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A STUDY OF COFFICIENTS DERIVED FROM ETA FUNCTIONS†

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ABSTRACT. The main purpose and motivation of this work is to investigate and provide some new results for coefficients derived from eta quotients related to 3. The result of this paper involve some restricted divisor numbers and their convolution sums. Also, our results give relation between the coefficients derived from infinite product, infinite sum and the convolution sum of restricted divisor functions.

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1. Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ will be denoted by the set of natural numbers, the set of non-negative integers and the ring of integers, respectively.

For $d, m, N \in \mathbb{N}$ and $r, s \in \mathbb{N}_0$, we define some divisor functions as follows:

$$\sigma_s(N) := \sum_{d|N} d^s, \ \ \hat{\sigma}(N) := \sum_{\substack{d|N \\ d \equiv \frac{N}{d} \pmod 3}} d - \frac{1}{2} \sum_{\substack{d|N \\ d - \frac{N}{d} \equiv \pm 1 \pmod 3}} d,$$

$$\bar{\sigma_s}(N) := \sum_{d|N} \chi(d) d^s \ \ \tilde{\sigma_s}(N) := \sum_{d|N} \chi(N/d) d^s,$$

$$E_r(N; m) := \sum_{\substack{d|N \\ d \equiv r \pmod m}} 1 - \sum_{\substack{d|N \\ d \equiv -r \pmod m}} 1,$$

$$\lambda(N) := E_1(N; 3) - 3E_1(\frac{N}{3}; 3),$$

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where

$$\chi(d) = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 3, \\ -1 & \text{if } d \equiv 2 \pmod 3, \\ 0 & \text{if } d \equiv 0 \pmod 3. \end{cases}$$

We also make use of the following convention:

$$\sigma_s(N) = E_1(N;3) = 0 \text{ if } N \notin \mathbb{Z} \text{ or } N < 0,\ \sigma(N) := \sigma_1(N) = \sum_{d|N} d.$$

In addition, in this article we do not consider $\sigma_s(0)$. Let

$$[a]_q := \eta\left(a\tau\right) = e^{\pi i a\tau/12} \prod_{n\geq 1} \left(1 - e^{2\pi i n a\tau}\right).$$

Here, we may write $q = e^{2\pi i\tau}$, where $Im(\tau) > 0$.

For all positive integers N we have

$$\sum_{j=1}^{N-1} E_1(j;3)E_1(N-j;3) = \frac{1}{3}(\sigma'(N) - E_1(N;3)),$$

where $\sigma'(N) = \sum_{d|N} d$ and d is not a multiple of 3([7]).

The convolution sum

$$\sum_{k=1}^{N-1} \sigma_1(k)\sigma_1(N-k)$$

first appeared in letter from Besge to Liouville in 1862([23]). The evaluation of such sums also appear in the works of Glaisher [14]–[17], Ramanujan [12]. See [13] and [23] for basic information on this area. In fact, the study of divisor numbers has been studied many mathematicians([6], [9], [13]).

From here, we introduce the basic notations for infinite sums and infinite products. Let us define

$$B_k(q) := \prod_{n\geq 1} \frac{(1-q^n)^{2k}}{(1+q^n+q^{2n})^k} = \sum_{N\geq 0} \mathfrak{b}_k(N)q^N, \tag{1}$$

$$C(q) := q\prod_{n\geq 1}(1-q^{3n})^8 = \sum_{N\geq 0} c(N)q^N, \tag{2}$$

$$D(q) := q\prod_{n\geq 1}(1-q^n)^3(1-q^{3n})^7 = \sum_{N\geq 1} d(N)q^N \tag{3}$$

and

$$E(q) := q\prod_{n\geq 1}(1-q^n)^6(1-q^{3n})^6 = \sum_{N\geq 1} e(N)q^N. \tag{4}$$

As is well known, [5], $B_1(q) = \sum_{n,m=-\infty}^{\infty} \omega^{n-m} q^{n^2+nm+m^2}$ with $\omega = e^{\frac{2\pi i}{3}}$. In [7] and [8], Farkas found formulas for convolution sums of $E_r(N;m)$. In this article, we obtain formulas for convolution sums of $\lambda(N)$. More precisely, we prove the following theorems.

Theorem 1.1. *If $N \geq 2$, then*

$$\sum_{t=1}^{N-1} \lambda(t)\lambda(N-t) =$$
$$\begin{cases} \frac{2}{3}\left(E_1(N;3) - \sigma(N)\right) & \textit{if } N \equiv 1 \pmod 3, \\ \frac{1}{3}\sigma(N) & \textit{if } N \equiv 2 \pmod 3, \\ \frac{4}{3}\left(-E_1(n;3) + \sigma(n)\right) & \textit{if } N = 3^e n,\ e \geq 1, \quad (3,n) = 1. \end{cases}$$

Remark 1.1. Farkas proved Theorem 1.1 in [7, Theorem 1] using the theta function.

Theorem 1.2. *For $a, b, c \in \mathbb{N}$. If $N \geq 3$, then*

$$\sum_{a+b+c=N} \lambda(a)\lambda(b)\lambda(c) =$$
$$\begin{cases} \frac{1}{3}\left(E_1(N;3) - 2\sigma(N) + \bar{\sigma_2}(N)\right) & \textit{if } N \equiv 1 \pmod 3, \\ \frac{1}{3}\left(\sigma(N) + \bar{\sigma_2}(N)\right) & \textit{if } N \equiv 2 \pmod 3, \\ \frac{1}{3}\left(-2E_1(n;3) + 4\sigma(n) + \bar{\sigma_2}(n)\right) & \textit{if } N = 3^e n,\ e \geq 1, \quad (3,n) = 1. \end{cases}$$

Theorem 1.3. *For $a, b, c, d \in \mathbb{N}$. If $N \geq 4$, then*

$$\sum_{a+b+c+d=N} \lambda(a)\lambda(b)\lambda(c)\lambda(d) =$$
$$\begin{cases} \frac{1}{27}\left(4E_1(N;3) - 12\sigma(N) + 12\bar{\sigma_2}(N) - \sigma_3(N) - 3c(N)\right) & \textit{if } N \equiv 1 \pmod 3, \\ \frac{1}{27}\left(6\sigma(N) + 12\bar{\sigma_2}(N) + 2\sigma_3(N)\right) & \textit{if } N \equiv 2 \pmod 3, \\ \frac{4}{27}\left(-2E_1(n;3) + 6\sigma(n) + 3\bar{\sigma_2}(n) - \frac{1}{13}\left(3^{3e+1} + 10\right)\sigma_3(n)\right) & \textit{if } N = 3^e n,\ e \geq 1, \\ & \quad (3,n) = 1. \end{cases}$$

Corollary 1.4. *If $N \geq 2$, then*

$$\sum_{t=1}^{N-1} \hat{\sigma}(t)\hat{\sigma}(N-t) =$$
$$\begin{cases} \frac{1}{12}(4\sigma(N) - \sigma_3(N) - 3c(N)) & \textit{if } N \equiv 1 \pmod 3, \\ \frac{1}{6}(-\sigma(N) + \sigma_3(N)) & \textit{if } N \equiv 2 \pmod 3, \\ \frac{1}{3}(-2\sigma(n) - \frac{1}{13}(3^{3e+1} + 10)\sigma_3(n)) & \textit{if } N = 3^e n,\ e \geq 1, \quad (3,n) = 1. \end{cases}$$

