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THE CONNECTIVITY AND THE MODIFIED SECOND MULTIPLICATIVE ZAGREB INDEX OF GRAPHS†

JIANWEI DU*, XIAOLING SUN

Abstract. Zagreb indices and their modified versions of a molecular graph are important descriptors which can be used to characterize the structural properties of organic molecules from different aspects. In this work, we investigate some properties of the modified second multiplicative Zagreb index of graphs with given connectivity. In particular, we obtain the maximum values of the modified second multiplicative Zagreb index with fixed number of cut edges, or cut vertices, or edge connectivity, or vertex connectivity of graphs. Furthermore, we characterize the corresponding extremal graphs.

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1. Introduction

Topological indices are mathematical descriptors reflecting some structural characteristics of organic molecules on the molecular graph, and they play an important role in chemistry, pharmacology, etc. (see [12,13,18]). The famous Zagreb indices, first introduced by Gutman and Trinajstić [14], are used to examine the structure dependence of total π-electron energy on molecular orbital. The first Zagreb index M_1 and the second Zagreb index M_2 of a graph G are defined as:

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2, \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v),$$

where $d_G(u)$ is the degree of vertex u.

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These two classical topological indices (M_1 and M_2) and their variations have been applied in studying heterosystems, ZE-isomerism, chirality and complexity of molecule, etc. Todeschini et al. [19] presented a version of Zagreb indices which nowadays are called multiplicative Zagreb indices, and they are expressed as:

$$\Pi_1(G) = \prod_{v\in V(G)} d_G(v)^2, \quad \Pi_2(G) = \prod_{uv\in E(G)} d_G(u)d_G(v).$$

Recently, Gutman, Eliasi and Iranmanesh, respectively [8, 11] introduced the modified first multiplicative Zagreb index (also called the multiplicative sum Zagreb index) of a graph defined as

$$\Pi_1^*(G) = \prod_{uv\in E(G)} (d_G(u) + d_G(v)).$$

Relevant results on the modified first multiplicative Zagreb index can be found in [2,5,7,8,11,22].

In 2016, Basavanagoud et al. [3] introduced another multiplicative version called the modified second multiplicative Zagreb index (denoted by Π_2^*) and defined as

$$\Pi_2^*(G) = \prod_{uv\in E(G)} (d_G(u) + d_G(v))^{(d_G(u)+d_G(v))}.$$

Basavanagoud et al. [3] studied several derived graphs. Wang et al. [20] determined the maximal and minimal modified multiplicative Zagreb indices of graphs with vertex connectivity or edge connectivity at most k.

In this work, we only deal with simple connected graphs. Let $G = (V(G), E(G))$ be the graph having vertex set $V(G)$ and edge set $E(G)$. Given a graph G, we use $G - x$ or $G - xy$ to denote the graph that arises from G by deleting the vertex $x \in V(G)$ or the edge $xy \in E(G)$. Similarly, $G + xy$ is a graph that arises from G by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$. Let $E' \subseteq E(G)$, we use $G - E'$ to denote the subgraph of G obtained by deleting the edges of E'. For $X \subseteq V(G)$, $G - X$ denotes the subgraph of G obtained by deleting the vertices of X and the edges incident with them. A block of a graph is a maximum connected subgraph with no cut vertex. If a block has at most one cut vertex in the graph as a whole, we call it an endblock. A clique of a graph G is a subset $W \subset V(G)$ such that $G[W]$ is complete. As usual, we use P_n, K_n and S_n to denote the paths, the complete graphs and the stars on n vertices, respectively.

Let $P_r = x_0x_1\cdots x_r$ ($r \geq 1$) be a path of graph G with $d_G(x_1) = \cdots = d_G(x_{r-1}) = 2$ (unless $r = 1$). If $d_G(x_0), d_G(x_r) \geq 3$, then P_r is called an internal path of G; if $d_G(x_0) \geq 3, d_G(x_r) = 1$, then P_r is called a pendant path of G. $G_1 \cup G_2$ denotes the vertex-disjoint union of the graphs G_1 and G_2, and $G_1 \vee G_2$ denotes the graph arising from $G_1 \cup G_2$ by adding all possible edges between the vertices of G_1 and the vertices of G_2. We denote by $\gamma(G) = |E(G)| - |V(G)| + 1$

the cyclomatic number of graph G. The k cyclic graph is the graph whose cyclomatic number is k. For $\gamma(G) = 0$, G is a tree.

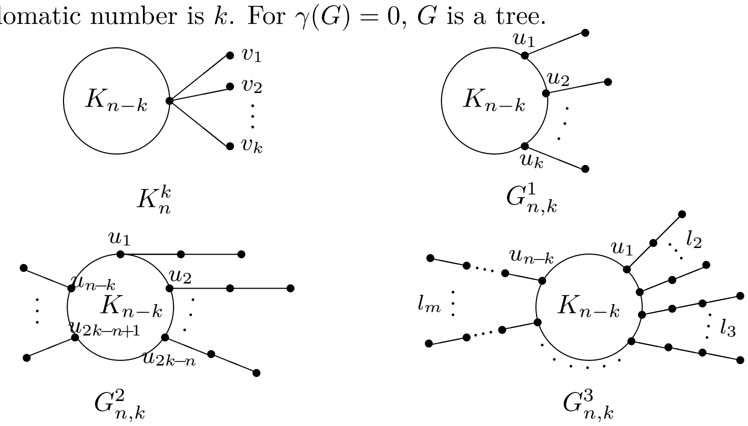

Figure 1. K_n^k, $G^1_{n,k}$, $G^2_{n,k}$ and $G^3_{n,k}$.

Let K_n^k (as shown in Figure 1) be the graph obtained by identifying one vertex of K_{n-k} with the central vertex of star S_{k+1}. Let $G^1_{n,k}$ (as shown in Figure 1) be the graph arising from K_{n-k} by attaching at most one pendant edge to each of its vertices, where $0 < k \leq \frac{n}{2}$. Let $G^2_{n,k}$ (as shown in Figure 1) be the graph arising from K_{n-k} by attaching one pendant path of length 2 to $2k - n$ vertices $(u_1, \cdots, u_{2k-n})$ of K_{n-k}, and attaching one pendant edge to the other $2n - 3k$ vertices $(u_{2k-n+1}, \cdots, u_{n-k})$ of K_{n-k}, where $\frac{n}{2} < k \leq \frac{2n}{3}$. Let $G^3_{n,k}$ (as shown in Figure 1) be the graph obtained from K_{n-k} by attaching exactly one pendant path of length greater than 1 to each vertex of K_{n-k}, where $\frac{2n}{3} < k \leq n - 3$, $l_2 + l_3 + \cdots + l_m = n - k$ and $2l_2 + 3l_3 + \cdots + ml_m = k$ (l_t is the number of paths with length t, $t = 2, 3, \cdots, m$). We can see [4] for other terminologies and notations.

There are many papers on the topological indices and the connectivity of graphs, such as [1,6,7,9,10,15-17,20,21,23]. Inspired by this, we go on studying the mathematical properties of the connectivity and the modified multiplicative Zagreb indices of graphs. The authors of this paper obtained some results on the connectivity and the modified first multiplicative Zagreb index of graphs [7]. The values of the modified second multiplicative Zagreb index are usually more difficult to determine. In this work, we present the maximum values of the modified second multiplicative Zagreb index with fixed number of cut edges, or cut vertices, or edge connectivity, or vertex connectivity of a graph. Furthermore, we characterize the corresponding extremal graphs.

2. Preliminaries

By the definition of Π_2^*, the following Lemma 2.1 is immediate.

Lemma 2.1. *Let $G = (V(G), E(G))$ be a simple connected graph. Then*
(*i*) *For each $e \in E(G)$, $\Pi_2^*(G) > \Pi_2^*(G - e)$;*

(ii) For each $e = uv \notin E(G)$, $u, v \in V(G)$, $\Pi_2^*(G) < \Pi_2^*(G+e)$.

Lemma 2.2. *Let* $l(x) = \frac{(x+a)^{x+a}}{x^x}$, *where* $x \geq 1$ *and* $a \geq 1$. *Then* $l(x)$ *is increasing for* $x \geq 1$.

