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PRECONDITIONED AOR ITERATIVE METHODS FOR SOLVING MULTI-LINEAR SYSTEMS WITH $\mathcal{M}$-TENSOR†

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Abstract. Some problems in engineering and science can be equivalently transformed into solving multi-linear systems. In this paper, we propose two preconditioned AOR iteration methods to solve multi-linear systems with -tensor. Based on these methods, the general conditions of preconditioners are given. We give the convergence theorem and comparison theorem of the two methods. The results of numerical examples show that methods we propose are more effective.

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1. Introduction

We consider the multi-linear system

$$\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}, \tag{1.1}$$

where $\mathcal{A} = (a_{ii_2\cdots i_m})$ is an order m dimension n tensor, $\mathbf{x}$ and $\mathbf{b}$ are n dimensional vectors. The tensor-vector product $\mathcal{A}\mathbf{x}^{m-1}$ is defined by

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m}, i = 1, 2, \cdots, n. \tag{1.2}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$. There are some practical applications of the multi-linear systems in engineering and science fields [1,2], for instance, numerical partial differential equations [3], tensor complementarity problems [4], data mining [5], tensor absolute value equations [6] and so on.

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Ding and Wei [3] proved the multi-linear system (1.1) always has a unique positive solution if $\mathcal{A}$ is a strong $\mathcal{M}$-tensor and $\mathbf{b}$ is a positive vector. In [12], the authors gave the definition of tensor splitting $\mathcal{A} = \mathcal{E} - \mathcal{F}$. The tensor splitting method for solving the system (1.1) is defined by

$$\mathbf{x}_k = [M(\mathcal{E})^{-1}\mathcal{F}\mathbf{x}_{k-1}^{m-1} + M(\mathcal{E})\mathbf{b}]^{[\frac{1}{m-1}]}, \qquad k = 1, 2, \cdots, n\ ,$$

According to this method, Li et al. [13] proposed preconditioned multi-linear systems based on the preconditioned technique of linear systems. Cui et al. [14] proposed a new preconditioner to solve the system (1.1).

Li, Liu and Vong [13] considered the preconditioner

$$P_\alpha = I + S_\alpha = \begin{bmatrix} 1 & -\alpha_1 a_{12\cdots 2} & 0 & \cdots & 0 \\ 0 & 1 & -\alpha_2 a_{23\cdots 3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1} a_{n-1,n\cdots n} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Cui, Li and Song [14] proposed a new preconditioner x

$$P_{\max} = (I + S_{\max}),$$

where $S_{\max}$ is defined by

$$S_{\max} = (s_{i,k_i}^m) = \begin{cases} -a_{i,k_i,\cdots k_i}, & i = 1, \cdots n-1\ , k_i > i, \\ 0, & otherwise, \end{cases}$$

where $k_i = \min\{j|\max_j|a_{ij,\cdots j}|, i < n, j > i\}$.

In this paper, $\mathcal{A}$ is a strong $\mathcal{M}$-tensor and $\mathbf{b} > \mathbf{0}$. Without loss of generality, we assume that the all diagonal entries of$\mathcal{A}$are 1. The preconditioned multi-linear system is $P\mathcal{A}\mathbf{x}^{m-1} = P\mathbf{b}$ where P is a nonsingular and nonnegative matrix with unit diagonal entries. Let $\widehat{\mathcal{A}} = P\mathcal{A}$ and $\widehat{\mathbf{b}} = P\mathbf{b}$.

2. AOR iterative method

2.1. Proposed method.

The order m dimension n unit tensor is denoted by $\mathcal{I}_m$. The majorization matrix of tensor $\mathcal{A}$ is denoted by $M(\mathcal{A})$ and $M(\mathcal{A})_{ij} = a_{ij\cdots j},\ i, j = 1, 2, \cdots, n$. Let$\widehat{\mathcal{A}} = \widehat{\mathcal{D}} - \widehat{\mathcal{L}} - \widehat{\mathcal{F}}$, where$\widehat{\mathcal{D}} = \widehat{D}\mathcal{I}_m$, $\widehat{\mathcal{L}} = \widehat{L}\mathcal{I}_m$, $\widehat{D}$ and $-\widehat{L}$are the diagonal and strictly lower triangular parts of $M(\widehat{\mathcal{A}})$, respectively. The matrix P is nonsingular and nonnegative with unit diagonal entries. Then

$$\widehat{\mathcal{A}} = (\hat{a}_{ii_2\cdots i_m}) = \sum_{k=1}^{n} p_{ik} a_{ki_2\cdots i_m}.$$

We define

$$M(\mathcal{A}) = I - L - U,$$

$$P = I + P_1 + P_2,$$
$$P_1U = E_1 + F_1 + G_1,$$
$$P_2L = E_2 + F_2 + G_2,$$

where E_1 and E_2 are diagonal matrixes, L, P_1, F_1 and F_2 are strictly lower triangular matrixes, U, P_2, G_1 and G_2 are strictly upper triangular matrixes. It is obvious that the matrices mentioned above are nonnegative.