Theorem 1.5. *For $a, b, c, d, e \in \mathbb{N}$. If $N \geq 5$, then we have the following theorem:*
a) If $N \equiv 1 \pmod 3$, then

$$\sum_{a+b+c+d+e=N} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e) = \frac{1}{81}(5E_1(N;3) - 20\sigma(N) + 30\bar{\sigma_2}(N) - 5\sigma_3(N) - 15c(N) + 5d(N)).$$

b) If $N \equiv 2 \pmod 3$*, then*

$$\sum_{a+b+c+d+e=N} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e) = \frac{1}{81}(10\sigma(N) + 30\bar{\sigma_2}(N) + 10\sigma_3(N) + \bar{\sigma_4}(N) + 5d(N)).$$

c) If $N = 3^e n, e \geq 1$ *and* $(3,n) = 1$*, then*

$$\sum_{a+b+c+d+e=N} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e) = \frac{1}{81}(-10E_1(n;3) + 40\sigma(n) + 30\bar{\sigma_2}(n) - \frac{20}{13}\left(3^{3e+1} + 10\right)\sigma_3(n) + (3^{4e}\chi(n) - 1)\bar{\sigma_4}(n)).$$

Theorem 1.6. *For* $a,b,c,d,e,f \in \mathbb{N}$*. If* $N \geq 6$*, then we have the following theorem:*
a) If $N \equiv 1 \pmod 3$*, then*

$$\sum_{a+b+c+d+e+f=N} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f) = \frac{1}{81}(2E_1(N;3) - 10\sigma(N) + 20\bar{\sigma_2}(N) - 5\sigma_3(N) + \frac{1}{13}\sigma_5(N) - 15c(N) + 10d(N) - \frac{27}{13}e(N)).$$

b) If $N \equiv 2 \pmod 3$*, then*

$$\sum_{a+b+c+d+e+f=N} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f) = \frac{1}{81}(5\sigma(N) + 20\bar{\sigma_2}(N) + 10\sigma_3(N) + 2\bar{\sigma_4}(N) + \frac{1}{13}\sigma_5(N) + 10d(N) - \frac{27}{13}e(N)).$$

c) If $N = 3^e n, e \geq 1$ *and* $(3,n) = 1$*, then*

$$\sum_{a+b+c+d+e+f=N} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f) = \frac{1}{81}(-4E_1(n;3) + 20\sigma(n) + 20\bar{\sigma_2}(n) - \frac{20}{13}\left(3^{3e+1} + 10\right)\sigma_3(n) + 2(3^{4e}\chi(n) - 1)\bar{\sigma_4}(n) + \frac{1}{1573}(364 - 3^{5e+5})\sigma_5(n) - \frac{27}{13}e(N)).$$

Corollary 1.7. *For* $a,b,c \in \mathbb{N}$*. If* $N \geq 3$*, then we have the following corollary:*
a) If $N \equiv 1 \pmod 3$*, then*

$$\sum_{a+b+c=N} \hat{\sigma}(a)\hat{\sigma}(b)\hat{\sigma}(c) = \frac{1}{24}(2\sigma(N) - \sigma_3(N) - \frac{1}{13}\sigma_5(N) - 3c(N) + \frac{27}{13}e(N)).$$

b) If $N \equiv 2 \pmod 3$*, then*

$$\sum_{a+b+c=N} \hat{\sigma}(a)\hat{\sigma}(b)\hat{\sigma}(c) = \frac{1}{24}(-\sigma(N) + 2\sigma_3(N) - \frac{1}{13}\sigma_5(N) + \frac{27}{13}e(N)).$$

c) If $N = 3^e n, e \geq 1$ *and* $(3, n) = 1$*, then*

$$\sum_{a+b+c=N} \hat{\sigma}(a)\hat{\sigma}(b)\hat{\sigma}(c) = \frac{1}{24}(-4\sigma(n) - \frac{4}{13}(3^{3e+1} + 10)\sigma_3(n) - \frac{1}{1573}(364 - 3^{5e+5})\sigma_5(n) + \frac{27}{13}e(N)).$$

Corollary 1.8. *For* $N \geq 2$*, we have the following corollary:*
a) If $N \not\equiv 0 \pmod 3$*, then*

$$\sum_{t=1}^{N-1} \bar{\sigma_2}(t)\bar{\sigma_2}(N-t) = \frac{1}{9}(2\bar{\sigma_2}(N) + \frac{1}{13}\sigma_5(N) - \frac{27}{13}e(N)).$$

b) If $N = 3^e n, e \geq 1$ *and* $(3, n) = 1$*, then*

$$\sum_{t=1}^{N-1} \bar{\sigma_2}(t)\bar{\sigma_2}(N-t) = \frac{1}{9}(2\bar{\sigma_2}(n) + \frac{1}{1573}(364 - 3^{5e+5})\sigma_5(n) - \frac{27}{13}e(N)).$$

Remark 1.2. We can regard the sums of the left-hand sides of each theorem as zero for small N for which the summations are not defined. Actually, looking at the proofs of the main results, we can verify that all results hold for all $N \geq 1$.

2. Preliminaries

The approach in this section follows the method used in [18]. We recall the some identities and lemmas to prove our results. In [10, p.21], we find a curious identity

$$\prod_{n\geq 1} \frac{(1-q^n)^4}{(1-2q^n \cos u + q^{2n})^2} = 1 - 8\sin^2\frac{u}{2} \sum_{N\geq 1} q^N \sum_{\substack{nk=N \\ n,k\geq 1}} n\cos{(k-n)u}. \tag{5}$$

Putting $u = \frac{2\pi}{3}$ in (5), we get

$$\begin{aligned}\prod_{n\geq 1} \frac{(1-q^n)^4}{(1+q^n+q^{2n})^2} &= 1 - 8\sin^2\frac{\pi}{3} \sum_{N\geq 1} q^N \sum_{\substack{nk=N \\ n,k\geq 1}} n\cos{(k-n)}\frac{2\pi}{3} \\ &= 1 - 8\cdot\frac{3}{4} \sum_{N\geq 1} q^N \sum_{\substack{nk=N \\ n,k\geq 1}} n\cos{(k-n)}\frac{2\pi}{3} \\ &= 1 - 6 \sum_{N\geq 1} q^N \sum_{\substack{nk=N \\ n,k\geq 1}} n\cos{(k-n)}\frac{2\pi}{3}.\end{aligned} \tag{6}$$

For $k, n \in \mathbb{N}$, we obtain two cases:

$$n\cos{(k-n)}\frac{2\pi}{3} = \begin{cases} n & \text{if } (k-n)\equiv 0 \pmod 3, \\ -\frac{1}{2}n & \text{if } (k-n)\equiv \pm 1 \pmod 3. \end{cases} \tag{7}$$

By (6) and (7), we get

$$\begin{aligned}
&\prod_{n\geq 1}\frac{(1-q^n)^4}{(1+q^n+q^{2n})^2} \\
&= 1-6\sum_{N\geq 1}\left(\sum_{\substack{nk=N\\ n,k\geq 1}} n\cos{(k-n)}\frac{2\pi}{3}\right)q^N \\
&= 1-6\sum_{N\geq 1}\left(\sum_{\substack{d\mid n\\ d\equiv \frac{n}{d} \pmod 3}} d-\frac{1}{2}\sum_{\substack{d\mid n\\ d-\frac{n}{d}\equiv \pm 1 \pmod 3}} d\right)q^N \\
&= 1-6\sum_{N\geq 1}\hat{\sigma}(N)q^N.
\end{aligned}$$

By the above equation, we get the following lemma:

Lemma 2.1. *For $N\geq 1$, we get*

$$\mathfrak{b}_2(N) = -6\hat{\sigma}(N) = \begin{cases} -6\sigma(N) & \textit{if } N\equiv 1 \pmod 3, \\ 3\sigma(N) & \textit{if } N\equiv 2 \pmod 3, \\ 12\sigma(n) & \textit{if } N=3^e n,\ e\geq 1,\ (3,n)=1. \end{cases} \tag{8}$$

In order to prove the results, we need the following lemmas. It should be noted that the following lemmas was obtained by Fine in [10].

