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MULTIPLICATION FORMULA AND (w, q)-ALTERNATING POWER SUMS OF TWISTED q-EULER POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. In this paper, we define twisted q-Euler polynomials of the second kind and explore some properties. We find generating function of twisted q-Euler polynomials of the second kind. Also, we investigate twisted q-Raabe's multiplication formula and (w, q)-alternating power sums of twisted q-Euler polynomials of the second kind. At the end, we define twisted q-Hurwitz's type Euler zeta function of the second kind.

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1. Introduction

Bernoulli polynomials, Euler polynomials, and Genocchi polynomials are topics that have been studied a lot in mathematics. Furthermore, mathematician have also researched Bernoulli polynomials of the second kind, Euler polynomials of the second kind, and Genocchi polynomials of the second kind(see [1-18]). Among them, we will study twisted q-Euler polynomials of the second kind related to Euler polynomials of the second kind. First, to discuss twisted q-Euler polynomials of the second kind, which are the topic of this paper, we will introduce precedent researches about the second kind Euler polynomials, second kind q-Euler polynomials, twisted q-Bernoulli polynomials of the second kind, and zeta functions.

In [1, 15, 16, 17, 18], the second kind Euler polynomials $\tilde{E}_n(x)$ were introduced and generating function of the second kind Euler polynomials was defined as

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follows:

$$\frac{2e^t}{e^{2t}+1}=\sum_{n=0}^{\infty}\tilde{E}_n(x)\frac{t^n}{n!}=2\sum_{n=0}^{\infty}(-1)^n e^{(2n+1+x)t}.$$

When $x=0$, the numbers $\tilde{E}_n=\tilde{E}_n(0)$ are called the second kind Euler numbers. Furthermore, Ryoo [18] discussed second kind q-Euler polynomials $\tilde{E}_{n,q}(x)$. It is defined as follows:

$$\tilde{E}_{n,q}(x)=\int_{\mathbb{Z}_p}[2y+1+x]_q^n d\mu_{-1}(y).$$

We explain notations, which are used in [18]: $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. For any natural number n, q-number is defined as follows:

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\cdots+q^{n-1}.$$

He examined some properties of second kind q-Euler polynomials, using the fermionic p-adic integral on $\mathbb{Z}_p$. Among them, we represent three properties:

Theorem 1.1. *Let $q\in\mathbb{C}_p$ with $|q-1|_p<1$. Then*

$$\tilde{E}_{n,q}(x)=2\sum_{n=0}^{\infty}(-1)^n[2n+1+x]_q^n.$$

Theorem 1.2. *Let n be a nonnegative integer. Then*

$$\tilde{E}_{n,q^{-1}}(-x)=(-1)^n q^n\tilde{E}_{n,q}(x).$$

Theorem 1.3. *Let n be a nonnegative integer. Then*

$$\tilde{E}_{n,q}(2)+\tilde{E}_{n,q}=2.$$

According to [4], twisted q-Bernoulli polynomials of the second kind were defined as following generating function:

$$\sum_{n=0}^{\infty}\tilde{B}_{n,q,w}(x)\frac{t^n}{n!}=-t\sum_{n=0}^{\infty}w^n q^{n+x}e^{[2n+1+x]_q t},$$

The numbers $\tilde{B}_{n,q,w}=\tilde{B}_{n,q,w}(0)$ are called twisted q-Bernoulli numbers of the second kind when $x=0$. Also, [4] investigated another generating function of twisted q-Bernoulli polynomials of the second kind, which is as follows:

$$\sum_{n=0}^{\infty}\tilde{B}_{n,q,w}(x)\frac{t^n}{n!}=-te^{\frac{t}{1-q}}\sum_{n=0}^{\infty}\left(\frac{q}{1-q}\right)^n\frac{(-1)^n q^{(n+1)x}}{1-wq^{2n+1}}\frac{t^n}{n!}.$$

Theorem 1.4. *Let n be a nonnegative integer. Then*

$$\tilde{B}_{n,q,w}(x)=\sum_{k=0}^{n}\binom{n}{k}\tilde{B}_{k,q,w}q^{kx}[x]_q^{n-k}. \tag{1.1}$$