Proof. Let $L(x) = \ln l(x) = (x+a)\ln(x+a) - x\ln x$. Then $L'(x) = \ln \frac{x+a}{x} > 0$. Thus $l(x)$ is increasing for $x \geq 1$. □

Lemma 2.3. *Let* n_1, n_2, s *be positive integers, where* $n_2 \geq n_1 \geq 2$ *and* $s \geq 1$. *Then*

$$\frac{(2n_1+2s-4)^{(2n_1+2s-4)\binom{n_1-1}{2}}(2n_2+2s)^{(2n_2+2s)\binom{n_2+1}{2}}}{(2n_1+2s-2)^{(2n_1+2s-2)\binom{n_1}{2}}(2n_2+2s-2)^{(2n_2+2s-2)\binom{n_2}{2}}} > 1.$$

Proof. Let $f(x) = (x^2+x)(2x+2s)\ln(2x+2s) - (x^2-x)(2x+2s-2)\ln(2x+2s-2)$ be a real function in x, where $x \geq 1$. Then

$$\begin{aligned} f'(x) =& 2[(2x+1)(x+s)+x^2+x]\ln(2x+2s) \\ & -2[(2x-1)(x+s-1)+x^2-x]\ln(2x+2s-2)+4x. \end{aligned}$$

Since $(2x+1)(x+s)+x^2+x > (2x-1)(x+s-1)+x^2-x$, then

$$\begin{aligned} f'(x) >& 2[(2x-1)(x+s-1)+x^2-x]\ln(2x+2s) \\ & -2[(2x-1)(x+s-1)+x^2-x]\ln(2x+2s-2) \\ =& 2[(2x-1)(x+s-1)+x^2-x]\ln\frac{(2x+2s)}{(2x+2s-2)} > 0. \end{aligned}$$

Thus $f(n_2) > f(n_1-1)$, that is,

$$\begin{aligned} & (n_2^2+n_2)(2n_2+2s)\ln(2n_2+2s) - (n_2^2-n_2)(2n_2+2s-2)\ln(2n_2+2s-2) \\ & > (n_1^2-n_1)(2n_1+2s-2)\ln(2n_1+2s-2) \\ & \quad - ((n_1-1)^2-(n_1-1))(2n_1+2s-4)\ln(2n_1+2s-4) \\ \Longrightarrow & \frac{(n_1-1)(n_1-2)}{2}\ln(2n_1+2s-4)^{(2n_1+2s-4)} + \frac{(n_2+1)n_2}{2}\ln(2n_2+2s)^{(2n_2+2s)} \\ & > \frac{n_1(n_1-1)}{2}\ln(2n_1+2s-2)^{(2n_1+2s-2)} + \frac{n_2(n_2-1)}{2}\ln(2n_2+2s-2)^{(2n_2+2s-2)} \\ \Longrightarrow & \ln\left((2n_1+2s-4)^{(2n_1+2s-4)\frac{(n_1-1)(n_1-2)}{2}}(2n_2+2s)^{(2n_2+2s)\frac{(n_2+1)n_2}{2}}\right) \\ & > \ln\left((2n_1+2s-2)^{(2n_1+2s-2)\frac{n_1(n_1-1)}{2}}(2n_2+2s-2)^{(2n_2+2s-2)\frac{n_2(n_2-1)}{2}}\right) \\ \Longrightarrow & (2n_1+2s-4)^{(2n_1+2s-4)\frac{(n_1-1)(n_1-2)}{2}}(2n_2+2s)^{(2n_2+2s)\frac{(n_2+1)n_2}{2}} \\ & > (2n_1+2s-2)^{(2n_1+2s-2)\frac{n_1(n_1-1)}{2}}(2n_2+2s-2)^{(2n_2+2s-2)\frac{n_2(n_2-1)}{2}}. \end{aligned}$$

This finishes the proof. □

Lemma 2.4. *Let n, a, n_1, n_2 be positive integers, where $n_2 \geq n_1 \geq 2$, $n_1+n_2 < n$ and $a \geq n-1$. Then*

$$\frac{(a+n_1-1)^{(a+n_1-1)(n_1-1)}(a+n_2+1)^{(a+n_2+1)(n_2+1)}}{(a+n_1)^{(a+n_1)n_1}(a+n_2)^{(a+n_2)n_2}} > 1.$$

Proof. Let $g(x) = x(a+x)\ln(a+x) - (x-1)(a+x-1)\ln(a+x-1)$ be a real function in x, where $a \geq n-1, x \geq 2$. Then

$$\begin{aligned} g'(x) &= 1 + (2x+a)\ln(a+x) - (2x+a-2)\ln(a+x-1) \\ &> (2x+a-2)\ln(a+x) - (2x+a-2)\ln(a+x-1) \\ &= (2x+a-2)\ln\frac{a+x}{a+x-1} > 0. \end{aligned}$$

Thus $g(n_2+1) > g(n_1)$, that is,

$$\begin{aligned} & (n_2+1)(a+n_2+1)\ln(a+n_2+1) - n_2(a+n_2)\ln(a+n_2) \\ & > n_1(a+n_1)\ln(a+n_1) - (n_1-1)(a+n_1-1)\ln(a+n_1-1) \\ \Longrightarrow & \ln\left((a+n_2+1)^{(a+n_2+1)(n_2+1)}(a+n_1-1)^{(a+n_1-1)(n_1-1)}\right) \\ & > \ln\left((a+n_1)^{(a+n_1)n_1}(a+n_2)^{(a+n_2)n_2}\right) \\ \Longrightarrow & (a+n_2+1)^{(a+n_2+1)(n_2+1)}(a+n_1-1)^{(a+n_1-1)(n_1-1)} \\ & > (a+n_1)^{(a+n_1)n_1}(a+n_2)^{(a+n_2)n_2}. \end{aligned}$$

This completes the proof. □

3. Modified second multiplicative Zagreb index of graphs with fixed number of cut edges

We use $\mathbf{G}_E(n,k)$ to denote the n-vertex graphs with k cut edges.

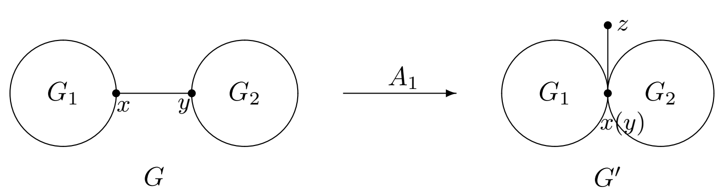

Figure 2. Transformation A_1.

Transformation A_1: Suppose G_1 and G_2 are graphs with $n_1 \geq 3$ and $n_2 \geq 2$ vertices, respectively, where G_1 is 2-edge connected. Suppose G is a graph, as shown in Figure 2, obtained from G_1 and G_2 by adding an edge from a vertex $x \in V(G_1)$ to a vertex $y \in V(G_2)$. Then xy is a non-pendant cut edge in G. Let G' be the graph obtained by identifying x of G_1 with y of G_2 and adding a pendant edge to $x(y)$, as shown in Figure 2.

Lemma 3.1. *Suppose G' and G are graphs in Figure 2. Then $\Pi_2^*(G') > \Pi_2^*(G)$.*

Proof. Denote $N_{G_1}(x) = \{x_1, x_2, \cdots, x_{d_1}\}$ and $N_{G_2}(y) = \{y_1, y_2, \cdots, y_{d_2}\}$. Since the function $(x+a)^{x+a}$ $(x \geq 1, a \geq 1)$ is increasing for x, by the definition of Π_2^* , it follows that

$$\frac{\Pi_2^*(G')}{\Pi_2^*(G)} = \frac{(d_1+d_2+2)^{(d_1+d_2+2)}\Big(\prod_{i=1}^{d_1}(d_{G_1}(x_i)+d_1+d_2+1)^{(d_{G_1}(x_i)+d_1+d_2+1)}\Big)}{(d_1+d_2+2)^{(d_1+d_2+2)}\Big(\prod_{i=1}^{d_1}(d_{G_1}(x_i)+d_1+1)^{(d_{G_1}(x_i)+d_1+1)}\Big)} \cdot \frac{\prod_{j=1}^{d_2}(d_{G_2}(y_j)+d_1+d_2+1)^{(d_{G_2}(y_j)+d_1+d_2+1)}}{\prod_{j=1}^{d_2}(d_{G_2}(y_j)+d_2+1)^{(d_{G_2}(y_j)+d_2+1)}} > 1.$$

The proof is completed. □

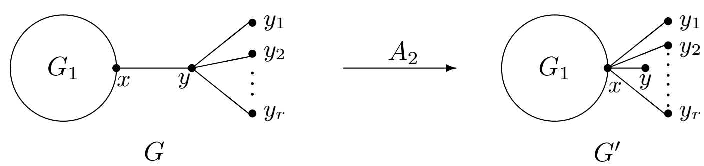

Figure 3. Transformation A_2.

Transformation A_2: Suppose G is a graph as shown in Figure 3, where xy is a non-pendant cut edge of G, G_1 is 2-edge connected, $d_G(x) \geq 2$, $N_G(y)/\{x\} = \{y_1, y_2, \cdots, y_r\}$ ($y_1, y_2, \cdots, y_r$ are pendant vertices). $G' = G - \{yy_1, yy_2, \cdots, yy_r\} + \{xy_1, xy_2, \cdots, xy_r\}$, as shown in Figure 3.