$$\widehat{\mathcal{D}} = (I - E_1 - E_2)\mathcal{I}_m,$$
$$\widehat{\mathcal{L}} = (L - P_1 + P_1L + F_1 + F_2)\mathcal{I}_m,$$
$$\widehat{\mathcal{F}} = (U - P_2 + P_2U + G_1 + G_2)\mathcal{I}_m + P\mathcal{F}.$$

Hadjidimos [18] proposed an accelerated overrelaxation (AOR) iterative method to solve linear systems. Based on this method, we propose the following AOR iterative algorithm:

$$\mathbf{x}_k{}^{[m-1]} = \left(\widehat{D} - r\widehat{L}\right)^{-1}[(1-w)\,\widehat{\mathcal{D}} + (w-r)\,\widehat{\mathcal{L}} + w\widehat{\mathcal{F}}]\mathbf{x}_{k-1}^{m-1} + w\left(\widehat{D} - r\widehat{L}\right)^{-1}\widehat{\mathbf{b}},$$

$k = 1, 2, \cdots, n$, where $\mathbf{x}_k{}^{[m-1]} = \left(x_1^{m-1}, x_2^{m-1}, \cdots, x_n^{m-1}\right)^T$, w and r are real parameters with $0 \le r \le w \le 1\,(w \neq 0)$. The iteration tensor of AOR iterative methods is

$$\widehat{\mathcal{T}}_{r,w} = \left(\widehat{D} - r\widehat{L}\right)^{-1}[(1-w)\,\widehat{\mathcal{D}} + (w-r)\,\widehat{\mathcal{L}} + w\widehat{\mathcal{F}}].$$

2.2. Convergence analysis of the proposed method.

First, we give the following lemma to show that $\widehat{\mathcal{A}}$ is a strong $\mathcal{M}$-tensor.

Lemma 2.1. *Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor. If $P = (p_{ij})$ is a nonsingular and nonnegative matrix with $p_{ii} = 1$ and*

$$\sum_{k=1}^{n} p_{ik}a_{kj\cdots j} \le 0 \;\; 1 \le i \neq j \le n,$$

then $\widehat{\mathcal{A}} = P\mathcal{A}$ is a strong $\mathcal{M}$-tensor.

Proof. When $(i_2, \cdots, i_m) \neq (j, \cdots, j)$, since $p_{ij} \ge 0$, we have $\sum\limits_{k=1}^{n} p_{ik}a_{ki_2\cdots i_m} \le 0$. Noticing that $\sum\limits_{k=1}^{n} p_{ik}a_{kj\cdots j} \le 0$ for $1 \le i \neq j \le n$, we obtain $\sum\limits_{k=1}^{n} p_{ik}a_{ki_2\cdots i_m} \le 0$ for $(i, i_2, \cdots, i_m) \neq (i, i, \cdots, i)$. Consequently, $\widehat{\mathcal{A}}$ is a $\mathcal{Z}$-tensor.

According to the theorem 3 in [15], we assume that $\mathbf{x} \ge \mathbf{0}$ and $\mathcal{A}\mathbf{x}^{m-1} > \mathbf{0}$. It is obvious that $P \ge 0$, then we have $P\mathcal{A}\mathbf{x}^{m-1} \ge 0$. Therefore, there exists $\mathbf{x} \ge \mathbf{0}$ such that $P\mathcal{A}\mathbf{x}^{m-1} \ge 0$. By the theorem 3 in [15], $\widehat{\mathcal{A}}$ is a strong $\mathcal{M}$-tensor. □

We give the splitting $\widehat{\mathcal{A}} = \widehat{\mathcal{M}} - \widehat{\mathcal{N}}$, where $\widehat{\mathcal{N}} = \frac{1}{w}[(1-w)\widehat{\mathcal{D}} + (w-r)\widehat{\mathcal{L}} + w\widehat{\mathcal{F}}]$and $\widehat{\mathcal{M}} = \frac{1}{w}(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}})$. By the definition and related theorems of tensor splitting in [12], we have the follow theorem 2.2.

Theorem 2.2. *Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor. If $P = (p_{ij})$ is a nonsingular and nonnegative matrix with $p_{ii} = 1$ and*

$$\sum_{k=1}^{n} p_{ik} a_{kj\cdots j} \le 0,\ 1 \le i \ne j \le n,$$

then $\widehat{\mathcal{A}} = \widehat{\mathcal{M}} - \widehat{\mathcal{N}}$ is a convergent regular splitting for $0 \le r \le w \le 1$ $(w \ne 0)$.