After Lerch [12] introduced the function $\phi(x, a, s)$, Lipschitz [11] and Apostol [3] studied the function $\phi(x, a, s)$, which is defined as follows(see [10, 5, 14, 6, 13]):

$$\phi(a, x, s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi ia}}{(n+x)^s}.$$

In [5, 6, 13, 14], Hurwitz-Lerch zeta function is defined by

$$\Phi(z, s, x) = \sum_{n=0}^{\infty} \frac{z^n}{(n+x)^s}.$$

The function $\phi(a, x, s)$ is related to Hurwitz-Lerch zeta function, which is a special case of Hurwitz-Lerch zeta function. It is $\phi(a, x, s) = \Phi(2\pi ia, s, x)$. [8] and [9] stduied Hurwitz's type Euler zeta function, which is defined as follows:

$$\zeta_E(s, x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}.$$

Hurwitz's type Euler zeta function is special case of Hurwitz-Lerch zeta function, which is $\zeta_E(s, x) = 2\Phi(-1, s, x)$. For $|z| < 1$, Hurwitz-Lerch Euler zeta function of the second kind is defined as follows:

$$\tilde{\Phi}_E(z, s, x) = \sum_{n=0}^{\infty} \frac{z^n}{(2n+1+x)^s}.$$

For $Re(s) > 0$, Hurwitz's type Euler zeta function of the second kind is defined as follows:

$$\tilde{\zeta}_E(s, x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1+x)^s}.$$

Hurwitz's type Euler zeta function of the second kind is special case of Hurwitz-Lerch Euler zeta function of the second kind, which is $\tilde{\zeta_E}(s, x) = 2\tilde{\Phi}_E(-1, s, x)$.

In this paper, we study twisted q-Euler polynomials of the second kind. To do so, we first define q-Euler polynomials of the second kind. Then we explore some properties using the definition. To be specific, in Section 2, we discuss twisted q-Euler polynomials of the second kind and explore some properties including addition formula, property of complement, and multiplication theorem. In Section 3, we find generating function of twisted q-Euler polynomials of the second kind. We also investigate multiplication formula using twisted q-Bernoulle polynomials of the second kind and twisted q-Euler polynomials of the second kind. Furthermore, we define (w, q)-alternating power sums relation of twisted q-Euler polynomials of the second kind. In Section 4, we define twisted q-Hurwitz's type Euler zeta function of the second kind in order to explore relation between twisted q-Euler polynomials of the second kind and twisted-type q-Hurwitz-Lerch Euler zeta function of the second kind.

2. Some properties of twisted q-Euler polynomials of the second kind

In this section, we define twisted q-Euler polynomials of the second kind and examine a few twisted q-Euler polynomials of the second kind. We also investigate some identities using definition of twisted q-Euler polynomials of the second kind, including addition formula, property of complement.

Definition 2.1. For $0 < q < 1$, we define twisted q-Euler polynomials of the second kind as the following generating function:

$$\sum_{n=0}^{\infty} \tilde{E}_{n,q,w}(x)\frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty}(-1)^n w^n q^{n+x} e^{[2n+1+x]_q t},$$

where w is r-th root of 1 with a positive integer r.

The numbers $\tilde{E}_{n,q,w} = \tilde{E}_{n,q,w}(0)$ are called twisted q-Euler numbers of the second kind. If $w = 1$ and $q \to 1$, then twisted q-Euler polynomials of the second kind are reduced to the second kind Euler polynomials. That is

$$\lim_{q\to 1} \tilde{E}_{n,q,1}(x) = \tilde{E}_n(x).$$

We can see the twisted q-Euler polynomials of the second kind using the generating function of twisted q-Euler polynomials of the second kind $\tilde{E}_{n,q,w}(x)$ as follows:

$$\tilde{E}_{n,q,w}(x) = [2]_q \sum_{m=0}^{\infty}(-1)^m w^m q^{n+x}[2m+q+x]_q^n. \tag{2.1}$$