Lemma 3.2. *Suppose G and G' are graphs in Figure 3. Then $\Pi_2^*(G') > \Pi_2^*(G)$.*

Proof. Denote $N_{G_1}(x) = \{x_1, x_2, \cdots, x_s\}$. By the definition of Π_2^*, it follows that

$$\frac{\Pi_2^*(G')}{\Pi_2^*(G)} = \frac{(r+s+2)^{(r+s+2)}\Big(\prod_{i=1}^{s}(d_{G_1}(x_i)+r+s+1)^{(d_{G_1}(x_i)+r+s+1)}\Big)}{(r+s+2)^{(r+s+2)}\Big(\prod_{i=1}^{s}(d_{G_1}(x_i)+s+1)^{(d_{G_1}(x_i)+s+1)}\Big)} \cdot \frac{\prod_{j=1}^{r}(s+r+2)^{(s+r+2)}}{\prod_{j=1}^{r}(r+2)^{(r+2)}} > 1.$$

This finishes the proof. □

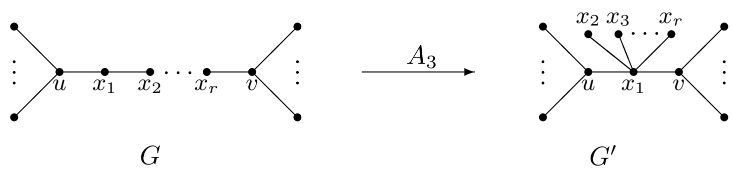

Figure 4. Transformation A_3.

Transformation A_3: Let $P = ux_1x_2\cdots x_rv$ $(r \geq 2)$ be an internal path in G, i.e., $d_G(x_i) = 2$ for $i = 1, 2, \cdots, r$, $d_G(u) \geq 2$ and $d_G(v) \geq 2$. Let $G' = G - \{x_2x_3, x_3x_4, \cdots, x_{r-1}x_r, x_rv\} + \{x_1x_3, x_1x_4, \cdots, x_1x_r, x_1v\}$, as shown in Figure 4.

Lemma 3.3. *Suppose G and G' are graphs in Figure 4. Then $\Pi_2^*(G') > \Pi_2^*(G)$.*

Proof. Denote $d_G(u) = s$ and $d_G(v) = t$. By the definition of Π_2^*, it follows that

$$\frac{\Pi_2^*(G')}{\Pi_2^*(G)} = \frac{(r+s+1)^{(r+s+1)}(r+t+1)^{(r+t+1)}\left(\prod_{i=2}^{r}(r+2)^{(r+2)}\right)}{(s+2)^{(s+2)}(t+2)^{(t+2)}\left(\prod_{i=2}^{r}4^4\right)} > 1.$$

This finishes the proof. □

$\mathbf{G}_E(n, n-1)$ is a tree, we give a theorem below.

Theorem 3.4. *Suppose $G \in \mathbf{G}_E(n, n-1)$, i.e., G is a tree. Then*

$$\Pi_2^*(G) \leq n^{n(n-1)}$$

with equality if and only if $G \cong S_n$.

Proof. Repeating Transformation A_2, any tree T of size s attached to graph G can be changed into a star S_{s+1}. And the $\Pi_2^*(G)$ increases by Lemma 3.2. Then G with maxium Π_2^* must be a caterpillar. Considering Transformations A_1 and A_3, from Lemmas 3.1 and 3.3, we conclude that any caterpillar can be changed into star S_n with a larger Π_2^*. Thus the result follows immediately. □

Let $G \in \mathbf{G}_E(n, k)$. If $\gamma(G) \geq 1$, then $k \leq n - 3$. Thus, in what follows, we discuss the case of $1 \leq k \leq n - 3$ when $G \in \mathbf{G}_E(n, k)$.

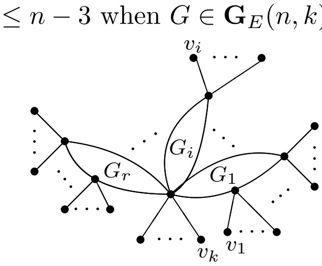

Figure 5. The graph G^*.

Remark 3.1. For any $G \in \mathbf{G}_E(n,k)$, if necessary, by repeating the graph transformation A_1 or A_2, any cut edges in G can changed into pendant edges. That is, if necessary, by a series of transformation A_1 or A_2, we can change G to G^* (as depicted in Figure 5), where $G_1, G_2, \cdots, G_r$ are 2-edge connected graphs.

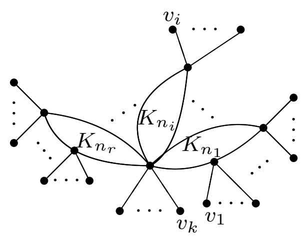

Figure 6. The graph H.

By Lemma 3.1, 3.2 and Remark 3.1, the following Lemma 3.5 is obtained immediately.

Lemma 3.5. *Suppose* $G \in \mathbf{G}_E(n,k)$. *Then* $\Pi_2^*(G) \le \Pi_2^*(G^*)$, *where* G^* *are graphs as depicted in Figure 5.*

Let K_{n_i} $(1 \le i \le r)$ be a clique which is obtained by adding edges in G_i $(1 \le i \le r)$ and changing G_i into complete subgraphs, where $G_1, G_2, \cdots, G_r$ in G^* are 2-edge connected graphs. By Lemma 2.1, we get the following Lemma 3.6.

Lemma 3.6. *Suppose* H *is the graph as depicted in Figure 6, where* K_{n_i} $(1 \le i \le r)$ *are cliques as above. Then* $\Pi_2^*(H) \ge \Pi_2^*(G^*)$.

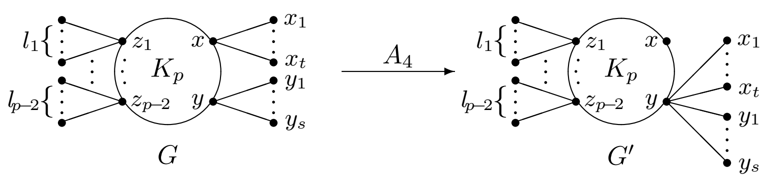

Figure 7. Transformation A_4.

Transformation A_4: Suppose G is a graph as depicted in Figure 7, $V(K_p) = \{x, y, z_1, \cdots, z_{p-2}\}$, each vertex on K_p either is of degree $p-1$ or has some pendant edges attached, where $p \ge 3$, $l_1, \cdots, l_{p-2} \ge 0$. $x_1, x_2, \cdots, x_t$ and $y_1, y_2, \cdots, y_s$ are pendant vertices adjacent to x and y, respectively, where $t, s \ge 1$. Let $G' = G - \{xx_1, xx_2, \cdots, xx_t\} + \{yx_1, yx_2, \cdots, yx_t\}$, as depicted in Figure 7.

Lemma 3.7. *Suppose* G' *and* G *are graphs in Figure 7. Then* $\Pi_2^*(G') > \Pi_2^*(G)$.

Proof. It is evident that $d_{G'}(x) = p-1$, $d_G(x) = p-1+t$, $d_{G'}(y) = p-1+t+s$, $d_G(y) = p-1+s$. By the definition of Π_2^* and Lemma 2.2, we find that

$$\begin{aligned}\frac{\Pi_2^*(G')}{\Pi_2^*(G)} =& \frac{(2p-2+t+s)^{(2p-2+t+s)}(p+t+s)^{(p+t+s)(t+s)}}{(2p-2+t+s)^{(2p-2+t+s)}(p+t)^{(p+t)t}(p+s)^{(p+s)s}} \\ &\cdot \frac{\prod_{i=1}^{p-2}\Big((d_G(z_i)+p-1)^{(d_G(z_i)+p-1)}(d_G(z_i)+p-1+t+s)^{(d_G(z_i)+p-1+t+s)}\Big)}{\prod_{i=1}^{p-2}\Big((d_G(z_i)+p-1+t)^{(d_G(z_i)+p-1+t)}(d_G(z_i)+p-1+s)^{(d_G(z_i)+p-1+s)}\Big)} \\ =& \Big(\frac{(p+t+s)^{(p+t+s)}}{(p+t)^{(p+t)}}\Big)^t \Big(\frac{(p+t+s)^{(p+t+s)}}{(p+s)^{(p+s)}}\Big)^s \\ &\cdot \prod_{i=1}^{p-2} \frac{\frac{(d_G(z_i)+p-1+t+s)^{(d_G(z_i)+p-1+t+s)}}{((d_G(z_i)+p-1+s)^{(d_G(z_i)+p-1+s)}}}{\frac{(d_G(z_i)+p-1+t)^{(d_G(z_i)+p-1+t)}}{(d_G(z_i)+p-1)^{(d_G(z_i)+p-1)}}} > 1.\end{aligned}$$

The proof is completed. □

Theorem 3.8. *Suppose $G \in \mathbf{G}_E(n,k)$, where $1 \le k \le n-3$. Then*

$$\Pi_2^*(G) \le n^{nk}(2n-k-2)^{(2n-k-2)(n-k-1)}(2n-2k-2)^{(2n-2k-2)\binom{n-k-1}{2}}$$

with equality if and only if $G \cong K_n^k$.