Lemma 2.2. *(*[10, p.79, p.85]*) Let N be a positive integer. Then*

$$\prod_{n\geq 1}\frac{(1-q^n)^3}{(1-q^{3n})} = \frac{[1]_q^3}{[3]_q} = 1-3\sum_{N\geq 1}\lambda(N)q^N \tag{9}$$

and

$$\frac{[1]_q^9}{[3]_q^3} = 1-9\sum_{N\geq 1}q^N\sum_{d\mid N}\chi(d)d^2.$$

In particular, if $N=3^e n$ with $e\geq 1$ and $(3,n)=1$, then we have the following lemma:

Lemma 2.3. *If $N=3^e n$ with $e\geq 1$ and $(3,n)=1$ then $\lambda(N)=-2E_1(n;3)$.*

Proof. It is easily checked that $E_1(3^e n;3)=E_1(n;3)$ with $(3,n)=1$. Thus the proof of Lemma 2.3 is proved. □

Lemma 2.4.

$$\frac{[1]_q^{12}}{[3]_q^4} = 1 + \sum_{N\geq 1}(-9c(N) + \frac{3}{2}\sigma_3\left(N\right) - 126\sigma_3\left(\frac{N}{3}\right) + \frac{729}{2}\sigma_3\left(\frac{N}{9}\right)$$
$$- \frac{9}{2}\chi\left(N\right)\sigma_3\left(N\right))q^N.$$

Proof. Viewing as $q = e^{2\pi i\tau}$ with τ in the complex upper half plane, $\frac{[1]_q^{12}}{[3]_q^4}$ is a holomorphic modular form of weight 4 for $\Gamma_0(9)$. The space of holomorphic modular forms $M_4(\Gamma_0(9))$ is a five-dimensional vector space by the dimension formula [21, Theorem 1.34]. Let $E_4(q)$ be the normalized Eisenstein series of weight 4

$$E_4(q) = 1 + 240\sum_{N\geq 1}\sigma_3(N)q^N.$$

Then the space $M_4(\Gamma_0(9))$ is spanned by $C(q)$, $E_4(q)$, $E_4(q^3)$, $E_4(q^9)$, and

$$\frac{E_4(\omega q) - E_4(\omega^2 q)}{240\sqrt{-3}} = \sum_{N\geq 1}\chi(N)\sigma_3(N)q^N, \tag{10}$$

where $\omega = e^{\frac{2\pi i}{3}}$. Now by comparing the coefficients of both sides in (10), we have a linear expression

$$\frac{[1]_q^{12}}{[3]_q^4} = -9C(q) + \frac{3}{2}\cdot\frac{E_4(q)}{240} - 126\cdot\frac{E_4(q^3)}{240} + \frac{729}{2}\cdot\frac{E_4(q^9)}{240}$$
$$- \frac{9}{2}\cdot\frac{E_4(\omega q) - E_4(\omega^2 q)}{240\sqrt{-3}}$$
$$= 1 + \sum_{N\geq 1}(-9c(N) + \frac{3}{2}\sigma_3\left(N\right) - 126\sigma_3\left(\frac{N}{3}\right) + \frac{729}{2}\sigma_3\left(\frac{N}{9}\right)$$
$$- \frac{9}{2}\chi\left(N\right)\sigma_3\left(N\right))q^N.$$

□

Lemma 2.5.

$$\frac{[1]_q^{15}}{[3]_q^5} = 1 + \sum_{N\geq 1}\left(-15d(N) - \frac{3}{2}\bar{\sigma_4}(N) + \frac{9}{2}\bar{\sigma_4}\left(\frac{N}{3}\right) + \frac{3}{2}\tilde{\sigma_4}(N)\right.$$
$$\left.-\frac{729}{2}\tilde{\sigma_4}\left(\frac{N}{3}\right)\right)q^N.$$

Proof. $\frac{[1]_q^{15}}{[3]_q^5}$ is a modular form in the space $M_5(\Gamma_0(9), (\frac{-3}{\cdot}))$. Since $(\frac{-3}{\cdot}) = (\frac{\cdot}{3})$, we can identify the Kronecker symbol $(\frac{-3}{\cdot})$ with χ. By the dimension formula [21, Theorem 1.34], we have $\dim M_5(\Gamma_0(9), \chi) = 6$. Note that

$$M_5(\Gamma_0(9), \chi) = S_5(\Gamma_0(9), \chi) \oplus \mathcal{E}_5(\Gamma_0(9), \chi),$$

where $S_5(\Gamma_0(9),\chi)$ is the space of cusp forms and $\mathcal{E}_5(\Gamma_0(9),\chi)$ is the Eisenstein subspace. The space $S_5(\Gamma_0(9),\chi)$ is a two-dimensional space spanned by $q^2\prod_{n\geq 1}(1-q^{3n})^7(1-q^{9n})^3$ and $q\prod_{n\geq 1}(1-q^n)^3(1-q^{3n})^7$. By [19, Theorem 4.5.2], the space $\mathcal{E}_5(\Gamma_0(9),\chi)$ is spanned by four forms $\bar{E_5}(q)$, $\bar{E_5}(q^3)$, $\tilde{E_5}(q)$, and $\tilde{E_5}(q^3)$, where

$$\bar{E_5}(q) = L(-4,\chi) + 2\sum_{N\geq 1}\bar{\sigma_4}(N)q^N,$$
$$\tilde{E_5}(q) = 2\sum_{N\geq 1}\tilde{\sigma_4}(N)q^N.$$

Here, $L(s,\chi)$ is the Dirichlet L-function. By [21, Proposition 1.51], we have $L(-4,\chi) = -\frac{B(5,\chi)}{5} = \frac{2}{3}$, where $B(k,\chi)$ is the generalized Bernoulli number. Now by comparing the coefficients, we get

$$\begin{aligned}\frac{[1]_q^{15}}{[3]_q^5} &= -15D(q) - \frac{3}{2}\cdot\frac{\bar{E_5}(q)}{2} + \frac{9}{2}\cdot\frac{\bar{E_5}(q^3)}{2} + \frac{3}{2}\cdot\frac{\tilde{E_5}(q)}{2} - \frac{729}{2}\cdot\frac{\tilde{E_5}(q^3)}{2}\\ &= 1 + \sum_{N\geq 1}\left(-15d(N) - \frac{3}{2}\bar{\sigma_4}(N) + \frac{9}{2}\bar{\sigma_4}\left(\frac{N}{3}\right) + \frac{3}{2}\tilde{\sigma_4}(N)\right.\\ &\quad \left.-\frac{729}{2}\tilde{\sigma_4}\left(\frac{N}{3}\right)\right)q^N.\end{aligned}$$

□

Lemma 2.6.

$$\frac{[1]_q^{18}}{[3]_q^6} = 1 + \sum_{N\geq 1}\left(-\frac{243}{13}e(N) + \frac{9}{13}\sigma_5(N) - \frac{6561}{13}\sigma_5\left(\frac{N}{3}\right)\right)q^N.$$