Here are a few twisted q-Euler polynomials of the second kind:

$$\tilde{E}_{0,q,w}(x) = \frac{[2]_q q^x}{wq+1},$$

$$\tilde{E}_{1,q,w}(x) = \frac{[2]_q q^{2x+1}(wq+1) - [2]_q(wq^3+1)}{(q-1)(wq+1)(wq^3+1)},$$

$$\begin{aligned}\tilde{E}_{2,q,w}(x) &= \frac{[2]_q q^{3x+2}(w^2q^4+wq^3+wq+1) - 2[2]_q q^{2x+1}(w^2q^6+wq^5+wq+1))}{(q-1)^2(wq+1)(wq^3+1)(wq^5+1)} \\ &\quad + \frac{[2]_q q^x(w^2q^8+wq^5+wq^3+1)}{(q-1)^2(wq+1)(wq^3+1)(wq^5+1)},\end{aligned}$$

$$\begin{aligned}\tilde{E}_{3,q,w}(x) &= \frac{[2]_q q^{4x+3}(w^3q^9+w^2q^8+w^2q^6+w^2q^4+wq^5+wq^3+wq+1)}{(q-1)^3(wq+1)(wq^3+1)(wq^5+1)(wq^7+1)} \\ &\quad - \frac{3[2]_q q^{3x+2}(w^3q^9+w^2q^{10}+w^2q^8+w^2q^4+wq^7+wq^3+wq+1)}{(q-1)^3(wq+1)(wq^3+1)(wq^5+1)(wq^7+1)} \\ &\quad + \frac{3[2]_q q^{2x+1}(w^3q^{13}+w^2q^{12}+w^2q^8+w^2q^6+wq^7+wq^5+wq+1)}{(q-1)^3(wq+1)(wq^3+1)(wq^5+1)(wq^7+1)} \\ &\quad - \frac{[2]_q q^x(w^3q^{15}+w^2q^{12}+w^2q^{10}+w^2q^8+wq^7+wq^5+wq^3+1)}{(q-1)^3(wq+1)(wq^3+1)(wq^5+1)(wq^7+1)}.\end{aligned}$$

Theorem 2.2. *Let m and n be the nonnegative integers. Then we have*

$$\tilde{E}_{n,q,w}(mx) = \sum_{k=0}^{n} \binom{n}{k} q^{(k+1)mx} \tilde{E}_{k,q,w} [mx]_q^{n-k}.$$

Proof. By using Definition 2.1, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \tilde{E}_{n.q.w}(mx) \frac{t^n}{n!} &= [2]_q \sum_{n=0}^{\infty} (-1)^n w^n q^{n+mx} e^{[2n+q+mx]_q t} \\
&= [2]_q \sum_{n=0}^{\infty} (-1)^n w^n q^{n+mx} e^{[2n+1]_q q^{mx} t + [mx]_q t} \\
&= \left(q^{mx} \sum_{n=0}^{\infty} \tilde{E}_{n,q,w} \frac{(q^{mx} t)^n}{n!} \right) \left(\sum_{n=0}^{\infty} [mx]_q^n \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} q^{(k+1)mx} \tilde{E}_{k,q,w} [mx]_q^{n-k} \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.2}$$

Therefore, we can compare the coefficients of $\frac{t^n}{n!}$ on both sides of the equation (2.2). This completes the proof. □

Let us put $m = 1$ in Theorem 2.2. Then we get the following corollary.

Corollary 2.3. *For a nonnegative integer n, we obtain*

$$\tilde{E}_{n,q,w}(x) = \sum_{k=0}^{n} \binom{n}{k} q^{(k+1)x} \tilde{E}_{n,q,w} [x]_q^{n-k}.$$