Proof. Assume that $G \in \mathbf{G}_E(n,k)$ has the maximum $\Pi_2^*(G)$. By Lemma 3.5 and 3.6, it follows that $\Pi_2^*(G) \le \Pi_2^*(H)$.

Next, we prove that $r = 1$. By contradiction. If $r \ge 2$, suppose without loss of generality that there exists an edge $e = xy \notin E(G)$, $x \in V(K_{n_i})$, $y \in V(K_{n_j})$, $1 \le i < j \le r$, and x, y is not the common vertex of K_{n_i} and K_{n_j}. By Lemma 2.1, it can be seen that $\Pi_2^*(G+e) > \Pi_2^*(G)$, a contradiction. So $r = 1$. Thus G is a graph obtained from K_{n-k} by attaching some pendant edges to some vertices of K_{n-k} (the number of all pendant edges of G is k). By Lemma 3.7, $G \cong K_n^k$. □

4. Modified second multiplicative Zagreb index of graphs with fixed number of cut vertices

We use $\mathbf{G}_V(n,k)$ to denote the n-vertex graphs with k cut vertices. Since $\mathbf{G}_V(n, n-2)$ is a path, thus, in this section, we always discuss the case of $1 \le k \le n-3$ when $G \in \mathbf{G}_V(n,k)$.

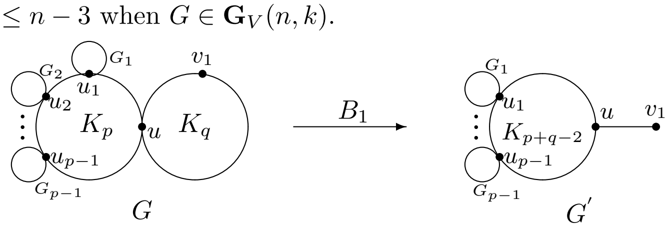

Figure 8. Transformation B_1.

Transformation B_1: Suppose G is a graph as depicted in Figure 8, K_p $(p \geq 2)$ and K_q $(q \geq 3)$ are two cliques of G, where K_q is an endblock. $V(K_p)$ and $V(K_q)$ have one cut vertex, say u, in common. $V(K_p) = \{u_1, u_2, \cdots, u_{p-1}, u\}$, $V(K_q) = \{v_1, v_2, \cdots, v_{q-1}, u\}$. G_i $(1 \leq i \leq p-1)$ is the subgraph attached to u_i $(1 \leq i \leq p-1)$ $(d_G(u_1) \geq 2$ when $p = 2)$. Let $G' = G - \{v_1v_2, v_1v_3, \cdots, v_1v_{q-1}\} + \{u_1v_2, u_1v_3, \cdots, u_1v_{q-1}\} + \cdots + \{u_{p-1}v_2, u_{p-1}v_3, \cdots, u_{p-1}v_{q-1}\}$, as depicted in Figure 8.

Lemma 4.1. *Suppose G' and G are graphs in Figure 8. Then* $\Pi_2^*(G') > \Pi_2^*(G)$.

Proof. Observe that $d_G(u) = d_{G'}(u) = p + q - 2$, $d_G(v_1) = q - 1$, $d_{G'}(v_1) = 1$, $d_{G'}(u_i) = d_G(u_i) + q - 2$ $(i = 1, 2, \cdots, p-1)$, $d_{G'}(v_j) = p + q - 3$ $(j = 2, 3, \cdots, q-1)$. For $x \in N_{G_i}(u_i)$, $d_{G'}(u_i) + d_{G'}(x) = d_G(u_i) + d_G(x) + q - 2 > d_G(u_i) + d_G(x)$, $i = 1, 2, \cdots, p-1$. Then $\frac{(d_{G'}(u_i)+d_{G'}(x))^{(d_{G'}(u_i)+d_{G'}(x))}}{(d_G(u_i)+d_G(x))^{(d_G(u_i)+d_G(x))}} > 1$, where $x \in N_{G_i}(u_i)$, $i = 1, 2, \cdots, p-1$.

If $p = 2$, $d_G(u_1) \geq 2$, by the definition of Π_2^* and Lemma 2.2, it follows that

$$
\begin{aligned}
\frac{\Pi_2^*(G')}{\Pi_2^*(G)} =& \frac{(d_{G'}(u) + d_{G'}(v_1))^{(d_{G'}(u)+d_{G'}(v_1))}(d_{G'}(u) + d_{G'}(u_1))^{(d_{G'}(u)+d_{G'}(u_1))}}{(d_G(u) + d_G(v_1))^{(d_G(u)+d_G(v_1))}(d_G(u) + d_G(u_1))^{(d_G(u)+d_G(u_1))}} \\
&\cdot \frac{\prod_{i=2}^{q-1}(d_{G'}(u_1)+d_{G'}(v_i))^{(d_{G'}(u_1)+d_{G'}(v_i))}\prod_{i=2}^{q-1}(d_{G'}(u)+d_{G'}(v_i))^{(d_{G'}(u)+d_{G'}(v_i))}}{\prod_{i=2}^{q-1}(d_G(v_1)+d_G(v_i))^{(d_G(v_1)+d_G(v_i))}\prod_{i=2}^{q-1}(d_G(u)+d_G(v_i))^{(d_G(u)+d_G(v_i))}} \\
&\cdot \frac{\prod_{x\in N_{G_1}(u_1)}(d_{G'}(u_1) + d_{G'}(x))^{(d_{G'}(u_1)+d_{G'}(x))}}{\prod_{x\in N_{G_1}(u_1)}(d_G(u_1) + d_G(x))^{(d_G(u_1)+d_G(x))}} \\
>& \frac{(q+1)^{(q+1)}(d_G(u_1)+2q-2)^{(d_G(u_1)+2q-2)}(d_G(u_1)+2q-3)^{(d_G(u_1)+2q-3)(q-2)}}{(2q-1)^{(2q-1)}(d_G(u_1)+q)^{(d_G(u_1)+q)}(2q-2)^{(2q-2)(q-2)}} \\
=& \frac{\frac{(d_G(u_1)+2q-2)^{(d_G(u_1)+2q-2)}}{(d_G(u_1)+q)^{(d_G(u_1)+q)}}}{\frac{(2q-1)^{(2q-1)}}{(q+1)^{(q+1)}}} \cdot \Big(\frac{(d_G(u_1)+2q-3)^{(d_G(u_1)+2q-3)}}{(2q-2)^{(2q-2)}}\Big)^{q-2} > 1.
\end{aligned}
$$