Proof. By lemma 2.1, $\widehat{\mathcal{A}}$ is a strong $\mathcal{M}$-tensor. It is obvious that $\widehat{\mathcal{F}} \ge \mathcal{O}$ and $\widehat{L} \ge \mathbf{0}$.Accordingly, $\widehat{\mathcal{N}} = \frac{1}{w}[(1-w)\widehat{\mathcal{D}} + (w-r)\widehat{\mathcal{L}} + w\widehat{\mathcal{F}}] \ge 0$ for $0 \le r \le w \le 1$ $(w \ne 0)$.By the Neumann series, we have

$$\begin{aligned} M\left(\widehat{\mathcal{M}}\right)^{-1} &= w\left(\widehat{D} - r\widehat{L}\right)^{-1} = w\left(I - r\widehat{D}^{-1}\widehat{L}\right)^{-1}\widehat{D}^{-1} \\ &= w\left[I + r\widehat{D}^{-1}\widehat{L} + \left(r\widehat{D}^{-1}\widehat{L}\right)^{2} + \left(r\widehat{D}^{-1}\widehat{L}\right)^{3} + \cdots + \left(r\widehat{D}^{-1}\widehat{L}\right)^{n-1}\right]\widehat{D}^{-1}. \end{aligned}$$

By the theorem 3 [15] and proposition 4 [15], we get $0 < \sum_{k=1}^{n} p_{ik} a_{ki\cdots i} \le 1$, i.e., $\widehat{D}^{-1} \ge I$. It is easy to know that $M\left(\widehat{\mathcal{M}}\right)^{-1} \ge \mathbf{0}$ and $\widehat{\mathcal{A}} = \widehat{\mathcal{M}} - \widehat{\mathcal{N}}$ is a regular splitting. By the lemma 3.16 in [12], $\widehat{\mathcal{A}} = \widehat{\mathcal{M}} - \widehat{\mathcal{N}}$ is a convergent splitting. □

According to the theorem 5.4 [12] and lemma 2.1, $\widehat{\mathcal{A}}$ is a strong $\mathcal{M}$-tensor and the AOR iterative method is convergent.

2.3. The comparison theorem.

Before the comparison theorem, we give the following lemma first.

Lemma 2.3. *Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor. Then there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, $\mathcal{A}(\varepsilon) = (a_{ii_2\cdots i_m}(\varepsilon))$ is also a strong $\mathcal{M}$-tensor, where*

$$a_{ii_2\cdots i_m}(\varepsilon) = \begin{cases} a_{ii_2\cdots i_m}, & a_{ii_2\cdots i_m} \ne 0, \\ -\varepsilon, & a_{ii_2\cdots i_m} = 0. \end{cases}$$

Proof. Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor, it is not hard to check that $\mathcal{A}(\varepsilon)$ is a $\mathcal{Z}$-tensor. By the theorem 3 in [15], there exists $\mathbf{x} \ge \mathbf{0}$ such that $\mathcal{A}\mathbf{x}^{m-1} > \mathbf{0}$. Thus,

$$\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} > 0 i = 1, 2, \cdots, n.$$

Assuming that $\delta=\sum_{i_2,\cdots,i_m=1}^{n} x_{i_2}\cdots x_{i_m}>0$, let

$$\varepsilon_0=\frac{1}{\delta}\min\left\{\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m},\ i=1,2,\cdots,n\right\}.$$

Then, we have $\varepsilon_0>0$ and

$$\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}-\delta\varepsilon_0\geq 0, i=1,2,\cdots,n.$$

For any $0<\varepsilon<\varepsilon_0$, we obtain

$$\begin{aligned}&\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}(\varepsilon)x_{i_2}\cdots x_{i_m}\\ &>\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}-\varepsilon\sum_{i_2,\cdots,i_m=1}^{n} x_{i_2}\cdots x_{i_m}\\ &\geq\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}-\delta\varepsilon_0\\ &\geq 0,\qquad\qquad i=1,2,\cdots n.\end{aligned}$$

Thus $\mathcal{A}\left(\varepsilon\right)\mathbf{x}^{m-1}>0$. By the theorem 3 in [15], we have $\mathcal{A}(\varepsilon)=(a_{ii_2\cdots i_m}(\varepsilon))$ is also a strong $\mathcal{M}$-tensor. □

Let $\mathcal{A}=\mathcal{I}_m-\mathcal{L}-\mathcal{F}$, where $\mathcal{L}=L\mathcal{I}_m$, $-L$ is the strictly lower triangular part of $M(\widehat{\mathcal{A}})$. We give the following splittings:

$$\mathcal{A}=\mathcal{M}-\mathcal{N}=\frac{1}{w}(\mathcal{I}_m-r\mathcal{L})-\frac{1}{w}[(1-w)\,\mathcal{I}_m+(w-r)\,\mathcal{L}+w\mathcal{F}],$$

$$\widehat{\mathcal{A}}=\widehat{\mathcal{M}}-\widehat{\mathcal{N}}=\frac{1}{w}(\widehat{\mathcal{D}}-r\widehat{\mathcal{L}})-\frac{1}{w}[(1-w)\,\widehat{\mathcal{D}}+(w-r)\,\widehat{\mathcal{L}}+w\widehat{\mathcal{F}}].$$

The iteration tensor:

$$\mathcal{T}_{r,w}=M(\mathcal{M})\mathcal{N}=(I-rL)^{-1}[(1-w)\,\mathcal{I}_m+(w-r)\,\mathcal{L}+w\mathcal{F}],$$

$$\widehat{\mathcal{T}}_{r,w}=M\left(\widehat{\mathcal{M}}\right)\widehat{\mathcal{N}}=\left(\widehat{D}-r\widehat{L}\right)^{-1}[(1-w)\,\widehat{\mathcal{D}}+(w-r)\,\widehat{\mathcal{L}}+w\widehat{\mathcal{F}}].$$

The comparison theorem of spectral radius between preconditioned AOR iterative method with and without preconditioner is given.