Theorem 2.4. *For a nonnegative integer n, we have*

$$wq^{-1} \tilde{E}_{n,q,w}(x+2) + \tilde{E}_{n,q,w}(x) = [2]_q q^x [1+x]_q^n.$$

Proof. By utilizing Definition 2.1, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(wq^{-1} \tilde{E}_{n,q,w}(x+2) + \tilde{E}_{n,q,w}(x) \right) \frac{t^n}{n!} \\
&= [2]_q wq^{-1} \sum_{n=0}^{\infty} (-1)^n w^n q^{n+x+2} e^{[2n+1+x+2]_q t} \frac{t^n}{n!} \\
&\quad + [2]_q \sum_{n=0}^{\infty} (-1)^n w^n q^{n+x} e^{[2n+1+x]_q t} \frac{t^n}{n!} \\
&= [2]_q q^x e^{[1+x]_q t} \\
&= \sum_{n=0}^{\infty} \left([2]_q q^x [1+x]_q^n \right) \frac{t^n}{n!}.
\end{aligned}$$

Hence, the proof is complete by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation above. □

Theorem 2.5. *For a natural number n, we have*

$$\frac{\partial}{\partial x}\tilde{E}_{n,q,w}(x) = \log q\tilde{E}_{n,q,w}(x) + \frac{q^{x+1}\log q}{q-1}n\tilde{E}_{n-1,q,wq^2}(x)$$

and $\frac{\partial}{\partial x}\tilde{E}_{0,q,w}(x) = \frac{[2]_q q^x \log q}{wq+1}$.

Proof. By using Definition 2.1, we get

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\tilde{E}_{n,q,w}(x)\frac{t^n}{n!}\right)\\
&= \frac{\partial}{\partial x}\left([2]_q\sum_{n=0}^{\infty}(-1)^n w^n q^{n+x} e^{[2n+1+x]_q t}\frac{t^n}{n!}\right)\\
&= \log q\sum_{n=0}^{\infty}\tilde{E}_{n,q,w}(x)\frac{t^n}{n!} + \frac{[2]_q q^{x+1} t\log q}{q-1}\sum_{n=0}^{\infty}(-1)^n (wq^2)^n q^{n+x} e^{[2n+1+x]_q t}\\
&= \log q\sum_{n=0}^{\infty}\tilde{E}_{n,q,w}(x)\frac{t^n}{n!} + \frac{q^{x+1}\log q}{q-1}\sum_{n=1}^{\infty}n\tilde{E}_{n-1,q,wq^2}(x)\frac{t^n}{n!}.
\end{aligned}$$

Thus, we compare the coefficients of $\frac{t^n}{n!}$ on both sides of the equation above. This completes the proof. □

Theorem 2.6. *For a nonnegative integer n, we obtain*

$$\tilde{E}_{n,q,w}(x+y) = \sum_{n=0}^{\infty}\binom{n}{k}q^{(k+1)y}\tilde{E}_{k,q,w}(x)[y]_q^{n-k}.$$

Proof. From Definition 2.1, we derive

$$\begin{aligned}
&\sum_{n=0}^{\infty}\tilde{E}_{n,q,w}(x+y)\frac{t^n}{n!}\\
&= [2]_q\sum_{n=0}^{\infty}(-1)^n w^n q^{n+x+y} e^{[2n+1+x+y]_q t}\\
&= [2]_q\sum_{n=0}^{\infty}(-1)^n w^n q^{n+x+y} e^{[2n+1+x+y]_q q^y t} e^{[y]_q t} \qquad (2.3)\\
&= \left(\sum_{n=0}^{\infty}q^y\tilde{E}_{n,q,w}(x)\frac{(q^y t)^n}{n!}\right)\left(\sum_{n=0}^{\infty}[y]_q^n\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}q^{(k+1)y}\tilde{E}_{k,q,w}(x)[y]_q^{n-k}\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, we compare the coefficients of $\frac{t^n}{n!}$ on both sides of the equation (2.3). This completes the proof. □

Theorem 2.7. *Let l be a natural number and n be a nonnegative integer. Then we obtain*

$$\tilde{E}_{n,q,w}(x)+(-1)^{l+1}(wq^{-1})^{l}\tilde{E}_{n,q,w}(x+2l)=[2]_q\sum_{k=0}^{l-1}(-1)^k w^k q^{k+x}[2k+1+x]_q^n.$$