If $p \geq 3$, we have

$$
\begin{aligned}
\frac{\Pi_2^*(G')}{\Pi_2^*(G)} =& \frac{(d_{G'}(u)+d_{G'}(v_1))^{(d_{G'}(u)+d_{G'}(v_1))}\prod_{i=1}^{p-1}(d_{G'}(u)+d_{G'}(u_i))^{(d_{G'}(u)+d_{G'}(u_i))}}{(d_G(u) + d_G(v_1))^{(d_G(u)+d_G(v_1))}\prod_{i=1}^{p-1}(d_G(u) + d_G(u_i))^{(d_G(u)+d_G(u_i))}} \\
&\cdot \frac{\prod_{1\leq i<j\leq p-1}(d_{G'}(u_i)+d_{G'}(u_j))^{(d_{G'}(u_i)+d_{G'}(u_j))}}{\prod_{1\leq i<j\leq p-1}(d_G(u_i) + d_G(u_j))^{(d_G(u_i)+d_G(u_j))}}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\frac{\prod_{i=2}^{q-1}(d_{G'}(u)+d_{G'}(v_i))^{(d_{G'}(u)+d_{G'}(v_i))}\prod_{2\le i<j\le q-1}(d_{G'}(v_i)+d_{G'}(v_j))^{(d_{G'}(v_i)+d_{G'}(v_j))}}{\prod_{i=2}^{q-1}(d_G(u)+d_G(v_i))^{(d_G(u)+d_G(v_i))}\prod_{2\le i<j\le q-1}(d_G(v_i)+d_G(v_j))^{(d_G(v_i)+d_G(v_j))}}\\
&\cdot\frac{\prod_{i=1}^{p-1}\prod_{j=2}^{q-1}(d_{G'}(u_i)+d_{G'}(v_j))^{(d_{G'}(u_i)+d_{G'}(v_j))}\prod_{i=1}^{p-1}\prod_{x\in N_{G_i}(u_i)}(d_{G'}(u_i)+d_{G'}(x))^{(d_{G'}(u_i)+d_{G'}(x))}}{\prod_{i=2}^{q-1}(d_G(v_1)+d_G(v_i))^{(d_G(v_1)+d_G(v_i))}\prod_{i=1}^{p-1}\prod_{x\in N_{G_i}(u_i)}(d_G(u_i)+d_G(x))^{(d_G(u_i)+d_G(x))}}\\
\ge&\frac{(p+q-1)^{(p+q-1)}\prod_{i=1}^{p-1}(d_G(u_i)+p+2q-4)^{(d_G(u_i)+p+2q-4)}}{(p+2q-3)^{(p+2q-3)}\prod_{i=1}^{p-1}(d_G(u_i)+p+q-2)^{(d_G(u_i)+p+q-2)}}\\
&\cdot\frac{\prod_{1\le i<j\le p-1}(d_G(u_i)+d_G(u_j)+2q-4)^{(d_G(u_i)+d_G(u_j)+2q-4)}}{\prod_{1\le i<j\le p-1}(d_G(u_i)+d_G(u_j))^{(d_G(u_i)+d_G(u_j))}}\\
&\cdot\frac{\prod_{i=2}^{q-1}(2p+2q-5)^{(2p+2q-5)}\prod_{2\le i<j\le q-1}(2p+2q-6)^{(2p+2q-6)}}{\prod_{i=2}^{q-1}(p+2q-3)^{(p+2q-3)}\prod_{2\le i<j\le q-1}(2q-2)^{(2q-2)}}\\
&\cdot\frac{\prod_{i=1}^{p-1}\prod_{j=2}^{q-1}(d_G(u_i)+p+2q-5)^{(d_G(u_i)+p+2q-5)}}{\prod_{i=2}^{q-1}(2q-2)^{(2q-2)}}\\
>&\frac{(p+q-1)^{(p+q-1)}\prod_{i=1}^{p-1}\prod_{j=2}^{q-1}(d_G(u_i)+p+2q-5)^{(d_G(u_i)+p+2q-5)}}{(p+2q-3)^{(p+2q-3)}\prod_{i=2}^{q-1}(2q-2)^{(2q-2)}}\\
\ge&(p+q-1)^{(p+q-1)}\frac{(d_G(u_1)+p+2q-5)^{(d_G(u_1)+p+2q-5)(q-2)}}{(p+2q-3)^{(p+2q-3)}}\\
&\cdot\Big(\frac{(d_G(u_2)+p+2q-5)^{(d_G(u_2)+p+2q-5)}}{(2q-2)^{(2q-2)}}\Big)^{q-2}>1.
\end{aligned}
$$

This completes the proof. □

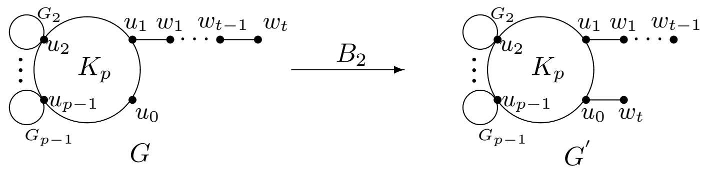

Figure 9. Transformation B_2.

Transformation B_2: Suppose G is a graph as depicted in Figure 9, K_p is a clique of G, where $p \geq 3$, $V(K_p) = \{u_0, u_1, \cdots, u_{p-1}\}$. $P = u_1 w_1 \cdots w_t$ $(t \geq 2)$ is a path attached to u_1. $N_G(u_0) = \{u_1, u_2, \cdots, u_{p-1}\}$, $N_G(u_1) = \{u_0, u_2, \cdots, u_{p-1}, w_1\}$. G_i $(2 \leq i \leq p-1)$ is the subgraph attached to u_i $(2 \leq i \leq p-1)$. Let $G' = G - w_{t-1}w_t + u_0 w_t$, as depicted in Figure 9.

Lemma 4.2. *Suppose G' and G are graphs in Figure 9. Then* $\Pi_2^*(G') > \Pi_2^*(G)$.

Proof. If $t = 2$, for $p \geq 3$, by Lemma 2.2, we have

$$\begin{aligned}\frac{\Pi_2^*(G')}{\Pi_2^*(G)} &= \frac{(2p)^{2p}(p+1)^{(p+1)}(p+1)^{(p+1)}\prod\limits_{i=2}^{p-1}(d_G(u_i)+p)^{(d_G(u_i)+p)}}{(2p-1)^{(2p-1)}(p+2)^{(p+2)}3^3\prod\limits_{i=2}^{p-1}(d_G(u_i)+p-1)^{(d_G(u_i)+p-1)}} \\ &= \frac{\frac{(2p)^{2p}}{(2p-1)^{(2p-1)}}}{\frac{(p+2)^{(p+2)}}{(p+1)^{(p+1)}}} \cdot \frac{(p+1)^{(p+1)}}{3^3}\prod_{i=2}^{p-1}\frac{(d_G(u_i)+p)^{(d_G(u_i)+p)}}{(d_G(u_i)+p-1)^{(d_G(u_i)+p-1)}} > 1.\end{aligned}$$

If $t \geq 3$, then

$$\begin{aligned}\frac{\Pi_2^*(G')}{\Pi_2^*(G)} &= \frac{(2p)^{2p}3^3(p+1)^{(p+1)}\prod\limits_{i=2}^{p-1}(d_G(u_i)+p)^{(d_G(u_i)+p)}}{(2p-1)^{(2p-1)}4^4 3^3\prod\limits_{i=2}^{p-1}(d_G(u_i)+p-1)^{(d_G(u_i)+p-1)}} \\ &= \frac{(2p)^{2p}}{(2p-1)^{(2p-1)}}\frac{(p+1)^{(p+1)}}{4^4}\prod_{i=2}^{p-1}\frac{(d_G(u_i)+p)^{(d_G(u_i)+p)}}{(d_G(u_i)+p-1)^{(d_G(u_i)+p-1)}} > 1.\end{aligned}$$

The proof is completed. □

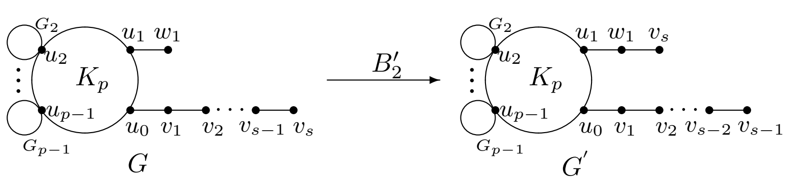

Figure 10. Transformation B_2'.

Transformation B_2': Suppose G is a graph as depicted in Figure 10, K_p is a clique of G, where $p \geq 3$. $V(K_p) = \{u_0, u_1, \cdots, u_{p-1}\}$. $P_1 = u_0 v_1 \cdots v_s$ $(s \geq 3)$ is a path attached to u_0 and $u_1 w_1$ is a pendant edge attached to u_1. $N_G(u_0) = \{u_1, u_2, \cdots, u_{p-1}, v_1\}$, $N_G(u_1) = \{u_0, u_2, \cdots, u_{p-1}, w_1\}$. G_i $(2 \leq i \leq p-1)$ is the subgraph attached to u_i $(2 \le i \le p-1)$. Let $G' = G - v_{s-1}v_s + w_1v_s$, as depicted in Figure 10.

Lemma 4.3. *Suppose G' and G are graphs in Figure 10. Then $\Pi_2^*(G') > \Pi_2^*(G)$.*

Proof. By Lemma 2.2, we notice that

$$\frac{\Pi_2^*(G')}{\Pi_2^*(G)} = \frac{(p+2)^{(p+2)}3^3 3^3}{(p+1)^{(p+1)}4^4 3^3} = \frac{\frac{(p+2)^{(p+2)}}{(p+1)^{(p+1)}}}{\frac{4^4}{3^3}} > 1.$$

This finishes the proof. □

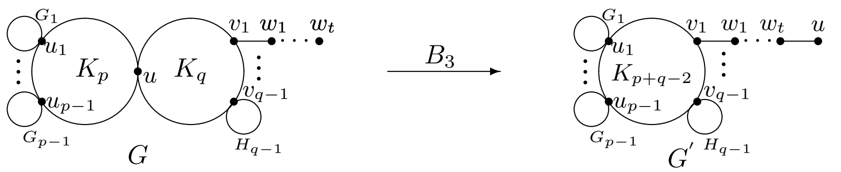

Figure 11. Transformation B_3.