Theorem 2.4. *Let $\mathcal{A}$ be an $\mathcal{M}$-tensor. If $P=(p_{ij})$ is a nonsingular and nonnegative matrix with $p_{ii}=1$ and*

$$\sum_{k=1}^{n} p_{ik}a_{kj\cdots j}\leq 0,\ 1\leq i\neq j\leq n.$$

we have $\widehat{\mathcal{T}}_{r,w} \le \mathcal{T}_{r,w} < 1$ *for* $0 \le r \le w \le 1\,(w \ne 0)$.

Proof. Since $\mathcal{A}$ is a $\mathcal{M}$-tensor, we can get $\mathcal{N} = \frac{1}{w}[(1-w)\,\mathcal{I}_m + (w-r)\,\mathcal{L} + w\mathcal{F}] \ge \mathcal{O}$. By Neumann series, we have $M(\mathcal{M})^{-1} \ge \mathbf{0}$. Thus, $M(\mathcal{M})^{-1}\mathcal{N}$ is a nonnegative tensor. By theorem 2.2, $M(\widehat{\mathcal{M}})^{-1}\widehat{\mathcal{N}}$ is a nonnegative tensor. According to theorem 1.3 in [22], there exists a nonnegative vector $\mathbf{x} \ne \mathbf{0}$ such that

$$M(\mathcal{M})^{-1}\mathcal{N}\mathbf{x}^{m-1} = \lambda\mathbf{x}^{[m-1]} \; \textit{and} \; 0 \le \rho(\mathcal{T}_{r,w}) = \lambda < 1,$$

or equivalently,

$$[(1-w)\mathcal{I}_m + (w-r)\mathcal{L} + w\mathcal{F}]\mathbf{x}^{m-1} = \lambda(\mathcal{I}_m - r\mathcal{L})\mathbf{x}^{m-1}.$$

Then we obtain

$$\mathcal{A}\mathbf{x} = M(\mathcal{M})(\mathcal{I}_m - M(\mathcal{M})^{-1}\mathcal{N})\mathbf{x}^{m-1} = (1-\lambda)M(\mathcal{M})\mathcal{I}_m\mathbf{x}^{m-1} = (1-\lambda)\mathcal{M}\mathbf{x}^{m-1}.$$

If $\lambda > 0$, we have $w - r + r\lambda \ne 0$ and $\mathcal{L}\mathbf{x}^{m-1} = \frac{(-1+w+\lambda)\mathcal{I}_m - w\mathcal{F}}{w-r+r\lambda}\mathbf{x}^{m-1}$.

Hence,

$$\begin{aligned}
& M(\widehat{\mathcal{M}})^{-1}\widehat{\mathcal{N}}\mathbf{x}^{m-1} - \lambda\mathbf{x}^{[m-1]} \\
&= M\left(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}}\right)^{-1}[(1-w)\,\widehat{\mathcal{D}} + (w-r)\,\widehat{\mathcal{L}} + w\widehat{\mathcal{F}} - \lambda(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}})]\mathbf{x}^{m-1} \\
&= M\left(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}}\right)^{-1}[(1-w-\lambda)\,\widehat{\mathcal{D}} + (w-r+r\lambda)\,\widehat{\mathcal{L}} + w\widehat{\mathcal{F}}]\mathbf{x}^{m-1} \\
&= M\left(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}}\right)^{-1}[(1-\lambda)\,\widehat{\mathcal{D}} + (r\lambda - r)\,\widehat{\mathcal{L}} - w\widehat{\mathcal{A}}]\mathbf{x}^{m-1} \\
&= M\left(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}}\right)^{-1}[(1-\lambda)\,\widehat{\mathcal{D}} + \mathrm{r}\,(\lambda-1)\,\widehat{\mathcal{L}} - wP\mathcal{A}]\mathbf{x}^{m-1} \\
&= M\left(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}}\right)^{-1}[(1-\lambda)\,\widehat{\mathcal{D}} - \mathrm{r}\,(1-\lambda)\,\widehat{\mathcal{L}} - (1-\lambda)P(\mathcal{I}_m - r\mathcal{L})]\mathbf{x}^{m-1} \\
&= (\lambda-1)M\left(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}}\right)^{-1}[-(I - E_1 - E_2)\mathcal{I}_m + \mathrm{r}(L - P_1 + P_1L + \\
&\quad F_1 + F_2)\mathcal{I}_m + (I + P_1 + P_2)\mathcal{I}_m - r(L + P_1L + P_2L)\mathcal{I}_m]\mathbf{x}^{m-1} \\
&= (\lambda-1)M\left(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}}\right)^{-1}[(E_1 + E_2 + r(F_1 + F_2) + (1-r)P_1 + P_2 \\
&\quad - rP_2L)\mathcal{I}_m]\mathbf{x}^{m-1} \\
&= (\lambda-1)M\left(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}}\right)^{-1}[(E_1 + E_2 + r(F_1 + F_2) + (1-r)P_1)\mathcal{I}_m \\
&\quad + P_2\mathcal{I}_m - rP_2\frac{(-1+w+\lambda)\mathcal{I}_m - w\mathcal{F}}{w-r+r\lambda}]\mathbf{x}^{m-1} \\
&= (\lambda-1)M\left(\widehat{\mathcal{D}} - r\widehat{\mathcal{L}}\right)^{-1}[(E_1 + E_2 + r(F_1 + F_2) + (1-r)P_1)\mathcal{I}_m \\
&\quad + wP_2\frac{(1-r)\mathcal{I}_m + r\mathcal{F}}{w-r+r\lambda}]\mathbf{x}^{m-1}.
\end{aligned} \tag{4.1}$$