Proof. By using Definition 2.1, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\tilde{E}_{n,q,w}(x)+(-1)^{l+1}(wq^{-1})^{l}\tilde{E}_{n,q,w}(x+2l)\right)\frac{t^n}{n!}\\&=[2]_q\sum_{n=0}^{\infty}(-1)^n w^n q^{n+x}e^{[2n+1+x]_q t}-[2]_q(wq^{-1})^l\sum_{n=0}^{\infty}(-1)^{n+l}w^n q^{n+2l+x}e^{[2n+1+2l+x]_q t}\\&=[2]_q\sum_{n=0}^{\infty}(-1)^n w^n q^{n+x}e^{[2n+1+x]_q t}-[2]_q\sum_{n=0}^{\infty}(-1)^{n+l}w^{n+l} q^{2n+l+x}e^{[2n+1+2l+x]_q t}\\&=\sum_{n=0}^{\infty}\left([2]_q\sum_{k=0}^{l-1}(-1)^k w^k q^{k+x}[2k+1+x]_q^n\right)\frac{t^n}{n!}.\end{aligned}\tag{2.4}$$

Therefore, the proof is complete from the equation (2.4). □

If $w=1$ and $q\to 1$ in Theorem 2.7, then we get the following corollary.

Corollary 2.8. *For a natural number l and a nonnegative integer n, we have*

$$\tilde{E}_n(x)+(-1)^{l+1}\tilde{E}_n(x+2l)=2\sum_{k=0}^{l-1}(-1)^k(2k+1+x)^n,$$

where the polynomials $\tilde{E}_n(x)$ are the second kind Euler polynomials.

3. Twisted q-Raabe's multiplication formula and (w,q)-alternating power sums of twisted q-Euler polynomials of the second kind

In this section, we explore another generating function of twisted q-Euler polynomials of the second kind in order to find multiplication formula. Also, we express twisted q-Euler numbers of the second kind using (w,q)-alternating power sums.

The following theorem is another generating function of twisted q-Euler polynomials of the second kind.

Theorem 3.1. *For $0<q<1$, we have*

$$\sum_{n=0}^{\infty}\tilde{E}_{n,q,w}(x)\frac{t^n}{n!}=[2]_q e^{\frac{t}{1-q}}\sum_{n=0}^{\infty}\left(\frac{q}{1-q}\right)^n\frac{(-1)^n q^{(n+1)x}}{1+wq^{2n+1}}\frac{t^n}{n!}.$$

Proof. By utilizing Definition 2.1, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \tilde{E}_{n,q,w}(x)\frac{t^n}{n!} \\
&= [2]_q \sum_{n=0}^{\infty}(-1)^k w^k q^{k+x} e^{[2k+1+x]_q t} \\
&= [2]_q e^{\frac{t}{1-q}} \sum_{n=0}^{\infty}\left(\frac{q}{1-q}\right)^n (-1)^n q^{(x+1)n+x} \sum_{k=0}^{\infty}(-1)^k w^k q^{k(2n+1)} \frac{t^n}{n!} \\
&= [2]_q e^{\frac{t}{1-q}} \sum_{n=0}^{\infty}\left(\frac{q}{1-q}\right)^n \frac{(-1)^n q^{(n+1)x}}{1+wq^{2n+1}} \frac{t^n}{n!}.
\end{aligned} \tag{3.1}$$

Thus, the proof is complete. □

Theorem 3.2. *For a nonnegative integer n, we have*

$$\tilde{E}_{n,q,w}(x) = \frac{[2]_q}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k q^{k(x+1)+x}}{1+wq^{2k+1}}.$$

Proof. We omit the proof. Because if we use Theorem 3.1 and Cauchy product, the proof is complete. □