Proof. $\frac{[1]_q^{18}}{[3]_q^6}$ is a modular form in the space $M_6(\Gamma_0(3))$, which is a three-dimensional space spanned by $E(q)$, $E_6(q)$, and $E_6(q^3)$, where $E_6(q)$ is the normalized Eisenstein series of weight 6

$$E_6(q) = 1 - 504\sum_{N\geq 1}\sigma_5(N)q^N.$$

By comparing the coefficients, we get

$$\begin{aligned}\frac{[1]_q^{18}}{[3]_q^6} &= -\frac{243}{13}E(q) + \frac{9}{13}\cdot\left(-\frac{E_6(q)}{504}\right) - \frac{6561}{13}\cdot\left(-\frac{E_6(q^3)}{504}\right)\\ &= 1 + \sum_{N\geq 1}\left(-\frac{243}{13}e(N) + \frac{9}{13}\sigma_5(N) - \frac{6561}{13}\sigma_5\left(\frac{N}{3}\right)\right)q^N.\end{aligned}$$

□

Lemma 2.7. *For $N \geq 1$, we get*

$$\bar{\sigma_4}(N) = \begin{cases} \tilde{\sigma_4}(N) & \text{if } N \equiv 1 \pmod 3, \\ -\tilde{\sigma_4}(N) & \text{if } N \equiv 2 \pmod 3. \end{cases}$$

Proof. If $N \equiv 1 \pmod 3$, then for any divisor d of N, we have $d^2 \equiv N \pmod 3$. Since $3 \nmid d$, we have $d \equiv \frac{N}{d} \pmod 3$. Thus $\chi(d) = \chi(N/d)$ for any divisor d. By definition, we obtain $\bar{\sigma_4}(N) = \tilde{\sigma_4}(N)$.

If $N \equiv 2 \pmod 3$, then for any divisor d of N, we have $d^2 \equiv -N \pmod 3$, so $d \equiv -\frac{N}{d} \pmod 3$. Thus $\chi(d) = \chi(-N/d) = -\chi(N/d)$, so we have $\bar{\sigma_4}(N) = -\tilde{\sigma_4}(N)$. □

Lemma 2.8. *For $N \geq 1$, we get*

$$\bar{\sigma_4}(3N) = \bar{\sigma_4}(N).$$

Proof. Let $S(N)$ be the set of all divisors of N. Write $N = 3^e n$ with $e \geq 0$ and $(3, n) = 1$. Then we have

$$S(3N) = S(N) \cup 3^{e+1} S(n).$$

For any d in $3^{e+1}S(n)$, $\chi(d) = 0$. Hence we have $\bar{\sigma_4}(3N) = \bar{\sigma_4}(N)$. □

Lemma 2.9. *For $N \geq 1$, we get*

$$\tilde{\sigma_4}(3N) = 81\tilde{\sigma_4}(N).$$

Proof. Let $S(N)$ be the set of all divisors of N. Write $N = 3^e n$ with $e \geq 0$ and $(3, n) = 1$. Then we have

$$S(N) = S(n) \cup 3S(n) \cup \cdots \cup 3^e S(n),$$
$$S(3N) = S(n) \cup 3S(n) \cup \cdots \cup 3^{e+1} S(n).$$

For any d in $S(n) \cup 3S(n) \cup \cdots \cup 3^{e-1}S(n)$, we get $\chi(N/d) = 0$. Thus we have

$$\tilde{\sigma_4}(N) = \sum_{i=1}^{k} (3^e d_i)^4 \chi(n/d_i),$$

where $S(n) = \{d_1, d_2, \cdots, d_k\}$. Similarly, for any d in $S(n) \cup 3S(n) \cup \cdots \cup 3^e S(n)$, we get $\chi(3N/d) = 0$. Thus we have

$$\tilde{\sigma_4}(3N) = \sum_{i=1}^{k} (3^{e+1} d_i)^4 \chi(n/d_i).$$

Accordingly, we obtain $\tilde{\sigma_4}(3N) = 3^4 \tilde{\sigma_4}(N)$. □

Lemma 2.10. *For $N \geq 1$ with $N \equiv 0 \pmod 3$, we have $d(N)=0$.*

Proof. Using Jacobi's Triple Product Identity [20, Chapter 8, Theorem 3.6], we have

$$q \prod_{n \geq 1} (1 - q^n)^3 = q \sum_{n=-\infty}^{\infty} (-1)^n n q^{\frac{n(n+1)}{2}}$$

$$= q\sum_{n\geq 0}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}$$
$$= \sum_{n\geq 0}(-1)^n(2n+1)q^{\frac{n^2+n+2}{2}}.$$

Note that for any nonnegative integer n, $\frac{n^2+n+2}{2} \not\equiv 0 \pmod 3$. We are done. □

3. Proof of the Theorems

In this section, we prove our results by using the above lemmas.
Proof of Theorem 1.1 Using Lemma 2.2, we get

$$\begin{aligned}
\prod_{n\geq 1}\frac{(1-q^n)^4}{(1+q^n+q^{2n})^2} &= \frac{[1]_q^6}{[3]_q^2} \\
&= \left(1-3\sum_{N\geq 1}\lambda(N)q^N\right)^2 \\
&= 1-6\sum_{N\geq 1}\lambda(N)q^N + 9\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)q^N \\
&= \sum_{N\geq 0}\mathfrak{b}_2(N)q^N.
\end{aligned} \tag{11}$$

By simple calculation, we get $\mathfrak{b}_2(0) = 1$ and $\mathfrak{b}_2(1) = -6\lambda(1) = -6$. For $N \geq 2$, we have three results:

Let $N \equiv 1 \pmod 3$. By Lemma 2.1, we have the following equation:

$$\mathfrak{b}_2(N) = -6\sigma(N) = -6\lambda(N) + 9\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t).$$

Therefore,

$$\begin{aligned}
\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t) &= \frac{2}{3}\left(\lambda(N)-\sigma(N)\right) \\
&= \frac{2}{3}\left(E_1(N;3)-\sigma(N)\right).
\end{aligned}$$

In the same way as above, if $N \equiv 2 \pmod 3$, then we have

$$\mathfrak{b}_2(N) = 3\sigma(N) = -6\lambda(N) + 9\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t).$$

However, $\lambda(N) = 0$ because if $N \equiv 2 \pmod 3$, then the number of $d \equiv r \pmod m$ and $d \equiv -r \pmod m$ are equal. Hence,

$$\sum_{t=1}^{N-1} \lambda(t)\lambda(N-t) = \frac{2}{3}\lambda(N) + \frac{1}{3}\sigma(N)$$
$$= \frac{1}{3}\sigma(N).$$

Finally, let $N = 3^e n$ with $e \geq 1$ and $(3, n) = 1$. By Lemma 2.1 and (11), we have

$$\mathfrak{b}_2(N) = 12\sigma(n) = -6\lambda(N) + 9\sum_{t=1}^{N-1} \lambda(t)\lambda(N-t).$$

Therefore,

$$\sum_{t=1}^{N-1} \lambda(t)\lambda(N-t) = \frac{2}{3}\lambda(N) + \frac{4}{3}\sigma(n)$$
$$= \frac{4}{3}\left(\sigma(n) - E_1(n;3)\right). \qquad \square$$

Proof of Theorem 1.2 To prove Theorem 1.2, we need the following lemma in [10, p.85].