Case 1: $\mathcal{A}$ is irreducible. It is easy to get that

$$\begin{aligned} M(\mathcal{M})^{-1}\mathcal{N} &= (I-rL)^{-1}[(1-w)\mathcal{I}_m + (w-r)\mathcal{L} + w\mathcal{F}] \\ &= [I + rL + (rL)^2 + \cdots + (rL)^{n-1}][(1-w)\mathcal{I}_m + (w-r)\mathcal{L} + w\mathcal{F}] \\ &\geq (1-w)\mathcal{I}_m + (w-r)\mathcal{L} + w\mathcal{F} + r(1-w)\mathcal{L} \\ &= (1-w)\mathcal{I}_m + w(1-r)\mathcal{L} + w\mathcal{F}. \end{aligned}$$

When $0 \leq r < 1$, since $\mathcal{A}$ is irreducible, $M(\mathcal{M})^{-1}\mathcal{N}$ is also irreducible. By theorem 1.4 in [21], it is easy to know that $\lambda > 0$ and the Perron vector $\mathbf{x} > \mathbf{0}$. From (4.1), we obtain $M(\widehat{\mathcal{M}})^{-1}\widehat{\mathcal{N}}\mathbf{x}^{m-1} \leq \lambda\mathbf{x}^{[m-1]}$. By lemma 3.2 in [13], we have

$$\rho(\widehat{\mathcal{T}}_{r,w}) \leq \lambda = \rho(\mathcal{T}_{r,w}).$$

When $r = w = 1$,

$$\rho(\widehat{\mathcal{T}}_{1,1}) = \lim_{r\to 1^-} \rho(\widehat{\mathcal{T}}_{r,1}) \leq \lim_{r\to 1^-} \rho(\mathcal{T}_{r,1}) = \rho(\mathcal{T}_{1,1}) < 1.$$

Case 2: $\mathcal{A}$ is reducible. By the lemma 2.3, there exists a positive number ε such that $\mathcal{A}(\varepsilon)$ is a irreducible $\mathcal{M}$-tensor. According to the proof above, we obtain

$$\rho(\widehat{\mathcal{T}}_{r,w}) = \lim_{\varepsilon\to 0} \rho(\widehat{\mathcal{T}}_{r,w}(\varepsilon)) \leq \lim_{\varepsilon\to 0} \rho(\mathcal{T}_{r,w}(\varepsilon)) = \rho(\mathcal{T}_{r,w}) < 1.$$

Thus, $\rho(M(\widehat{\mathcal{M}})^{-1}\widehat{\mathcal{N}}) \leq \rho(M(\mathcal{M})^{-1}\mathcal{N}) < 1$.The proof is completed. □

3. Modified AOR iterative method

We give the splitting $\widehat{\mathcal{A}} = \overline{\mathcal{U}} - \overline{\mathcal{L}}$, where

$$\overline{\mathcal{U}} = \begin{cases} \hat{a}_{ii_2\cdots i_m}, & i_2, \cdots, i_m \geq i \ , \\ 0, & otherwise. \end{cases}$$

We give the definition: $\mathcal{A}_i = (a_{ii_2\cdots i_m})_{i_2,i_3,\cdots,i_m=1}^n$, then

$$\mathcal{A}\mathbf{x}^{m-1} = \left(\mathcal{A}_1\mathbf{x}^{m-1}, \mathcal{A}_2\mathbf{x}^{m-1}, \cdots, \mathcal{A}_n\mathbf{x}^{m-1}\right)^T,$$

$i = 1, 2, \cdots, n$. Then we propose modified AOR iterative method as follows.