Theorem 3.3. *For a nonnegative integer n, we obtain*

$$(-1)^n wq^{n+1}\tilde{E}_{n,q,w}(x) = \tilde{E}_{n,q^{-1},w^{-1}}(-x).$$

Proof. By using Definition 3.2, we get

$$\begin{aligned}
(-1)^n wq^{n+1}\tilde{E}_{n,q,w}(x) &= (-1)^n wq^{n+1}\frac{[2]_q}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k q^{k(x+1)+x}}{q+wq^{2k+1}} \\
&= \frac{(-1)^n[2]_q}{(q^{-1}-1)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k wq^{2k+1+k(x-1)+x}}{1+wq^{2k+1}} \\
&= \frac{[2]_{q^{-1}}}{(1-q^{-1})^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k q^{-k(-x+1)-(-x)}}{1+w^{-1}q^{-(2k+1)}} \\
&= \tilde{E}_{n,q^{-1},w^{-1}}(-x).
\end{aligned}$$

Therefore, this completes the proof. □

Theorem 3.4. *Let m be an odd number and n be a nonnegative integer. Then we have*

$$\tilde{E}_{n,q,w}(mx) = \frac{[m]_q^n}{[m]_{-q}}\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}\tilde{E}_{n,q^m w^m}\left(x+\frac{2k+1-m}{m}\right).$$

Proof. Let m be an odd number. By using Definition 3.1, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\tilde{E}_{n,q,w}(mx)\frac{t^n}{n!}\\
&=[2]_q e^{\frac{[m]_q t}{1-q^m}}\sum_{n=0}^{\infty}\left(\frac{q}{1-q^m}\right)^n\frac{(-1)^n q^{(n+1)mx}}{1+(wq^{2n+1})^m}\sum_{k=0}^{m}(-1)^k(wq^{2n+1})^k\frac{([m]_q t)^n}{n!}\\
&=\frac{[2]_{q^m}}{[m]_{-q}}\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}e^{\frac{[m]_q t}{1-q^m}}\\
&\quad\times\sum_{n=0}^{\infty}\left(\frac{q^m}{1-q^m}\right)^n\frac{(-1)^n q^{m(n+1)(x+\frac{2k+1-m}{m})}}{1+(wq^{2n+1})^m}\frac{([m]_q t)^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\frac{[m]_q^n}{[m]_{-q}}\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}\tilde{E}_{n,q^m,w^m}\left(x+\frac{2k+1-m}{m}\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, the proof is complete by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equation above. □

We recall the definition of generating function of twisted q-Bernoulli polynomials of the second kind in introduction as follows(see [4]):

$$\sum_{n=0}^{\infty}\tilde{B}_{n,q,w}(x)\frac{t^n}{n!}=-te^{\frac{t}{1-q}}\sum_{n=0}^{\infty}\left(\frac{q}{1-q}\right)^n\frac{(-1)^n q^{(n+1)x}}{1-wq^{2n+1}}\frac{t^n}{n!}.$$

Theorem 3.5. *Let m be an even number and n be a natural numbers. Then we obtain*

$$n\tilde{E}_{n-1,q,w}(mx)=-[m]_q^{n-1}[2]_q\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}\tilde{B}_{n,q^m,w^m}\left(x+\frac{2k+1-m}{m}\right).$$

Proof. Let m be an even number. We get the following equation in a similar way to the proof in Theorem 3.4.

$$\begin{aligned}
&\sum_{n=0}^{\infty}\tilde{E}_{n,q,w}(mx)\frac{t^n}{n!}\\
&=[2]_q\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}e^{\frac{[m]_q t}{1-q^m}}\\
&\quad\times\sum_{n=0}^{\infty}\left(\frac{q^m}{1-q^m}\right)^n\frac{(-1)^n q^{m(n+1)(x+\frac{2k+1-m}{m})}}{1-(wq^{2n+1})^m}\frac{([m]_q t)^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(-\frac{[m]_q^n[2]_q}{[m]_q t}\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}\tilde{B}_{n,q^m,w^m}\left(x+\frac{2k+1-m}{m}\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

If we multiply t to both sides of the equation above, we get

$$\begin{aligned}&\sum_{n=1}^{\infty} n\tilde{E}_{n-1,q,w}(mx)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(-[2]_q[m]_q^{n-1}\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}\tilde{B}_{n,q^m,w^m}\left(x+\frac{2k+1-m}{m}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.2}$$