Transformation B_3: Suppose G is a graph as depicted in Figure 11, K_p $(p \ge 3)$ and K_q $(q \ge 3)$ are two cliques of G. $V(K_p)$ and $V(K_q)$ have one cut vertex, say u, in common. $V(K_p) = \{u_1, u_2, \cdots, u_{p-1}, u\}$, $V(K_q) = \{v_1, v_2, \cdots, v_{q-1}, u\}$. $P = v_1w_1 \cdots w_t$ $(t \ge 1)$ is a path attached to v_1 and $N_G(v_1) = \{u, v_2, \cdots, v_{q-1}, w_1\}$. G_i $(1 \le i \le p-1)$ is the subgraph attached to u_i $(1 \le i \le p-1)$ and H_j $(2 \le j \le q-1)$ is the subgraph attached to v_j $(2 \le j \le q-1)$. Let $G' = G - \{uu_1, uu_2, \cdots, uu_{p-1}, uv_1, uv_2, \cdots, uv_{q-1}\} + \{w_tu\} + \{u_1v_1, u_1v_2, \cdots, u_1v_{q-1}\} + \cdots + \{u_{p-1}v_1, u_{p-1}v_2, \cdots, u_{p-1}v_{q-1}\}$, as depicted in Figure 11.

Lemma 4.4. *Suppose G' and G are graphs in Figure 11. Then $\Pi_2^*(G') > \Pi_2^*(G)$.*

Proof. It can be seen that $d_G(u) = p+q-2$, $d_{G'}(u) = d_G(w_t) = 1$, $d_{G'}(w_t) = 2$, $d_G(v_1) = q$, $d_{G'}(v_1) = p+q-2$, $d_{G'}(u_i) = d_G(u_i) + q - 2$ $(i = 1, 2, \cdots, p-1)$, $d_{G'}(v_j) = d_G(v_j) + p - 2$ $(j = 2, \cdots, q-1)$. For $x \in N_{G_i}(u_i)$, $d_{G'}(u_i) + d_{G'}(x) = d_G(u_i) + d_G(x) + q - 2 > d_G(u_i) + d_G(x)$, $i = 1, 2, \cdots, p-1$. For $y \in N_{H_j}(v_j)$, $d_{G'}(v_j) + d_{G'}(y) = d_G(v_j) + d_G(y) + p - 2 > d_G(v_j) + d_G(y)$, $j = 2, 3, \cdots, q-1$.

If $t = 1$, by the definition of Π_2^*, it follows that

$$\frac{\Pi_2^*(G')}{\Pi_2^*(G)} \ge \frac{3^3(p+q)^{(p+q)} \prod_{i=1}^{p-1} (d_G(u_i)+p+2q-4)^{(d_G(u_i)+p+2q-4)}}{(q+1)^{(q+1)} \prod_{i=1}^{p-1} (d_G(u_i)+p+q-2)^{(d_G(u_i)+p+q-2)}} \cdot \frac{\prod_{1 \le i < j \le p-1} (d_G(u_i)+d_G(u_j)+2q-4)^{(d_G(u_i)+d_G(u_j)+2q-4)}}{\prod_{1 \le i < j \le p-1} (d_G(u_i)+d_G(u_j))^{(d_G(u_i)+d_G(u_j))}}$$

$$\cdot \frac{\prod_{i=2}^{q-1}(d_G(v_i)+2p+q-4)^{(d_G(v_i)+2p+q-4)}}{\prod_{i=2}^{q-1}(q+d_G(v_i))^{(q+d_G(v_i))}}$$
$$\cdot \frac{\prod_{2\le i<j\le q-1}(d_G(v_i)+d_G(v_j)+2p-4)^{(d_G(v_i)+d_G(v_j)+2p-4)}}{\prod_{2\le i<j\le q-1}(d_G(v_i)+d_G(v_j))^{(d_G(v_i)+d_G(v_j))}}$$
$$\cdot \frac{\prod_{i=1}^{p-1}\prod_{j=2}^{q-1}(d_G(u_i)+d_G(v_j)+p+q-4)^{(d_G(u_i)+d_G(v_j)+p+q-4)}}{(p+2q-2)^{(p+2q-2)}\prod_{i=2}^{q-1}(d_G(v_i)+p+q-2)^{(d_G(v_i)+p+q-2)}}$$
$$> \frac{3^3(p+q)^{(p+q)}}{(q+1)^{(q+1)}(p+2q-2)^{(p+2q-2)}}$$
$$\cdot \frac{\prod_{i=1}^{p-1}\prod_{j=2}^{q-1}(d_G(u_i)+d_G(v_j)+p+q-4)^{(d_G(u_i)+d_G(v_j)+p+q-4)}}{\prod_{i=2}^{q-1}(d_G(v_i)+p+q-2)^{(d_G(v_i)+p+q-2)}}$$
$$> \frac{3^3(p+q)^{(p+q)}(d_G(u_1)+d_G(v_2)+p+q-4)^{(d_G(u_1)+d_G(v_2)+p+q-4)}}{(q+1)^{(q+1)}(p+2q-2)^{(p+2q-2)}}$$
$$\ge \frac{3^3(p+q)^{(p+q)}(2p+2q-6)^{2p+2q-6}}{(q+1)^{(q+1)}(p+2q-2)^{(p+2q-2)}}$$

since $d_G(u_1) \ge p-1$ and $d_G(v_2) \ge q-1$. If $p \ge 4$, then $\frac{\Pi_2^*(G')}{\Pi_2^*(G)} > 1$. If $p = 3$, then $\frac{\Pi_2^*(G')}{\Pi_2^*(G)} > \frac{3^3(q+3)^{(q+3)}(2q)^{2q}}{(q+1)^{(q+1)}(2q+1)^{(2q+1)}}$. Let $h(q) = \frac{3^3(q+3)^{(q+3)}(2q)^{2q}}{(q+1)^{(q+1)}(2q+1)^{(2q+1)}}$, where $q \ge 3$. Then $\ln h(q) = 3\ln 3 + (q+3)\ln(q+3) + 2q\ln(2q) - (q+1)\ln(q+1) - (2q+1)\ln(2q+1)$ and $(\ln h(q))' = \ln\frac{(q+3)(2q)^2}{(q+1)(2q+1)^2}$. Note that $(q+3)(2q)^2 - (q+1)(2q+1)^2 = 4q^2 - 5q - 1 > 0$ for $q \ge 3$. Therefore $\frac{\Pi_2^*(G')}{\Pi_2^*(G)} > h(3) = \frac{3^3 6^6 6^6}{4^4 7^7} > 1$.

The case of $t \ge 2$ can be proved similarly. □

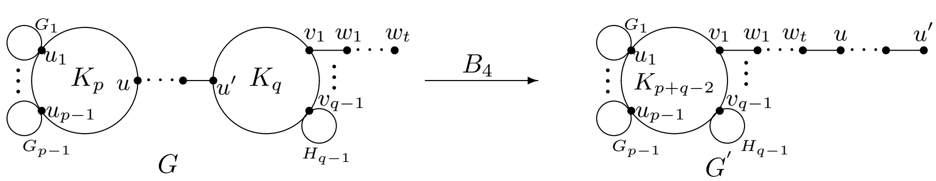

Figure 12. Transformation B_4.

Transformation B_4: Suppose G is a graph as depicted in Figure 12, K_p and K_q are two cliques of G, where $p, q \ge 3$. K_p connects K_q by an internal path $P = u \cdots u'$ with length $s \ge 1$. $V(K_p) = \{u_1, u_2, \cdots, u_{p-1}, u\}$,

$V(K_q) = \{v_1, v_2, \cdots, v_{q-1}, u'\}$. $P_{t+1} = v_1 w_1 \cdots w_t$ $(t \geq 1)$ is a path attached to v_1 and $N_G(v_1) = \{u', v_2, \cdots, v_{q-1}, w_1\}$. G_i $(1 \leq i \leq p-1)$ is the subgraph attached to u_i $(1 \leq i \leq p-1)$ and H_j $(2 \leq j \leq q-1)$ is the subgraph attached to v_j $(2 \leq j \leq q-1)$. Let $G^{'} = G - \{uu_1, uu_2, \cdots, uu_{p-1}, u'v_1, u'v_2, \cdots, u'v_{q-1}\} + \{w_t u\} + \{u_1 v_1, u_1 v_2, \cdots, u_1 v_{q-1}\} + \cdots + \{u_{p-1} v_1, u_{p-1} v_2, \cdots, u_{p-1} v_{q-1}\}$, as depicted in Figure 12.