Algorithm 3.1:

(1) We give $k_{\max}$ as the maximum iteration steps and the precision ε as the termination conditions. Then we take a positive initial vector $\mathbf{x}_0 = (x_1^{(0)}, x_2^{(0)}, \cdots x_n^{(0)})^T$ and let $k = 1$;

(2) While $k \leq k_{\max}$;

(3)$a_{ii\cdots i}\left[x_i^{(k)}\right]^{m-1} = a_{ii\cdots i}\left[x_i^{(k-1)}\right]^{m-1} + r\overline{\mathcal{L}}_i\left(\mathbf{x}_{k,i-1}^{m-1} - \mathbf{x}_{k-1}^{m-1}\right) + w\left(\widehat{b}_i - \widehat{\mathcal{A}}_i\mathbf{x}_{k-1}^{m-1}\right)$, $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots$, where $\mathbf{x}_{k,i} = (x_1^{(k)}, \cdots x_i^{(k)}, x_{i+1}^{(k-1)}, \cdots x_n^{(k-1)})^T$, $\mathbf{x}_{k,\ 0}$=$\mathbf{x}_{k-1}$, $\mathbf{x}_{k,\ n}$=$\mathbf{x}_k$;

(4) If $\left\|\mathcal{A}\mathbf{x}_k^{m-1} - b\right\|_2 < \varepsilon$,break and output $\mathbf{x}_k$;

(5) $k = k + 1$and back to step (2). Before giving the convergence theorem, we give following two lemmas.

Lemma 3.1. *Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor. If $0 < r \leq w \leq 1\,(w \neq 0)$, under the conditions of lemma 2.1, the vector sequence $\{\mathbf{x}_k\}$ generated by the algorithm 3.1 is increasing for any positive initial vector $\mathbf{x}_0 > \mathbf{0}$ with $\mathbf{0} < \mathcal{A}\mathbf{x}_0^{m-1} \leq \mathbf{b}$.*

Proof. By lemma 2.1, $\widehat{\mathcal{A}}$ is a strong $\mathcal{M}$-tensor. Assuming that $\mathbf{x}_{k-1} > \mathbf{0}$ and $\widehat{\mathcal{A}}\mathbf{x}_{k-1}^{m-1} \leq \widehat{\mathbf{b}}$, when $i = 1$, it is obvious that

$$\begin{aligned}
& a_{11\cdots1}\left[x_1^{(k)}\right]^{m-1} \\
&= a_{11\cdots1}\left[x_1^{(k-1)}\right]^{m-1} + r\overline{\mathcal{L}}_1\left(\mathbf{x}_{k,0}^{m-1} - \mathbf{x}_{k-1}^{m-1}\right) + w\left(\widehat{b}_1 - \widehat{\mathcal{A}}_1\mathbf{x}_{k-1}^{m-1}\right) \\
&= a_{11\cdots1}\left[x_1^{(k-1)}\right]^{m-1} + w\left(\widehat{b}_1 - \widehat{\mathcal{A}}_1\mathbf{x}_{k-1}^{m-1}\right) \\
&\geq a_{ii\cdots i}\left[x_1^{(k-1)}\right]^{m-1}.
\end{aligned}$$

When $i = j$, assuming that $\left[x_t^{(k-1)}\right]^{m-1} \leq \left[x_t^{(k)}\right]^{m-1}$, $t = 1, 2, \cdots, j$.
When $i = j+1$, it is obvious that $\mathbf{x}_{k,j} \geq \mathbf{x}_{k,0} = \mathbf{x}_{k-1}$. We have

$$\begin{aligned}
& a_{j+1,j+1,\cdots,j+1}\left[x_{j+1}^{(k)}\right]^{m-1} \\
&= a_{j+1,j+1,\cdots,j+1}\left[x_{j+1}^{(k-1)}\right]^{m-1} + r\overline{\mathcal{L}}_{j+1}\left(\mathbf{x}_{k,j}^{m-1} - \mathbf{x}_{k-1}^{m-1}\right) + w\left[b_{j+1} - \mathcal{A}_{j+1}\mathbf{x}_{k-1}^{m-1}\right] \\
&\geq a_{j+1,j+1,\cdots,j+1}\left[x_{j+1}^{(k-1)}\right]^{m-1} + w\left[b_{j+1} - \mathcal{A}_{j+1}\mathbf{x}_{k-1}^{m-1}\right] \\
&\geq a_{j+1,j+1,\cdots,j+1}\left[x_{j+1}^{(k\text{-}1)}\right]^{m-1}.
\end{aligned}$$

Therefore, $\left[x_i^{(k-1)}\right]^{m-1} \leq \left[x_i^{(k)}\right]^{m-1}$ for $i = 1, 2, \cdots, n$, we obtain $\mathbf{x}_k \geq \mathbf{x}_{k-1}$.
Noticing that $0 < r \leq w \leq 1$, then