Since the polynomial $\tilde{B}_{0,q,w}(x)=0$, we can compare the coefficients on both sides of the equation (3.2). Therefore, the proof is complete. □

In order to see the relation between (w,q)-alternating power sums $A_{i,q,w}(m,n)$ and twisted q-Euler polynomials of the second kind $\tilde{E}_{n,q,w}$, we define (w,q)-alternating power sums as follows:

$$A_{i,q,w}(m,n)=\sum_{k=0}^{m-1}(-1)^k w^k q^{k(2n-2i+1)}[2k]_q^i.$$

Theorem 3.6. *Let m be a natural number and n be a nonnegative integer. Then we have*

$$\tilde{E}_{n,q,w}=\sum_{l=0}^{n}\binom{n}{l}\frac{[m]_q^l}{[m]_{-q}}q^{l(1-m)}\tilde{E}_{n,q^m,w^m}\sum_{i=0}^{n-l}\binom{n-l}{i}[1-m]_q^{n-l-i}A_{i,q,w}(m,n),$$

where $A_{i,q,w}(m,n)=\sum_{k=0}^{m-1}(-1)^k w^k q^{k(2n-2i+1)}[2k]_q^i$ are (w,q)-alternating power sums.

Proof. Let us put $x=0$ in Theorem 3.4. By using Theorme 2.2, we have

$$\begin{aligned}\tilde{E}_{n,q,w}&=\frac{[m]_q^n}{[m]_{-q}}\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}\tilde{E}_{n,q^m w^m}\left(\frac{2k+1-m}{m}\right)\\&=\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}\sum_{l=0}^{n}\binom{n}{l}\frac{[m]_q^l}{[m]_{-q}}q^{(l+1)(2k+1-m)}\tilde{E}_{n.q^m,w^m}\\&\quad\times\left([2k]_q+q^{2k}[1-m]_q\right)^{n-l}\\&=\sum_{l=0}^{n}\binom{n}{l}\frac{[m]_q^l}{[m]_{-q}}q^{l(1-m)}\tilde{E}_{n,q^m,w^m}\\&\quad\times\sum_{i=0}^{n-l}\binom{n-l}{i}[1-m]_q^{n-l-i}\sum_{k=0}^{m-1}(-1)^k w^k q^{k(2n-2i+1)}[2k]_q^i.\end{aligned} \tag{3.3}$$

Therefore, if we apply (w,q)-alternating power sums to the right-hand side of the equation (3.3), the proof is complete. □

Theorem 3.7. *For a natural number m and a nonnegative integer n, we have*

$$\tilde{E}_{n,q,w} = -\frac{[2]_q}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}[m]_q^{l-1}q^{(l-1)(1-m)}\tilde{B}_{l,q^m,w^m} \times \sum_{i=0}^{n+1-l}\binom{n+1-l}{i}[1-m]_q^{n+1-l-i}A_{i,q,w}(m,n),$$

where $A_{i,q,w}(m,n) = \sum_{k=0}^{m-1}(-1)^k w^k q^{k(2n-2i+1)}[2k]_q^i$ are (w,q)-alternating power sums.

Proof. Let us take $x = 0$ in Theorem 3.5. By using equation (1.1), we have

$$\begin{aligned}\tilde{E}_{n,q,w} &= -\frac{[m]_q^n[2]_q}{n+1}\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}\tilde{B}_{n+1,q^m,w^m}\left(\frac{2k+1-m}{m}\right)\\ &= -\frac{[m]_q^{n-1}[2]_q}{n+1}\sum_{k=0}^{m-1}(-1)^k w^k q^{m-k-1}\\ &\quad\times\sum_{l=0}^{n+1}\binom{n+1}{l}q^{l(2k+1-m)}\tilde{B}_{l,q^m,w^m}\left[\frac{2k+1-m}{m}\right]_{q^m}^{n+1-l}\\ &= -\frac{[2]_q}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}[m]_q^{l-1}q^{(l-1)(1-m)}\tilde{B}_{l,q^m,w^m}\\ &\quad\times\sum_{i=0}^{n+1-l}\binom{n+1-l}{i}[1-m]_q^{n+1-l-i}A_{i,q,w}(m,n).\end{aligned}$$

Therefore, this completes the proof. □

4. Special cases of Zeta functions and their relation with twisted q-Euler polynomials of the second kind

In this section, we introduce twisted-type q-Hurwitz-Lerch Euler zeta function of the second kind and define twisted q-Hurwitz's type Euler zeta function of the second kind. We investigate relation between the zeta functions and twisted q-Euler polynomials of the second kind.