Lemma 4.5. *Suppose G' and G are graphs in Figure 12. Then $\Pi_2^*(G') > \Pi_2^*(G)$.*

Proof. We notice that $d_{G'}(u) = 2$, $d_G(u) = p$, $d_{G'}(u') = 1$, $d_G(u') = q$, $d_{G'}(w_t) = 2$, $d_G(w_t) = 1$, $d_{G'}(v_1) = p + q - 2$, $d_G(v_1) = q$, $d_{G'}(u_i) = d_G(u_i) + q - 2$ $(i = 1, 2, \cdots, p-1)$, $d_{G'}(v_j) = d_G(v_j) + p - 2$ $(j = 2, \cdots, q-1)$. For $x \in N_{G_i}(u_i)$, $d_{G'}(u_i) + d_{G'}(x) > d_G(u_i) + d_G(x)$, $i = 1, 2, \cdots, p-1$. For $y \in N_{H_j}(v_j)$, $d_{G'}(v_j) + d_{G'}(y) > d_G(v_j) + d_G(y)$, $j = 2, 3, \cdots, q-1$.

If $t = s = 1$, in view of the definition of Π_2^*, it can be concluded that

$$
\begin{aligned}
\frac{\Pi_2^*(G')}{\Pi_2^*(G)} \geq & \frac{4^4 3^3 (p+q)^{(p+q)} \prod_{i=1}^{p-1} (d_G(u_i)+p+2q-4)^{(d_G(u_i)+p+2q-4)}}{(p+q)^{(p+q)} (q+1)^{(q+1)} (2q)^{2q} \prod_{i=1}^{p-1} (d_G(u_i)+p)^{(d_G(u_i)+p)}} \\
& \cdot \frac{\prod_{1 \leq i < j \leq p-1} (d_G(u_i)+d_G(u_j)+2q-4)^{(d_G(u_i)+d_G(u_j)+2q-4)}}{\prod_{1 \leq i < j \leq p-1} (d_G(u_i)+d_G(u_j))^{(d_G(u_i)+d_G(u_j))}} \\
& \cdot \frac{\prod_{i=2}^{q-1} (d_G(v_i)+2p+q-4)^{(d_G(v_i)+2p+q-4)}}{\prod_{i=2}^{q-1} (q+d_G(v_i))^{(q+d_G(v_i))}} \\
& \cdot \frac{\prod_{2 \leq i < j \leq q-1} (d_G(v_i)+d_G(v_j)+2p-4)^{(d_G(v_i)+d_G(v_j)+2p-4)}}{\prod_{2 \leq i < j \leq q-1} (d_G(v_i)+d_G(v_j))^{(d_G(v_i)+d_G(v_j))}} \\
& \cdot \frac{\prod_{i=1}^{p-1} \prod_{j=2}^{q-1} (d_G(u_i)+d_G(v_j)+p+q-4)^{(d_G(u_i)+d_G(v_j)+p+q-4)}}{\prod_{i=2}^{q-1} (d_G(v_i)+q)^{(d_G(v_i)+q)}} \\
> & \frac{4^4 3^3}{(q+1)^{(q+1)} (2q)^{2q}} \cdot \frac{\prod_{i=1}^{p-1} \prod_{j=2}^{q-1} (d_G(u_i)+d_G(v_j)+p+q-4)^{(d_G(u_i)+d_G(v_j)+p+q-4)}}{\prod_{i=2}^{q-1} (d_G(v_i)+q)^{(d_G(v_i)+q)}}.
\end{aligned}
$$

If $q = 3$, then

$$\frac{\Pi_2^*(G')}{\Pi_2^*(G)} > \frac{4^4 3^3 (d_G(u_1) + d_G(v_2) + p - 1)^{(d_G(u_1)+d_G(v_2)+p-1)}}{4^4 6^6} > 1.$$

If $q > 3$, then

$$\begin{aligned}\frac{\Pi_2^*(G')}{\Pi_2^*(G)} >& \frac{4^4 3^3 (d_G(u_1)+d_G(v_2)+p+q-4)^{(d_G(u_1)+d_G(v_2)+p+q-4)}}{(q+1)^{(q+1)}(2q)^{2q}} \\ &\cdot (d_G(u_1)+d_G(v_3)+p+q-4)^{(d_G(u_1)+d_G(v_3)+p+q-4)} \\ \geq& \frac{4^4 3^3 (2p+2q-6)^{(2p+2q-6)}(2p+2q-6)^{(2p+2q-6)}}{(q+1)^{(q+1)}(2q)^{2q}} > 1.\end{aligned}$$

The cases of $t, s > 1$; $t = 1, s > 1$ or $t > 1, s = 1$ can be proved similarly as the case of $t = s = 1$, and we omit the details. □

Let $G \in \mathbf{G}_V(n,k)$. In order to get the maximum $\Pi_2^*(G)$, we first provide a definition and a notation. Suppose K_p ($p \geq 3$) and K_q ($q \geq 3$) are two cliques in G. If K_p connects K_q by a path P (perhaps $|E(P)| = 0$, namely K_p and K_q has a vertex in common which is a cut vertex of G) such that P doesn't intersect some other cliques K_r with $r \geq 3$, we call K_p and K_q are adjacent. Denote $\mathbf{G}_{n,k} = \{G \mid G \in \mathbf{G}_V(n,k)$ is the graph obtained from K_{n-k} by attaching at most one pendant path to each of its vertices$\}$. Clearly, $\{G^1_{n,k}, G^2_{n,k}, G^3_{n,k}\} \subset \mathbf{G}_{n,k}$.

Theorem 4.6. *Suppose $G \in \mathbf{G}_V(n,k)$, where $1 \leq k \leq n-3$. Then*

(i) if $1 \leq k \leq \frac{n}{2}$, $\Pi_2^*(G) \leq (n-k+1)^{(n-k+1)k}(2n-2k)^{(2n-2k)\binom{k}{2}}(2n-2k-2)^{(2n-2k-2)\binom{n-2k}{2}}(2n-2k-1)^{(2n-2k-1)k(n-2k)}$ with equality if and only if $G \cong G^1_{n,k}$;

(ii) if $\frac{n}{2} < k \leq \frac{2n}{3}$, $\Pi_2^*(G) \leq (n-k+2)^{(n-k+2)(2k-n)}(2n-2k)^{(2n-2k)\binom{n-k}{2}}(n-k+1)^{(n-k+1)(2n-3k)}3^{3(2k-n)}$ with equality if and only if $G \cong G^2_{n,k}$;

(iii) if $\frac{2n}{3} < k \leq n-3$, $\Pi_2^*(G) \leq (n-k+2)^{(n-k+2)(n-k)}(2n-2k)^{(2n-2k)\binom{n-k}{2}}3^{3(n-k)}4^{4(3k-2n)}$ with equality if and only if $G \cong G^3_{n,k}$.

Proof. Suppose $G \in \mathbf{G}_V(n,k)$ has the maximum Π_2^*. First some claims will be given.

Claim 1. Each cut vertex of G connects exactly two blocks, and all blocks of G are cliques.

Proof. By contradiction. Assume that x is a cut vertex in G, and $G - x = \bigcup_{i=1}^r G_i$, where $r \geq 3$. Choose $y \in V(G_2)\backslash\{x\}$ and $z \in V(G_r)\backslash\{x\}$, and let $G^* = G + yz$. Clearly, $G^* \in \mathbf{G}_V(n,k)$. By Lemma 2.1, it follows that $\Pi_2^*(G) < \Pi_2^*(G^*)$, a contradiction. Thus, we get that each cut vertex connects exactly two blocks of G. Moreover, by Lemma 2.1, we can conclude that all blocks in G are cliques. □

By Claim 1, the following Claim 2 is obtained.

Claim 2. If two cliques K_p, K_q with $p, q \geq 3$ of G are adjacent, then the path, say P, connecting K_p and K_q is either $|E(P)| = 0$ or an internal path.

Claim 3. Let K_q be an endblock of G. Then $q = 2$.

Proof. We prove this claim by contradiction. Suppose that $q \geq 3$. Let K_p ($p \geq 2$) be a clique such that K_p connects K_q by a cut vertex, say u. By Claim 1, u is not the cut vertex of some other cliques. By Lemma 4.1, G can be changed to G' by transformation B_1 with a larger Π_2^*, which contradicts the choice of G. Hence, $q = 2$. □

By Claim 1, we suppose that $K_{n_1}, K_{n_2}, \cdots, K_{n_r}$ are all of the cliques in G.

Claim 4. Let $K_{n_1}, K_{n_2}, \cdots, K_{n_r}$ be all of the cliques in G. Then there is only one clique K_{n_i} with $n_i \geq 3$.

