \theta)^{++}$.

The result (2) of the above lemma can be generalized as given in the following. If $\{\mu_i : i \in \Delta\}$ is a family of co-fuzzy annihilator filters of L, then

$$(\bigcap_{i\in\Delta} \mu_i)^{++} = \bigcap_{i\in\Delta} \mu_i.$$

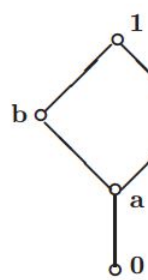
Theorem 4.10. *A map $\alpha : FF(L) \longrightarrow FF(L)$ defined by $\alpha(\mu) = \mu^{++}$, $\forall \mu \in FF(L)$ is a closure operator on $FF(L)$. That is,*

(1) $\mu \subseteq \alpha(\mu)$,
(2) $\alpha(\alpha(\mu)) = \alpha(\mu)$,
(3) $\mu \subseteq \theta \Rightarrow \alpha(\mu) \subseteq \alpha(\theta)$, *for any two fuzzy filters μ, θ of L.*

Co-fuzzy annihilator filters are simply the closed elements with respect to the closure operator α.

We know that $(FF(L), \vee, \wedge)$ is a distributive lattice. The set of all co-fuzzy annihilator filters of L is not a sublattice of all fuzzy filters of L. For, consider the following example.

Example 4.11. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given below.



Consider the fuzzy filters μ and θ of L defined as:

$$\theta(1) = \theta(b) = 1,\ \theta(a) = \theta(c) = \theta(0) = 0 \text{ and}$$
$$\mu(1) = \mu(c) = 1,\ \mu(a) = \mu(b) = \mu(0) = 0.$$

Then we can easily verified that μ and θ are co-fuzzy annihilator filters of L. But the fuzzy filter $\eta = \mu \vee \theta$ is not a co-fuzzy annihilator filter of L. Thus $FF^+(L)$ is not a sublattice of $FF(L)$. However, in the following theorem, it is proved that $FF^+(L)$ forms a complete Boolean algebra.

Lemma 4.12. *If $\mu, \theta \in FF^+(L)$, the supremum of μ and θ is given by:*

$$\mu \underline{\vee} \theta = (\mu^+ \cap \theta^+)^+.$$

Proof. First, we need to show $\mu\underline{\vee}\theta$ is a co-fuzzy annihilator filter. Clearly $\mu\underline{\vee}\theta$ is a fuzzy filter of L. Now, $(\mu\underline{\vee}\theta)^{++} = (\mu^{+} \cap \theta^{+})^{+++}$ and by Lemma 4.3(5) we get that $(\mu\underline{\vee}\theta)^{++} = \mu\underline{\vee}\theta$. Thus $\mu\underline{\vee}\theta$ is a co-fuzzy annihilator filter of L.

Now we proceed to show that $\mu\underline{\vee}\theta$ is the least upper bound of $\{\mu, \theta\}$. Since $\mu, \theta \subseteq (\mu \vee \theta)^{++}$, it yields $(\mu^{+} \cap \theta^{+})^{+}$ is an upper bound of $\{\mu, \theta\}$. Let η be any upper bound of $\{\mu, \theta\}$ in $FF^{+}(L)$. Then $\mu \vee \theta \subseteq \eta$. This implies that $(\mu \vee \theta)^{++} \subseteq \eta$. Thus $\mu\underline{\vee}\theta$ is the supremum of μ and θ in $FF^{+}(L)$. □

In the following theorem, we prove that the class of all co-fuzzy annihilator filters forms a complete Boolean algebra.