Using Theorem 1.1 and Lemma 2.1, we have

$$\frac{[1]_q^9}{[3]_q^3} = 1 - 9\sum q^N \sum_{d|n} d^2\chi(d) = 1 - 9\sum_{N\geq 1} \bar{\sigma_2}(N)q^N$$
$$= \left(1 - 3\sum_{N\geq 1} \lambda(N)q^N\right)^3$$
$$= 1 - 9\sum_{N\geq 1} \lambda(N)q^N + 27\sum_{N\geq 2}\left(\sum_{t=1}^{N-1} \lambda(t)\lambda(N-t)\right)q^N$$
$$- 27\sum_{\substack{a+b+c=N \\ N\geq 3}} \lambda(a)\lambda(b)\lambda(c)q^N$$
$$= \sum_{N\geq 0} \mathfrak{b}_3(N)q^N.$$

By simple calculation, we get $\mathfrak{b}_3(0) = 1$, $\mathfrak{b}_3(1) = -9\lambda(1) = -9$ and $\mathfrak{b}_3(2) = -9\lambda(2) + 27\lambda(1)\lambda(1) = 27$. For $N \geq 3$,

$$\mathfrak{b}_3(N) = -9\bar{\sigma_2}(N)$$
$$= -9\lambda(N) + 27\left(\sum_{t=1}^{N-1} \lambda(t)\lambda(N-t)\right) - 27\sum_{a+b+c=N} \lambda(a)\lambda(b)\lambda(c).$$

Therefore,

$$\bar{\sigma_2}(N)=\lambda(N)-3\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)+3\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c).$$

By Theorem 1.1, we have the three results:
If $N\equiv 1 \pmod 3$, then

$$\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)=\frac{1}{3}\left(E_1(N;3)-2\sigma(n)+\bar{\sigma_2}(N)\right).$$

If $N\equiv 2 \pmod 3$, then

$$\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)=\frac{1}{3}\left(\sigma(n)+\bar{\sigma_2}(N)\right).$$

If $N=3^e n$ with $e\geq 1$ and $(3,n)=1$, then

$$\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)=\frac{1}{3}\left(-2E_1(n;3)+4\sigma(n)+\bar{\sigma_2}(n)\right). \quad \square$$

Proof of Theorem 1.3 By Lemma 2.4, we obtain

$$\begin{aligned}\frac{[1]_q^{12}}{[3]_q^4}&=1+\sum_{N\geq 1}(-9c(N)+\frac{3}{2}\sigma_3(N)-126\sigma_3\left(\frac{N}{3}\right)+\frac{729}{2}\sigma_3\left(\frac{N}{9}\right)\\&\quad-\frac{9}{2}\chi(N)\sigma_3(N))q^N\\&=1-12\sum_{N\geq 1}\lambda(N)q^N+54\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)q^N\\&\quad-108\sum_{\substack{a+b+c=N\\N\geq 3}}\lambda(a)\lambda(b)\lambda(c)q^N+81\sum_{\substack{a+b+c+d=N\\N\geq 4}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)q^N\\&=\sum_{N\geq 0}\mathfrak{b}_4(N)q^N.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathfrak{b}_4(N)&=-9c(N)+\frac{3}{2}\sigma_3(N)-126\sigma_3\left(\frac{N}{3}\right)+\frac{729}{2}\sigma_3\left(\frac{N}{9}\right)\\&\quad-\frac{9}{2}\chi(N)\sigma_3(N)\\&=-12\sum_{N\geq 1}\lambda(N)+54\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)\\&\quad-108\sum_{\substack{a+b+c=N\\N\geq 3}}\lambda(a)\lambda(b)\lambda(c)+81\sum_{\substack{a+b+c+d=N\\N\geq 4}}\lambda(a)\lambda(b)\lambda(c)\lambda(d).\end{aligned}$$

By simple calculation, we get $\mathfrak{b}_4(0) = 1$, $\mathfrak{b}_4(1) = -12$, $\mathfrak{b}_4(2) = 54$ and $\mathfrak{b}_4(3) = -84$. To prove Theorem 1.3 for $N \geq 4$, we have to prove it in the three cases.

For $N \equiv 1 \pmod 3$, we use Theorem 1.1 and 1.2. Then we have,

$$\sum_{\substack{a+b+c+d=N \\ N\geq 4}} \lambda(a)\lambda(b)\lambda(c)\lambda(d) = \frac{1}{27}(4E_1(N;3) - 12\sigma(N) + 12\bar{\sigma_2}(N) - \sigma_3(N) - 3c(N)).$$

For $N \equiv 2 \pmod 3$, we use Theorem 1.1 and 1.2. Therefore,

$$\sum_{\substack{a+b+c+d=N \\ N\geq 4}} \lambda(a)\lambda(b)\lambda(c)\lambda(d) = \frac{1}{27}\left(6\sigma(N) + 12\bar{\sigma_2}(N) + 2\sigma_3(N)\right).$$

To prove last case, we prove it in the two cases. For $N = 3^e n$ with $e \geq 1$ and $(3, n) = 1$, we use Theorem 1.1, 1.2 and Lemma 2.3. If $e = 1$, then we have

$$\sum_{\substack{a+b+c+d=N \\ N\geq 4}} \lambda(a)\lambda(b)\lambda(c)\lambda(d) = \frac{4}{27}\left(6\sigma(n) + 3\bar{\sigma_2}(n) - 2E_1(n;3) - 7\sigma_3(n)\right).$$

If $e \geq 2$, we use an well-known identity in [23], then we have

$$\sum_{\substack{a+b+c+d=N \\ N\geq 4}} \lambda(a)\lambda(b)\lambda(c)\lambda(d) = \frac{4}{27}(6\sigma(n) + 3\bar{\sigma_2}(n) - 2E_1(n;3) - \frac{1}{13}\left(3^{3e+1} + 10\right)\sigma_3(n)).$$

□

Proof of Corollary 1.4 Using Lemma 2.2, we get

$$\sum_{N\geq 0} \mathfrak{b}_4(N)q^N = 1 - 12\sum_{N\geq 1}\lambda(N)q^N + 54\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)q^N - 108\sum_{\substack{a+b+c=N \\ N\geq 3}}\lambda(a)\lambda(b)\lambda(c)q^N + 81\sum_{\substack{a+b+c+d=N \\ N\geq 4}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)q^N.$$

Meanwhile, using Lemma 2.1, we have

$$\begin{aligned}\sum_{N\geq 0}\mathfrak{b}_4(N)q^N &= \left(\sum_{N\geq 0}\mathfrak{b}_2(N)q^N\right)^2 = \left(1 - 6\sum_{N\geq 1}\hat{\sigma}(N)q^N\right)^2 \\ &= 1 - 12\sum_{N\geq 1}\hat{\sigma}(N)q^N + 36\sum_{N\geq 2}\left(\sum_{k=1}^{N-1}\hat{\sigma}(k)\hat{\sigma}(N-k)\right)q^N \\ &= 1 + 2\sum_{N\geq 1}\mathfrak{b}_2(N)q^N + 36\sum_{N\geq 2}\left(\sum_{k=1}^{N-1}\hat{\sigma}(k)\hat{\sigma}(N-k)\right)q^N.\end{aligned}$$

Since by Lemma 2.2

$$\sum_{N\geq 1}\mathfrak{b}_2(N)q^N = -6\sum_{N\geq 1}\lambda(N)q^N + 9\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)q^N,$$

we obtain for $N \geq 4$,

$$\begin{aligned}
&-12\lambda(N) + 54\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t) - 108\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)\\
&+81\sum_{a+b+c+d=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\\
&= -12\lambda(N) + 18\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t) + 36\sum_{k=1}^{N-1}\hat{\sigma}(k)\hat{\sigma}(N-k).
\end{aligned}$$

