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ON A NEW APPLICATION OF QUASI POWER INCREASING SEQUENCES

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Abstract. In the present paper, a theorem on $\varphi - \mid C, \alpha; \delta \mid_k$ summability of an infinite series is obtained by using a quasi β-power increasing sequence.

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1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants K and M such that $Kc_n \leq b_n \leq Mc_n$ (see [1]). A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that $Kn^\beta \gamma_n \geq m^\beta \gamma_m$ holds for all $n \geq m \geq 1$ (see [13]). Every almost increasing sequence is a quasi β-power increasing sequence for any non-negative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta \lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. Let $\sum a_n$ be an infinite series with partial sums (s_n). By u_n^α and t_n^α, we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, that is (see [9])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \tag{1}$$

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where

$$A_n^{\alpha}=O(n^{\alpha}),\quad A_0^{\alpha}=1\quad \texttt{and}\quad A_{-n}^{\alpha}=0\quad \texttt{for}\quad n>0.$$

Let (ω_n^{α}) be a sequence defined by (see [21])

$$\omega_n^{\alpha}=\begin{cases}|t_n^{\alpha}|, & \alpha=1\\ \max_{1\leq v\leq n}|t_v^{\alpha}|, & 0<\alpha<1\end{cases} \tag{2}$$

Let (φ_n) be a sequence of positive numbers. The series $\sum a_n$ is said to be summable $\varphi-\mid C,\alpha;\delta\mid_k$, $k\geq 1$, $\alpha>-1$, $\delta\geq 0$, if (see [23])

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\mid u_n^{\alpha}-u_{n-1}^{\alpha}\mid^k<\infty.$$

Because of the equality $t_n^{\alpha}=n\left(u_n^{\alpha}-u_{n-1}^{\alpha}\right)$ (see [12]), the definition of the $\varphi-\mid C,\alpha;\delta\mid_k$ summability can be given as

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}n^{-k}\mid t_n^{\alpha}\mid^k<\infty.$$

On taking $\varphi_n=n$ in above definition, we get the definition of $\mid C,\alpha;\delta\mid_k$ summability (see [11]). If we take $\varphi_n=n$ and $\delta=0$, then $\varphi-\mid C,\alpha;\delta\mid_k$ summability is the same as $\mid C,\alpha\mid_k$ summability (see [10]). Also, if we take $\varphi_n=n$, $\delta=0$ and $\alpha=1$, then $\varphi-\mid C,\alpha;\delta\mid_k$ summability is the same as $\mid C,1\mid_k$ summability (see [10]).

The following theorem on $\mid C,\alpha\mid_k$ summability has been proved by Bor and Srivastava (see [7]).

Theorem 1.1. *Let (X_n) be an almost increasing sequence and let there be sequences (μ_n) and (λ_n) such that*

$$\mid\Delta\lambda_n\mid\leq\mu_n, \tag{3}$$

$$\mu_n\to 0\quad as\quad n\to\infty, \tag{4}$$

$$\sum_{n=1}^{\infty}n\mid\Delta\mu_n\mid X_n<\infty, \tag{5}$$

$$\mid\lambda_n\mid X_n=O(1)\quad as\quad n\to\infty. \tag{6}$$

If the sequence (ω_n^{α}) defined by (2) satisfies the condition

$$\sum_{n=1}^{m}\frac{1}{n}(\omega_n^{\alpha})^k=O(X_m)\quad as\quad m\to\infty, \tag{7}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C,\alpha\mid_k$, $k\geq 1$ and $0<\alpha\leq 1$.

2. Main Result

One can see some papers on generalized Cesàro summability ([3]-[6], [14]-[20], [22]). The aim of this paper is to obtain a theorem which generalizes Theorem 1.1 by using a quasi β-power increasing sequence in the following form.

Theorem 2.1. *Let* $(\lambda_n) \in \mathcal{BV}$ *and* (X_n) *be a quasi* β*-power increasing sequence. Let the conditions (3)-(6) be satisfied. If there is an* $\epsilon > 0$ *such that the sequence* $(n^{\epsilon-k}\varphi_n^{\delta k+k-1})$ *is non-increasing and the sequence* (ω_n^α) *defined by (2) satisfies the condition*

$$\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} n^{-k} (\omega_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty, \tag{8}$$

then the series $\sum a_n\lambda_n$ *is summable* $\varphi - \mid C, \alpha; \delta \mid_k$, $k \geq 1$, $\delta \geq 0$, $0 < \alpha \leq 1$ *and* $\alpha k + \epsilon > 1$.

3. Lemmas

To prove Theorem 2.1, we need the lemmas given below.