Theorem 4.13. *The set $FF^{+}(L)$ of all co-fuzzy annihilator filters of L forms a complete Boolean algebra.*

Proof. For $\mu, \theta \in FF^{+}(L)$, define

$$\mu \wedge \theta = \mu \cap \theta \text{ and } \mu\underline{\vee}\theta = (\mu^{+} \cap \theta^{+})^{+}.$$

Then clearly $\mu \cap \theta$, $\mu\underline{\vee}\theta \in FF^{+}(L)$. Thus $\langle FF^{+}(L), \cap, \underline{\vee}\rangle$ is a lattice. Since $(\chi_{\{1\}})^{+} = \chi_L$ and $(\chi_L)^{+} = \chi_{\{1\}}$, then $\chi_{\{1\}}$ and χ_L are the least and the greatest elements of $FF^{+}(L)$ respectively. Hence $\langle FAI(L), \cap, \underline{\vee}\rangle$ is a bounded lattice.

Let $\mu \in FF^{+}(L)$. Then $\mu^{+} \in FF^{+}(L)$ and $\mu \cap \mu^{+} = \chi_{\{1\}}$, $\mu\underline{\vee}\mu^{+} = \chi_L$. Thus μ^{+} is a complement of μ.

Let μ, θ, $\eta \in FF^{+}(L)$. We prove that $\mu\underline{\vee}(\theta \cap \eta) = (\mu\underline{\vee}\theta) \cap (\mu\underline{\vee}\eta)$. To prove our claim it's enough to show that $(\mu\underline{\vee}\theta) \cap \eta \subseteq \mu\underline{\vee}(\theta \cap \eta)$. Since $\mu \cap \mu^{+} = \chi_{\{1\}}$, we have that $\mu \cap \eta \cap (\mu^{+} \cap (\theta \cap \eta)^{+}) = \chi_{\{1\}}$. So that $\eta \cap (\mu^{+} \cap (\theta \cap \eta)^{+}) \subseteq \mu^{+}$. Similarly, $\theta \cap \eta \cap (\mu^{+} \cap (\theta \cap \eta)^{+}) \subseteq \chi_{\{1\}}$ implies that $\eta \cap (\mu^{+} \cap (\theta \cap \eta)^{+}) \subseteq \theta^{+}$. Then $\eta \cap (\mu^{+} \cap (\theta \cap \eta)^{+}) \subseteq \mu^{+} \cap \theta^{+}$. Thus

$$\eta \cap (\mu^{+} \cap (\theta \cap \eta)^{+}) \cap (\mu^{+} \cap \theta^{+})^{+} = \chi_{\{1\}}.$$

That is, $(\mu^{+} \cap (\theta \cap \eta)^{+}) \cap (\eta \cap (\mu^{+} \cap \theta^{+})^{+}) = \chi_{\{1\}}$. So $(\mu^{+} \cap (\theta \cap \eta)^{+}) \subseteq (\eta \cap (\mu^{+} \cap \theta^{+})^{+})^{+}$. Hence $(\mu\underline{\vee}\theta) \cap \eta \subseteq \mu\underline{\vee}(\theta \cap \eta)$ and so $FF^{+}(L)$ is a distributive lattice. Therefore $\langle FF^{+}(L), \cap, \underline{\vee}\rangle$ is a Boolean algebra.

Next we prove the completeness. Let $\{\mu_i : i \in \Delta\}$ be a family of $FF^{+}(L)$. Then $(\bigcap_{i\in\Delta} \mu_i)^{++} = \bigcap_{i\in\Delta} \mu_i$.
Thus $\langle FF^{+}(L), \wedge, \underline{\vee}, +, \chi_{\{1\}}, \chi_L\rangle$ is a complete Boolean algebra. □

Definition 4.14. A fuzzy filter μ of L is called co-dense if $\mu^{+} = \chi_{\{1\}}$.

Conclusion

In this work, we introduced the concept of relative co-fuzzy annihilator filters of a distributive lattice. We characterized distributive lattice with the help of relative co-fuzzy annihilator filters. It is proved that the set of all fuzzy filters of a distributive lattice forms a Heything algebra. Furthermore, we studied co-fuzzy annihilator filters of a distributive lattice and we proved that the set of all co-fuzzy annihilator filters forms a complete Boolean algebra. Our future work will focus on μ-fuzzy filters of a distributive lattice.

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