Finally, for $N \geq 4$, we have

$$\begin{aligned}
\sum_{k=1}^{N-1}\hat{\sigma}(k)\hat{\sigma}(N-k) = &\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t) - 3\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)\\
&+\frac{9}{4}\sum_{a+b+c+d=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d).
\end{aligned}$$

Now using Theorem 1.1, 1.2 and 1.3, we have the desired result for $N \geq 4$. In fact, since

$$\hat{\sigma}(1)^2 = 1 = -\frac{1}{6}\sigma(2) + \frac{1}{6}\sigma_3(2),$$
$$2\hat{\sigma}(1)\hat{\sigma}(2) = -3 = -\frac{2}{3}\sigma(1) - \frac{1}{39}(3^4+10)\sigma_3(1),$$

the result is true for all $N \geq 2$. □

Proof of Theorem 1.5 Using Lemma 2.5, we get

$$\begin{aligned}
\frac{[1]_q^{15}}{[3]_q^5} &= 1 + \sum_{N\geq 1}(-15d(N) - \frac{3}{2}\bar{\sigma_4}(N) + \frac{9}{2}\bar{\sigma_4}\left(\frac{N}{3}\right) + \frac{3}{2}\tilde{\sigma_4}(N)\\
&\quad - \frac{729}{2}\tilde{\sigma_4}\left(\frac{N}{3}\right))q^N\\
&= 1 - 15\sum_{N\geq 1}\lambda(N)q^N + 90\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)q^N\\
&\quad - 270\sum_{\substack{a+b+c=N\\ N\geq 3}}\lambda(a)\lambda(b)\lambda(c)q^N + 405\sum_{\substack{a+b+c+d=N\\ N\geq 4}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)q^N\\
&\quad - 243\sum_{\substack{a+b+c+d+e=N\\ N\geq 5}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)q^N
\end{aligned}$$

$$= \sum_{N\geq 0} \mathfrak{b}_5(N)q^N.$$

Therefore,

$$\begin{aligned}\mathfrak{b}_5(N) &= -15d(N) - \frac{3}{2}\bar{\sigma_4}(N) + \frac{9}{2}\bar{\sigma_4}\left(\frac{N}{3}\right) + \frac{3}{2}\tilde{\sigma_4}(N) - \frac{729}{2}\tilde{\sigma_4}\left(\frac{N}{3}\right)\\ &= -15\sum_{N\geq 1}\lambda(N) + 90\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)\\ &\quad - 270\sum_{\substack{a+b+c=N\\ N\geq 3}}\lambda(a)\lambda(b)\lambda(c) + 405\sum_{\substack{a+b+c+d=N\\ N\geq 4}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\\ &\quad - 243\sum_{\substack{a+b+c+d+e=N\\ N\geq 5}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e).\end{aligned}$$

By simple calculation, we get $\mathfrak{b}_5(0) = 1$, $\mathfrak{b}_5(1) = -15$, $\mathfrak{b}_5(2) = 90$, $\mathfrak{b}_5(3) = -240$ and $\mathfrak{b}_5(4) = 30$. To prove Theorem 1.5 for $N \geq 5$, we have to prove it in the three cases.

For $N \equiv 1 \pmod 3$, we use Theorem 1.1, 1.2 and 1.3. Then, we have

$$\begin{aligned}&\sum_{\substack{a+b+c+d+e=N\\ N\geq 5}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e) = \frac{1}{81}(5E_1(N;3) - 20\sigma(N) + 30\bar{\sigma_2}(N)\\ &- 5\sigma_3(N) - 15c(N) + 5d(N) + \frac{1}{2}\bar{\sigma_4}(N) - \frac{1}{2}\tilde{\sigma_4}(N))\\ &= \frac{1}{81}\left(5E_1(N;3) - 20\sigma(N) + 30\bar{\sigma_2}(N) - 5\sigma_3(N) - 15c(N) + 5d(N)\right).\end{aligned}$$

The last equality holds by Lemma 2.7.

For $N \equiv 2 \pmod 3$, we use Theorem 1.1, 1.2 and 1.3. Therefore,

$$\begin{aligned}&\sum_{\substack{a+b+c+d+e=N\\ N\geq 5}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e) =\\ &\frac{1}{81}\left(10\sigma(N) + 30\bar{\sigma_2}(N) + 10\sigma_3(N) + 5d(N) + \frac{1}{2}\bar{\sigma_4}(N) - \frac{1}{2}\tilde{\sigma_4}(N)\right)\\ &= \frac{1}{81}\left(10\sigma(N) + 30\bar{\sigma_2}(N) + 10\sigma_3(N) + 5d(N) + \bar{\sigma_4}(N)\right).\end{aligned}$$

The last equality holds by Lemma 2.7.

To prove last case, we prove it in the two cases. For $N = 3^e n$ with $e \geq 1$ and $(3, n) = 1$, we use Theorem 1.1, 1.2, 1.3 and Lemma 2.3. Then, we have

$$\begin{aligned}&\sum_{\substack{a+b+c+d+e=N\\ N\geq 5}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e) = \frac{1}{81}(-10E_1(n;3) + 40\sigma(n) + 30\bar{\sigma_2}(n)\\ &- \frac{20}{13}\left(3^{3e+1} + 10\right)\sigma_3(n) + 5d(N) + \frac{1}{2}\bar{\sigma_4}(N) - \frac{3}{2}\bar{\sigma_4}(3^{e-1}n) - \frac{1}{2}\tilde{\sigma_4}(N)\end{aligned}$$

$$+\frac{243}{2}\tilde{\sigma_4}(3^{e-1}n)).$$

Using Lemma 2.7, 2.8 and 2.9, we have

$$\begin{aligned}
&\bar{\sigma_4}(3^en)-3\bar{\sigma_4}(3^{e-1}n)-\tilde{\sigma_4}(3^en)+243\tilde{\sigma_4}(3^{e-1}n)\\
&=\bar{\sigma_4}(n)-3\bar{\sigma_4}(n)-81^e\tilde{\sigma_4}(n)+243\cdot 81^{e-1}\tilde{\sigma_4}(n)\\
&=-2\bar{\sigma_4}(n)+2\cdot 81^e\tilde{\sigma_4}(n)\\
&=-2\bar{\sigma_4}(n)+2\cdot 81^e\chi(n)\bar{\sigma_4}(n)\\
&=2(-1+3^{4e}\chi(n))\bar{\sigma_4}(n).
\end{aligned}$$

Finally, using Lemma 2.10, we have

$$\begin{aligned}
\sum_{\substack{a+b+c+d+e=N\\ N\geq 5}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)&=\frac{1}{81}(-10E_1(n;3)+40\sigma(n)\\
&+30\bar{\sigma_2}(n)-\frac{20}{13}\left(3^{3e+1}+10\right)\sigma_3(n)+(3^{4e}\chi(n)-1)\bar{\sigma_4}(n)).
\end{aligned}$$