$$\begin{aligned}
& a_{ii\cdots i}\left[x_i^{(k)}\right]^{m-1} \\
&= a_{ii\cdots i}\left[x_i^{(k-1)}\right]^{m-1} + r\overline{\mathcal{L}}_i\left(\mathbf{x}_{k,i-1}^{m-1} - \mathbf{x}_{k-1}^{m-1}\right) + w\left[\widehat{b}_i - \widehat{\mathcal{A}}_i\mathbf{x}_{k-1}^{m-1}\right] \\
&\leq a_{ii\cdots i}\left[x_i^{(k-1)}\right]^{m-1} + \overline{\mathcal{L}}_i\left(\mathbf{x}_{k,i-1}^{m-1} - \mathbf{x}_{k-1}^{m-1}\right) + \left[\widehat{b}_i - \widehat{\mathcal{A}}_i\mathbf{x}_{k-1}^{m-1}\right] \\
&= a_{ii\cdots i}\left[x_i^{(k-1)}\right]^{m-1} + \overline{\mathcal{L}}_i\mathbf{x}_{k,i-1}^{m-1} + \widehat{b}_i - \overline{\mathcal{U}}_i\mathbf{x}_{k-1}^{m-1} \\
&\leq a_{ii\cdots i}\left[x_i^{(k)}\right]^{m-1} + \overline{\mathcal{L}}_i\mathbf{x}_k^{m-1} + \widehat{b}_i - \overline{\mathcal{U}}_i\mathbf{x}_k^{m-1} \\
&\leq a_{ii\cdots i}\left[x_i^{(k)}\right]^{m-1} + \left[\widehat{b}_i - \widehat{\mathcal{A}}_i\mathbf{x}_k^{m-1}\right].
\end{aligned}$$

we get $\widehat{\mathcal{A}}\mathbf{x}_k^{m-1} \leq \mathbf{b}$. Since $P > \mathbf{0}$ and $\mathbf{0} < \mathcal{A}\mathbf{x}_0^{m-1} \leq \mathbf{b}$, it is easy to know $\widehat{\mathcal{A}}\mathbf{x}_0^{m-1} \leq \widehat{\mathbf{b}}$. By the mathematical induction, we have $\mathbf{x}_k^{[m-1]} \geq \mathbf{x}_{k-1}^{[m-1]}$ $(k = 1, 2, \cdots)$. The vector sequence $\{\mathbf{x}_k\}$ is increasing. ∎

Lemma 3.2. *Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor. If $0 \le r \le w \le 1\,(w \ne 0)$, under the conditions of lemma 2.1, the vector sequence $\{\mathbf{x}_k\}$ generated by the algorithm 3.1 is bounded above for any positive initial vector $\mathbf{x}_0 > \mathbf{0}$ with $\mathbf{0} < \mathcal{A}\mathbf{x}_0^{m-1} \le \mathbf{b}$.*

Proof. By lemma 2.1, $\widehat{\mathcal{A}}$ is a strong $\mathcal{M}$-tensor. Let $\widehat{\mathcal{A}} = \widehat{\mathcal{D}} - \widehat{\mathcal{B}}$, where $\widehat{\mathcal{B}}$ is a nonnegative tensor and $\widehat{\mathcal{D}} = \widehat{D}\mathcal{I}_m$. Since $0 < \mathcal{A}\mathbf{x}_0^{m-1} \le b$, it is easy to know $\mathbf{0} \le \widehat{\mathcal{A}}\mathbf{x}_0^{m-1} \le \widehat{\mathbf{b}}$. Let

$$\begin{cases} \widehat{\mathcal{B}}\mathbf{x}_0^{m-1} \le \alpha\widehat{\mathcal{D}}\mathbf{x}_0^{m-1}, & 0 < \alpha < 1, \\ \widehat{\mathbf{b}} \le \beta\widehat{\mathcal{D}}\mathbf{x}_0^{m-1}. \end{cases}$$

It is easy to know $\widehat{\mathcal{B}}\mathbf{x}_0^{m-1} + \mathbf{b} \le (\alpha+\beta)\,\widehat{\mathcal{D}}\mathbf{x}_0^{m-1}$. By lemma 2.4, we can get $\mathbf{x}_k > \mathbf{0}$ and $\widehat{\mathcal{A}}\mathbf{x}_k^{m-1} \le \widehat{\mathbf{b}}$ for $k = 1, 2, \cdots$. Therefore,

$$a_{ii\cdots i}\left[x_i^{(k)}\right]^{m-1} = \widehat{\mathcal{D}_i}\mathbf{x}_k^{m-1} \le \widehat{\mathcal{B}}_i\mathbf{x}_k^{m-1} + b_i.$$

We assume that $\mathbf{x}_{k-1}^{[m-1]} \le \left(\alpha^{(k-1)n} + \alpha^{(k-1)n-1}\beta + \cdots + \alpha\beta + \beta\right)\mathbf{x}_0^{[m-1]}$, $k > 1$. For the k-step iteration: when $i = 1$, we have

$$\begin{aligned}
&a_{11\cdots 1}\left[x_1^{(k)}\right]^{m-1} \\
&= a_{11\cdots 1}\left[x_1^{(k-1)}\right]^{m-1} + r\overline{\mathcal{L}}_1\left(\mathbf{x}_{k,0}^{m-1} - \mathbf{x}_{k-1}^{m-1}\right) + w\left(\widehat{b}_1 - \widehat{\mathcal{A}}_1\mathbf{x}_{k-1}^{m-1}\right) \\
&\le \widehat{\mathcal{D}}_1\mathbf{x}_{k-1}^{m-1} + \left(\widehat{b}_1 - \widehat{\mathcal{A}}_1\mathbf{x}_{k-1}^{m-1}\right) \le \widehat{\mathcal{B}}_1\mathbf{x}_{k-1}^{m-1} + b_1 \\
&\le \left(\alpha^{(k-1)n} + \alpha^{(k-1)n-1}\beta + \cdots + \alpha\beta + \beta\right)\widehat{\mathcal{B}}_1\mathbf{x}_0^{m-1} + \widehat{b}_1 \\
&\le \left(\alpha^{(k-1)n+1} + \alpha^{(k-1)n}\beta + \cdots + \alpha\beta + \beta\right)a_{11\cdots 1}\left[x_1^{(0)}\right]^{m-1}.
\end{aligned}$$