Proof. To the contrary, suppose that there are two cliques K_p, K_q ($K_p, K_q \in \{K_{n_1}, K_{n_2}, \cdots, K_{n_r}\}$ and $p \neq q$) such that K_p is adjacent to K_q, where $p, q \geq 3$. By Claim 3, it can be seen that K_p and K_q are not endblocks. Furthermore, by Claim 1, we can choose two such blocks such that at least one of them has a pendant path attached to one of its vertex. Suppose without loss of generality that K_q is one of such cliques which has a pendant path, say $P_{t+1} = v_1 w_1 \cdots w_t$ ($t \geq 1$), attached on $v_1 \in V(K_q)$. By Claim 2 , we can see that K_p connects K_q by a cut vertex u or an internal path $P = u \cdots u'$ with length $s \geq 1$. By Lemma 4.4 or 4.5, G can be changed to G' by transformation B_3 or B_4 with a larger Π_2^*, a contradiction. □

Claim 5. Suppose K_p is the only clique with $p \geq 3$. Then $p = n - k$.

Proof. In view of Claim 1 and Claim 4, it can be concluded that there exist $k+1$ cliques in G and among them, k cliques are isomorphic to K_2. Furthermore, $G \in \mathbf{G}_V(n, k)$, and each cut vertex belongs to two cliques, we can get immediately that $2k + p - k = n$. As a result, $p = n - k$. □

Claim 6. Let $H \in \mathbf{G}_{n,k}$. Then $\Pi_2^*(H) \leq \Pi_2^*(G_{n,k}^1)$ or $\Pi_2^*(H) \leq \Pi_2^*(G_{n,k}^2)$ or $\Pi_2^*(H) \leq \Pi_2^*(G_{n,k}^3)$.

Proof. Let $H \in \mathbf{G}_{n,k}$ such that H has the maximum Π_2^*. If $H \cong G_{n,k}^1$ or $G_{n,k}^3$ or $G_{n,k}^2$, the claim holds. Otherwise, $H \in \mathbf{G}_{n,k} \backslash \{G_{n,k}^1, G_{n,k}^2, G_{n,k}^3\}$. Then H satisfies the following (i) or (ii).

(i) There is a vertex of K_{n-k} with no pendant path attached in H, and H has a pendant path with length equal or more than 2;

(ii) H has a pendant edge and a pendant path of length greater than 2.

By Lemma 4.2 or 4.3, H can be changed to H' by transformation B_2 or B_2' with a larger Π_2^*, which contradicts the assumption of H. □

By Claim 4 and 5, it follows that $G \in \mathbf{G}_{n,k}$. By Claim 6, it follows that $\Pi_2^*(G) \leq \Pi_2^*(G^1_{n,k})$ when $1 \leq k \leq \frac{n}{2}$, $\Pi_2^*(G) \leq \Pi_2^*(G^2_{n,k})$ when $\frac{n}{2} < k \leq \frac{2n}{3}$ and $\Pi_2^*(G) \leq \Pi_2^*(G^3_{n,k})$ when $\frac{2n}{3} < k \leq n-3$. □

5. Modified second multiplicative Zagreb index of graphs with fixed vertex connectivity or edge connectivity

Lemma 5.1. *Let $G \cong K_s \vee (K_{n_1} \cup K_{n_2})$ and $G' \cong K_s \vee (K_{n_1-1} \cup K_{n_2+1})$, where $n_1 + n_2 = n - s$, $n_2 \geq n_1 \geq 2$. Then*

$$\Pi_2^*(G') > \Pi_2^*(G).$$

Proof. By the definition of Π_2^*, it follows that

$$\begin{aligned}\frac{\Pi_2^*(G')}{\Pi_2^*(G)} =& \frac{(2n_1+2n_2+2s-2)^{(2n_1+2n_2+2s-2)\binom{s}{2}}}{(2n_1+2n_2+2s-2)^{(2n_1+2n_2+2s-2)\binom{s}{2}}}\\ &\cdot \frac{(2n_1+2s-4)^{(2n_1+2s-4)\binom{n_1-1}{2}}(2n_2+2s)^{(2n_2+2s)\binom{n_2+1}{2}}}{(2n_1+2s-2)^{(2n_1+2s-2)\binom{n_1}{2}}(2n_2+2s-2)^{(2n_2+2s-2)\binom{n_2}{2}}}\\ &\cdot \frac{(2n_1+n_2+2s-3)^{(2n_1+n_2+2s-3)s(n_1-1)}(2n_2+n_1+2s-1)^{(2n_2+n_1+2s-1)s(n_2+1)}}{(2n_1+n_2+2s-2)^{(2n_1+n_2+2s-2)sn_1}(2n_2+n_1+2s-2)^{(2n_2+n_1+2s-2)sn_2}}\\ =& \frac{(2n_1+2s-4)^{(2n_1+2s-4)\frac{(n_1-1)(n_1-2)}{2}}(2n_2+2s)^{(2n_2+2s)\frac{(n_2+1)n_2}{2}}}{(2n_1+2s-2)^{(2n_1+2s-2)\frac{n_1(n_1-1)}{2}}(2n_2+2s-2)^{(2n_2+2s-2)\frac{n_2(n_2-1)}{2}}}\\ &\cdot\left(\frac{(n+s-3+n_1)^{(n+s-3+n_1)(n_1-1)}(n+s-1+n_2)^{(n+s-1+n_2)(n_2+1)}}{(n+s-2+n_1)^{(n+s-2+n_1)n_1}(n+s-2+n_2)^{(n+s-2+n_2)n_2}}\right)^s.\end{aligned}$$

By Lemma 2.3 and 2.4 ($a = n+s-2$), we have $\frac{\Pi_2^*(G')}{\Pi_2^*(G)} > 1$. □

Theorem 5.2. *Suppose G is a graph of order $n \geq 4$ with vertex connectivity $\kappa < n-1$. Then $\Pi_2^*(G) \leq (\kappa+n-1)^{(\kappa+n-1)\kappa}(2n-2)^{(2n-2)\binom{\kappa}{2}}(2n-3)^{(2n-3)\kappa(n-\kappa-1)}(2n-4)^{(2n-4)\binom{n-\kappa-1}{2}}$ with equality if and only if $G \cong K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$.*

Proof. Choose G such that G has the maximum Π_2^* among all graphs of order n with vertex connectivity κ. Assume that X is a vertex cut with $|X| = \kappa$ of G such that $G - X$ has κ components, say $G_1, G_2, \cdots, G_\kappa$, where $\kappa \geq 2$. Let $n_1 = |V(G_1)|$ and $n_2 = |V(G_2 \cup \cdots \cup G_\kappa)|$. It is clear that G is a spanning sub-graph of $K_\kappa \vee (K_{n_1} \cup K_{n_2})$. By Lemma 2.1, $\Pi_2^*(G) \leq \Pi_2^*(K_\kappa \vee (K_{n_1} \cup K_{n_2}))$. Moreover, by Lemma 5.1, $G \cong K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$. □

Theorem 5.3. *Suppose G is a graph of order $n \geq 4$ with edge connectivity $\lambda < n-1$. Then $\Pi_2^*(G) \leq (\lambda+n-1)^{(\lambda+n-1)\lambda}(2n-2)^{(2n-2)\binom{\lambda}{2}}(2n-3)^{(2n-3)\lambda(n-\lambda-1)}(2n-4)^{(2n-4)\binom{n-\lambda-1}{2}}$ with equality if and only if $G \cong K_\lambda \vee (K_1 \cup K_{n-\lambda-1})$.*

Proof. Choose G such that G has the maximum Π_2^* among all graphs on n vertices with edge connectivity λ. Suppose the vertex connectivity of G is κ. It follows that $\kappa \leq \lambda < n-1$. By Theorem 5.2, we have $G \cong K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$. Furthermore, $K_\kappa \vee (K_1 \cup K_{n-\kappa-1})$ is a spanning sub-graph of $K_\lambda \vee (K_1 \cup K_{n-\lambda-1})$ for $\kappa \leq \lambda$, in view of Lemma 2.1, the theorem holds immediately. □

Remark 5.1. In [20], Wang et al. determined the graph $\mathbf{K}_n^k$ (obtained by adding a vertex to a clique K_{n-1} and joining the vertex to exactly $k \leq n-1$ vertices of K_{n-1}) which has the maximum modified multiplicative Zagreb indices in graphs with vertex connectivity or edge connectivity at most k. The values of modified multiplicative Zagreb indices of $\mathbf{K}_n^k$ in [20] are wrong. It is clear that $\mathbf{K}_n^k \cong K_k \vee (K_1 \cup K_{n-k-1})$. When we obtain the maximum modified multiplicative Zagreb indices in graphs with vertex connectivity or edge connectivity at most k, it seems easier to read writing like this section.

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Jianwei Du received Ph.D. from North University of China. Since 2006 he has been at North University of China. His research interests include matrix theory and graph theory.

School of Science, North University of China, Taiyuan 030051, China.
e-mail: jianweidu@nuc.edu.cn

Xiaoling Sun received Ph.D. from at North University of China. Since 2006 she has been at North University of China. Her research interests include matrix theory and graph theory.

School of Science, North University of China, Taiyuan 030051, China.
e-mail: sunxiaoling@nuc.edu.cn