Choi and Kim [4] defined twisted-type q-Hurwitz-Lerch Euler zeta function of the second kind as follows:

$$\tilde{\Phi}_{E,q}(w,s,x) = \sum_{n=0}^{\infty}\frac{w^n q^{n+x}}{[2n+1+x]_q^s}.$$

Definition 4.1. For $Re(s) > 0$, we define twisted q-Hurwitz's type Euler zeta function of the second kind as follows:

$$\tilde{\zeta}_{E,q,w}(s,x) = [2]_q\sum_{n=0}^{\infty}\frac{(-1)^n w^n q^{n+x}}{[2n+1+x]_q^s}.$$

Twisted q-Hurwitz's type Euler zeta function of the second kind $\tilde{\zeta}_{E,q,w}(s.x)$ is special case of twisted-type q-Hurwitz-Lerch Euler zera function of the second kind $\tilde{\Phi}_{E,q}(w,s,x)$ as follows:

$$\tilde{\zeta}_{E,q,w}(s.x) = [2]_q \tilde{\Phi}_{E,q}(-w,s,x).$$

If $w = 1$ and $q \to 1$, then twisted q-Hurwitz's type Euler zeta function of the second kind $\tilde{\zeta}_{E,q,w}(s.x)$ is reduced to zeta function of second kind $\tilde{\zeta}_E(s,x)$. That is

$$\lim_{q\to 1} \tilde{\zeta}_{E,q,1}(s,x) = \tilde{\zeta}_E(s,x).$$

Let us differentiate both sides of Definition 2.1 with respect to t. Then we get the following identity:

$$\tilde{E}_{k,q,w}(x) = [2]_q \sum_{n=0}^{\infty} (-1)^n w^n q^{n+x} [2n+1+x]_q^k.$$

We get the following theorem that is relation between twisted q-Euler polynomials of the second kind $\tilde{E}_{k,q,w}(x)$ and twisted-type q-Hurwitz-Lerch Euler zeta function of the second kind $\tilde{\Phi}_{E,q}(w,s,x)$.

Theorem 4.2. *Let k be a nonnegative integer. Then we obtain*

$$\tilde{E}_{k,q,w}(x) = [2]_q \tilde{\Phi}_{E,q}(-w,-k,x).$$

Let $w = 1$ and $q \to 1$ in Theorem 4.2. Then we get the following corollary.

Corollary 4.3. *For a nonnegative integer k, we have*

$$\tilde{E}_k(x) = 2\tilde{\Phi}_E(-1,-k,x),$$

where the polynomials $\tilde{E}_k(x)$ are the second kind Euler polynomials and the function $\tilde{\Phi}_E(z,s,x)$ is Hurwitz-Lerch Euler zeta function of the second kind.

The following theorem is relation between twisted q-Euler polynomials of the second kind $\tilde{E}_{k,q,w}(x)$ and twisted q-Hurwitz's type Euler zeta function of the second kind $\tilde{\zeta}_{E,q,w}(s.x)$.

Theorem 4.4. *Let k be a nonnegative integer. Then we have*

$$\tilde{E}_{k,q,w}(x) = \tilde{\zeta}_{E,q,w}(-k,x).$$

If we apply $w = 1$ and $q \to 1$ in Theorem 4.4, then we get the following corollary.

Corollary 4.5. *For a nonnegative integer k, we obtain*

$$\tilde{E}_k(x) = \tilde{\zeta}_E(-k,x).$$

where the polynomials $\tilde{E}_k(x)$ are the second kind Euler polynomials and the function $\tilde{\zeta}_E(s,x)$ is Hurwitz's type Euler zeta function of the second kind.

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