Lemma 3.1. [8] *If* $0 < \alpha \leq 1$ *and* $1 \leq v \leq n$, *then*

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p \right|. \tag{9}$$

Lemma 3.2. [13] *Under the conditions on* (X_n), (μ_n) *and* (λ_n) *as taken in the statement of Theorem 2.1, we have*

$$n\mu_n X_n = O(1) \quad as \quad n \to \infty \tag{10}$$

$$\sum_{n=1}^{\infty} \mu_n X_n < \infty. \tag{11}$$

Proof of Theorem 2.1. Let $0 < \alpha \leq 1$. Let (M_n^α) be the nth (C, α) mean of the sequence $(na_n\lambda_n)$. So, we have

$$M_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v$$

by (1). Applying Abel's transformation, we get

$$M_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v.$$

Here, using Lemma 3.1, we have

$$\begin{aligned} |M_n^\alpha| &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid \left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p\right| + \frac{\mid \lambda_n \mid}{A_n^\alpha}\left|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v\right| \\ &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha \omega_v^\alpha \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \omega_n^\alpha \\ &= M_{n,1}^\alpha + M_{n,2}^\alpha . \end{aligned}$$

To complete the proof, we need to show

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} n^{-k} \mid M_{n,r}^\alpha \mid^k < \infty, \quad \text{for} \quad r=1,2.$$

First, applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\begin{aligned} &\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} n^{-k} \mid M_{n,1}^\alpha \mid^k \\ = &\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} n^{-k} \left| \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha \omega_v^\alpha \mid \Delta\lambda_v \mid \right|^k \\ \leq &\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} n^{-k} (A_n^\alpha)^{-k} \sum_{v=1}^{n-1} (A_v^\alpha)^k (\omega_v^\alpha)^k \mid \Delta\lambda_v \mid \left\{ \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid \right\}^{k-1} . \end{aligned}$$

Here, using the fact that $(\lambda_n) \in \mathcal{BV}$, we get $\sum \mid \Delta\lambda_v \mid < \infty$. Also, by using (3), we have

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} n^{-k} \mid M_{n,1}^\alpha \mid^k = O(1)\sum_{v=1}^{m} v^{\alpha k} (\omega_v^\alpha)^k \mu_v \sum_{n=v+1}^{m+1} \frac{\varphi_n^{\delta k+k-1} n^{\epsilon-k}}{n^{\alpha k+\epsilon}} .$$

Then, we get

$$\begin{aligned} &\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} n^{-k} \mid M_{n,1}^\alpha \mid^k \\ = &O(1)\sum_{v=1}^{m} v^{\alpha k} (\omega_v^\alpha)^k \mu_v \varphi_v^{\delta k+k-1} v^{\epsilon-k} \int_v^\infty \frac{dx}{x^{\alpha k+\epsilon}} \\ = &O(1)\sum_{v=1}^{m-1} \Delta(v\mu_v) \sum_{r=1}^{v} \varphi_r^{\delta k+k-1} r^{-k} (\omega_r^\alpha)^k + O(1) m\mu_m \sum_{v=1}^{m} \varphi_v^{\delta k+k-1} v^{-k} (\omega_v^\alpha)^k \\ = &O(1)\sum_{v=1}^{m-1} v|\Delta\mu_v| X_v + O(1)\sum_{v=1}^{m-1} \mu_{v+1} X_{v+1} + O(1) m\mu_m X_m \\ = &O(1) \quad as \quad m \to \infty, \end{aligned}$$

by Abel's transformation and the conditions (8), (5), (11) and (10).

For $r = 2$,

$$\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} n^{-k} \mid M_{n,2}^{\alpha} \mid^k = \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} n^{-k} \left|\lambda_n\right|^k (\omega_n^{\alpha})^k.$$

Since $|\lambda_n| X_n = O(1)$ by (6), we write $|\lambda_n|^{k-1} = O(1)$. So, we get

$$\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} n^{-k} \mid M_{n,2}^{\alpha} \mid^k = \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} n^{-k} \left|\lambda_n\right| (\omega_n^{\alpha})^k.$$

Then applying Abel's transformation and using the conditions (8), (3), (11) and (6), we get

$$\begin{aligned} & \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} n^{-k} \mid M_{n,2}^{\alpha} \mid^k \\ = \quad & O(1) \sum_{n=1}^{m-1} \Delta \left|\lambda_n\right| \sum_{v=1}^{n} \varphi_v^{\delta k+k-1} v^{-k} (\omega_v^{\alpha})^k + O(1) \left|\lambda_m\right| \sum_{v=1}^{m} \varphi_v^{\delta k+k-1} v^{-k} (\omega_v^{\alpha})^k \\ = \quad & O(1) \sum_{n=1}^{m-1} \left|\Delta\lambda_n\right| X_n + O(1) \left|\lambda_m\right| X_m \\ = \quad & O(1) \sum_{n=1}^{m-1} \mu_n X_n + O(1) \left|\lambda_m\right| X_m \\ = \quad & O(1) \quad as \quad m \to \infty. \end{aligned}$$

So the proof of Theorem 2.1 is completed. □

If we take (X_n) as a positive non-decreasing sequence in Theorem 2.1, then we get a known result (see [23]). Also, if we take (X_n) as an almost increasing sequence, $\varphi_n = n$, $\delta = 0$ and $\epsilon = 1$ in Theorem 2.1, then we get Theorem 1.1. Furthermore, if we take (X_n) as a positive non-decreasing sequence, $\varphi_n = n$, $\delta = 0$, $\alpha = 1$ and $\epsilon = 1$ in Theorem 2.1, then we get a known theorem on $|C,1|_k$ summability (see [2]).

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