Proof of Theorem 1.6

Using Lemma 2.6, we get

$$\begin{aligned}
\frac{[1]_q^{18}}{[3]_q^6}&=1+\sum_{N\geq 1}\left(-\frac{243}{13}e(N)+\frac{9}{13}\sigma_5(N)-\frac{6561}{13}\sigma_5\left(\frac{N}{3}\right)\right)q^N\\
&=1-18\sum_{N\geq 1}\lambda(N)q^N+135\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)q^N\\
&\quad-540\sum_{\substack{a+b+c=N\\ N\geq 3}}\lambda(a)\lambda(b)\lambda(c)q^N+1215\sum_{\substack{a+b+c+d=N\\ N\geq 4}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)q^N\\
&\quad-1458\sum_{\substack{a+b+c+d+e=N\\ N\geq 5}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)q^N\\
&\quad+729\sum_{\substack{a+b+c+d+e+f=N\\ N\geq 6}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f)q^N\\
&=\sum_{N\geq 0}\mathfrak{b}_6(N)q^N.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathfrak{b}_6(N)&=-\frac{243}{13}e(N)+\frac{9}{13}\sigma_5(N)-\frac{6561}{13}\sigma_5\left(\frac{N}{3}\right)\\
&=1-18\sum_{N\geq 1}\lambda(N)+135\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)
\end{aligned}$$

$$- 540 \sum_{\substack{a+b+c=N \\ N\geq 3}} \lambda(a)\lambda(b)\lambda(c) + 1215 \sum_{\substack{a+b+c+d=N \\ N\geq 4}} \lambda(a)\lambda(b)\lambda(c)\lambda(d)$$
$$- 1458 \sum_{\substack{a+b+c+d+e=N \\ N\geq 5}} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)$$
$$+ 729 \sum_{\substack{a+b+c+d+e+f=N \\ N\geq 6}} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f).$$

By simple calculation, we get $\mathfrak{b}_6(0) = 1$, $\mathfrak{b}_6(1) = -18$, $\mathfrak{b}_6(2) = 135$, $\mathfrak{b}_6(3) = -504$,$\mathfrak{b}_6(4) = 657$ and $\mathfrak{b}_6(5) = 2062$. To prove Theorem 1.5 for $N \geq 6$, we have to prove it in the three cases.

For $N \equiv 1 \pmod 3$, we use Theorem 1.1, 1.2, 1.3 and 1.5. Then, we have

$$\sum_{\substack{a+b+c+d+e+f=N \\ N\geq 6}} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f) = \frac{1}{81}(2E_1(N;3) - 10\sigma(N)$$
$$+ 20\bar{\sigma_2}(N) - 5\sigma_3(N) - 15c(N) + 10d(N) - \frac{27}{13}e(N) + \frac{1}{13}\sigma_5(N)).$$

For $N \equiv 2 \pmod 3$, we use Theorem 1.1, 1.2, 1.3 and 1.5. Then, we have

$$\sum_{\substack{a+b+c+d+e+f=N \\ N\geq 6}} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f) = \frac{1}{81}(5\sigma(N) + 20\bar{\sigma_2}(N)$$
$$+ 10\sigma_3(N) + 10d(N) + 2\bar{\sigma_4}(N) - \frac{27}{13}e(N) + \frac{1}{13}\sigma_5(N)).$$

To prove last case, we prove it in the two cases. For $N = 3^e n$ with $e \geq 1$ and $(3, n) = 1$, we use Theorem 1.1, 1.2, 1.3 and 1.5. Then, we have

$$\sum_{\substack{a+b+c+d+e+f=N \\ N\geq 6}} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f) = \frac{1}{81}(-4E_1(n;3) + 20\sigma(n)$$
$$+ 20\bar{\sigma_2}(n) - \frac{20}{13}\left(3^{3e+1} + 10\right)\sigma_3(n) + 2(3^{4e}\chi(n) - 1)\bar{\sigma_4}(n)$$
$$- \frac{27}{13}e(N) + \frac{1}{13}\sigma_5(N) - \frac{729}{13}\sigma_5(3^{e-1}n)).$$

Using

$$\sigma_5(3^e n) - 729\sigma_5(3^{e-1}n) = (\sigma_5(3^e) - 729\sigma_5(3^{e-1}))\sigma_5(n)$$
$$= \frac{1}{121}(364 - 3^{5e+5})\sigma_5(n),$$

we have

$$\sum_{\substack{a+b+c+d+e+f=N \\ N\geq 6}} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f) = \frac{1}{81}(-4E_1(n;3) + 20\sigma(n)$$
$$+ 20\bar{\sigma_2}(n) - \frac{20}{13}\left(3^{3e+1} + 10\right)\sigma_3(n) + 2(3^{4e}\chi(n) - 1)\bar{\sigma_4}(n)$$

$$-\frac{27}{13}e(N)+\frac{1}{1573}(364-3^{5e+5})\sigma_5(n)).$$

Proof of Corollary 1.7 Using Lemma 2.2, we get

$$\begin{aligned}\sum_{N\geq 0}\mathfrak{b}_6(N)q^N &= 1-18\sum_{N\geq 1}\lambda(N)q^N+135\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)q^N\\ &-540\sum_{\substack{a+b+c=N\\N\geq 3}}\lambda(a)\lambda(b)\lambda(c)q^N\\ &+1215\sum_{\substack{a+b+c+d=N\\N\geq 4}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)q^N\\ &-1458\sum_{\substack{a+b+c+d+e=N\\N\geq 5}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)q^N\\ &+729\sum_{\substack{a+b+c+d+e+f=N\\N\geq 6}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f)q^N.\end{aligned}$$

Meanwhile, using Lemma 2.1, we have

$$\begin{aligned}\sum_{N\geq 0}\mathfrak{b}_6(N)q^N &= \left(\sum_{N\geq 0}\mathfrak{b}_2(N)q^N\right)^3=\left(1-6\sum_{N\geq 1}\hat{\sigma}(N)q^N\right)^3\\ &=1-18\sum_{N\geq 1}\hat{\sigma}(N)q^N+108\sum_{N\geq 2}\left(\sum_{k=1}^{N-1}\hat{\sigma}(k)\hat{\sigma}(N-k)\right)q^N\\ &-216\sum_{\substack{a+b+c=N\\N\geq 3}}\hat{\sigma}(a)\hat{\sigma}(b)\hat{\sigma}(c)\\ &=1+3\sum_{N\geq 1}\mathfrak{b}_2(N)q^N+108\sum_{N\geq 2}\left(\sum_{k=1}^{N-1}\hat{\sigma}(k)\hat{\sigma}(N-k)\right)q^N\\ &-216\sum_{\substack{a+b+c=N\\N\geq 3}}\hat{\sigma}(a)\hat{\sigma}(b)\hat{\sigma}(c).\end{aligned}$$

By Lemma 2.2 and Corollary 1.4,

$$\sum_{N\geq 1}\mathfrak{b}_2(N)q^N=-6\sum_{N\geq 1}\lambda(N)q^N+9\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)q^N$$

and

$$\sum_{k=1}^{N-1}\hat{\sigma}(k)\hat{\sigma}(N-k)=\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)-3\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)$$

$$+\frac{9}{4}\sum_{a+b+c+d=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d).$$

So we obtain for $N\geq 6$,

$$\begin{aligned}
&-18\lambda(N)+135\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t))-540\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)\\
&+1215\sum_{a+b+c+d=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\\
&-1458\sum_{a+b+c+d+e=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\\
&+729\sum_{a+b+c+d+e+f=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f)\\
&=-18\lambda(N)+135\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)-324\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)\\
&\quad+243\sum_{a+b+c+d+e=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)-216\sum_{a+b+c=N}\hat{\sigma}(a)\hat{\sigma}(b)\hat{\sigma}(c).
\end{aligned}$$

Finally, for $N\geq 6$, we have

$$\begin{aligned}
\sum_{a+b+c=N}\hat{\sigma}(a)\hat{\sigma}(b)\hat{\sigma}(c)=&\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)\\
&-\frac{9}{2}\sum_{a+b+c+d=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\\
&+\frac{27}{4}\sum_{a+b+c+d+e=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\\
&-\frac{27}{8}\sum_{a+b+c+d+e+f=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f).
\end{aligned}$$

For $N=3,4,5$, the result is also true.

Proof of Corollary 1.8 Using Lemma 2.2, we get

$$\begin{aligned}
\sum_{N\geq 0}\mathfrak{b}_6(N)q^N=1&-18\sum_{N\geq 1}\lambda(N)q^N+135\sum_{N\geq 2}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)q^N\\
&-540\sum_{\substack{a+b+c=N\\N\geq 3}}\lambda(a)\lambda(b)\lambda(c)q^N\\
&+1215\sum_{\substack{a+b+c+d=N\\N\geq 4}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)q^N\\
&-1458\sum_{\substack{a+b+c+d+e=N\\N\geq 5}}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)q^N
\end{aligned}$$

$$+ 729 \sum_{\substack{a+b+c+d+e+f=N \\ N\geq 6}} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f)q^N.$$

Meanwhile, using Lemma 2.2, we have

$$\begin{aligned}\sum_{N\geq 0} \mathfrak{b}_6(N)q^N &= \left(\sum_{N\geq 0} \mathfrak{b}_3(N)q^N\right)^2 = \left(1 - 9\sum_{N\geq 1} \bar{\sigma_2}(N)q^N\right)^2 \\ &= 1 - 18\sum_{N\geq 1} \bar{\sigma_2}(N)q^N + 81\sum_{N\geq 2}\left(\sum_{k=1}^{N-1} \bar{\sigma_2}(k)\bar{\sigma_2}(N-k)\right)q^N \\ &= 1 + 2\sum_{N\geq 1} \mathfrak{b}_3(N)q^N + 81\sum_{N\geq 2}\left(\sum_{k=1}^{N-1} \bar{\sigma_2}(k)\bar{\sigma_2}(N-k)\right)q^N.\end{aligned}$$

Since by Lemma 2.2

$$\begin{aligned}\sum_{N\geq 1} \mathfrak{b}_3(N)q^N = &-9\sum_{N\geq 1}\lambda(N)q^N + 27\sum_{N\geq 1}\left(\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t)\right)q^N \\ &- 27\sum_{N\geq 3}\left(\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)\right)q^N,\end{aligned}$$

we obtain for $N \geq 6$,

$$\begin{aligned}&- 18\lambda(N) + 135\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t) - 540\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c) \\ &+ 1215\sum_{a+b+c+d=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d) \\ &- 1458\sum_{a+b+c+d+e=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e) \\ &+ 729\sum_{a+b+c+d+e+f=N}\lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f) \\ = &-18\lambda(N) + 54\sum_{t=1}^{N-1}\lambda(t)\lambda(N-t) \\ &- 54\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c) + 81\sum_{k=1}^{N-1}\bar{\sigma_2}(k)\bar{\sigma_2}(N-k).\end{aligned}$$

Finally, for $N \geq 6$, we have

$$\sum_{k=1}^{N-1}\bar{\sigma_2}(k)\bar{\sigma_2}(N-k) = \sum_{t=1}^{N-1}\lambda(t)\lambda(N-t) - 6\sum_{a+b+c=N}\lambda(a)\lambda(b)\lambda(c)$$

$$+ 15 \sum_{a+b+c+d=N} \lambda(a)\lambda(b)\lambda(c)\lambda(d) - 18 \sum_{a+b+c+d+e=N} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)$$
$$+ 9 \sum_{a+b+c+d+e+f=N} \lambda(a)\lambda(b)\lambda(c)\lambda(d)\lambda(e)\lambda(f).$$

Now using the main theorems, we have the desired result for $N \geq 6$. For $N = 2, 3, 4, 5$, the result is also true.

Conclusion: The study of convolution sums for restricted divisor numbers is very interesting study in elementary number theory and combinatorics. Many mathematicians approach from various perspectives and re-interpret the meaning of each field. From this point of view, this paper can be understood as a field of theory to understand the number of restricted divisor numbers using infinite product and sums. A study of eta quotients is a frequently used functions in the modular form theory, and it is also a curious attempt to utilize these restricted divisor numbers. By using well-known properties, we derived various interesting results of coefficients derived from eta quotients related to 3. Consequently, the results of this paper may potentially be used, not only in number theory, but also in other areas.

References

1. C. Baiocchi and A. Capelo, *Variational and Quasi Variational Inequalities*, J. Wiley and Sons, New York, 1984.
2. D. Chan and J.S. Pang, *The generalized quasi variational inequality problems*, Math. Oper. Research **7** (1982), 211-222.
3. C. Belly, *Variational and Quasi Variational Inequalities*, J. Appl. Math. and Computing **6** (1999), 234-266.
4. D. Pang, *The generalized quasi variational inequality problems*, J. Appl. Math. and Computing **8** (2002), 123-245.
5. J.M. Borwein, P.B. Borwein, F.G. Garvan, *Some cubic modular identities of Ramanujan*, Trans. Amer. Math. Soc. **343** (1994), 35-47.
6. L. Euler, *Introductio in analysin infinitorum*, Lausanne: Marcum-Michaelem Bousquet, **1** (1748), 1-320.
7. H.M. Farkas, *On an arithmetical function*, Ramanujan J. **8** (2004), 309-315.
8. H.M. Farkas, *On an arithmetical function II*, Contemp. Math. **382** (2005), 121-130.
9. C.G.J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Regiomonti, fratrum Borntrager, 1829.
10. N.J. Fine, *Basic hyper-geometric series and applications*, American Mathematical Society, Providence, Rhode Island, 1988.
11. G.E. Andrew, *The theory of partitions*, Encyclopedia of Mathematics and its Applications, Addison-Wesley, Reading, Mass., 1976.
12. S. Ramanujan, *Collected papers*, AMS Chelsea Publishing, Province RI, USA, 2000.
13. L.E. Dickson, *History of the Theory of Numbers*, Chelsea Publ. Co., New York, 1952.
14. J.W.L. Glaisher, *On the square of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **14** (1884), 56-163.
15. J.W.L. Glaisher, *On certain sums of products of quantities depending upon the divisors of a number*, Mess. Math. **15** (1885), 1-20.
16. J.W.L. Glaisher, *On the representations of a number as the sum of two,four,six,eight,ten and twelve squares*, Quart. J. Pure and Appl. Math. **38** (1907), 1-62.

17. J.W.L. Glaisher, *On the representations of a number as the sum of fourteen and sixteen squares*, Quart. J. Pure and Appl. Math. **38** (1907), 178-236.
18. J. Hwang, Y. Li, D. Kim, *Arithmetic properties derived from coefficients of certain eta quotients*, J. Inequal. Appl. **2020** (2020), 1-23.
19. F. Diamond, J. Shurman, *A first course in modular forms*, Springer-Verlag, New York, 2005.
20. L.K. Hua, *Introduction to number theory*, Springer-Verlag, Berlin-New York, 1982.
21. K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, American Mathematical Society, Province RI, 2004.
22. L. Pehlivan, K.S. Williams, *The power series expansion of certain infinite products* $q^r\prod_{n=1}^{\infty}(1-q^n)^{a_1}(1-q^{2n})^{a_2}\cdots(1-q^{mn})^{a_m}$, Ramanujan Journal **33** (2014), 23-53.
23. K.S. Williams, *Number Theory in the Spirit of Liouville*, Cambridge University Press, New York, 2011.

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