we obtain $x_1^{(k)} \le \sqrt[m-1]{\left(\alpha^{(k-1)n+1} + \alpha^{(k-1)n}\beta + \cdots + \alpha\beta + \beta\right)}x_1^{(0)}$.
When $i = j$, assuming that $x_t^{(k)} \le \sqrt[m-1]{\left(\alpha^{(k-1)n+t} + \alpha^{(k-1)n+t-1}\beta + \cdots + \beta\right)}x_t^{(0)}$, $t = 1, 2, \cdots, j$.
When $i = j+1$, we have

$$\begin{aligned}
&a_{j+1,\cdots,j+1}\left[x_{j+1}^{(k)}\right]^{m-1} \\
&= a_{j+1,\cdots,j+1}\left[x_{j+1}^{(k-1)}\right]^{m-1} + r\overline{\mathcal{L}}_{j+1}\left(\mathbf{x}_{k,j}^{m-1} - \mathbf{x}_{k-1}^{m-1}\right) + w\left[\widehat{b}_{j+1} - \widehat{\mathcal{A}}_{j+1}\mathbf{x}_{k-1}^{m-1}\right] \\
&\le \widehat{\mathcal{D}}_{j+1}\mathbf{x}_{k-1}^{m-1} + \left(\widehat{b}_{j+1} - \widehat{\mathcal{A}}\;_1\mathbf{x}_{k,j}^{m-1}\right) = \widehat{\mathcal{B}}_{j+1}\mathbf{x}_{k,j}^{m-1} + \widehat{b}_{j+1} \\
&\le \left(\alpha^{(k-1)n} + \alpha^{(k-1)n-1}\beta + \cdots + \alpha\beta + \beta\right)\widehat{\mathcal{B}}_{j+1}\mathbf{x}_0^{m-1} + \widehat{b}_{j+1} \\
&\le \left(\alpha^{(k-1)n+1} + \alpha^{(k-1)n}\beta + \cdots + \alpha\beta + \beta\right)a_{j+1,\cdots,j+1}\left[x_{j+1}^{(0)}\right]^{m-1}.
\end{aligned}$$

We obtain $x_{j+1}^{(k)} \le \sqrt[m-1]{\alpha^{(k-1)n+j+1} + \alpha^{(k-1)n+j}\beta + \cdots + \alpha\beta + \beta}x_{j+1}^{(0)}$.

According to the initial conditions, we can get

$$\mathbf{x}_1^{[m-1]} \leq \left(\alpha^n + \alpha^{n-1}\beta + \cdots + \alpha\beta + \beta\right) \mathbf{x}_0^{[m-1]}.$$

by calculating. By the mathematical induction, we have

$$\mathbf{x}_k^{[m-1]} \leq \left(\alpha^{kn} + \alpha^{kn-1}\beta + \cdots + \alpha\beta + \beta\right) \mathbf{x}_0^{[m-1]},$$

for $k = 1, 2, \cdots$. Let $k \to \infty$, then $\lim\limits_{k\to\infty} \left(\alpha^{kn} + \alpha^{kn-1}\beta + \cdots + \alpha\beta + \beta\right) = \frac{\beta}{1-\alpha}$. Therefore, the vector sequence $\{\mathbf{x}_k\}$ is bounded above. □

According to the two lemmas, we have the following convergence theorem.

Theorem 3.3. *Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor. If $0 \leq r \leq w \leq 1\,(w \neq 0)$, under the conditions of lemma 2.1, then the vector sequence $\{\mathbf{x}_k\}$ generated by the algorithm 3.1 converges to the only positive limit for any positive initial vector $\mathbf{x}_0 > \mathbf{0}$ with $\mathbf{0} < \mathcal{A}\mathbf{x}_0^{m-1} \leq \mathbf{b}$.*

Proof. By the lemma 3.1 and lemma 3.2, the vector sequence $\{\mathbf{x}_k\}$ is increasing and bounded above. Consequently, $\{\mathbf{x}_k\}$ converges to the only positive limit. □

4. Numerical examples

All numerical examples will be done in MATLAB R2018b on a personal computer with Intel(R) Core (TM) i5-7300HQ CPU @2.50GHz and 8.00GB RAM. In the section, "IT" and "CPU" denote the number of iteration steps and the CPU time, respectively. In the numerical examples, we set the maximum number of iterative steps to 1000 and the precision to 10-11. w is from 0.1 to 2 and the interval is 0.1, then $(w, r)_{opt}$ denotes the optimal parameters of AOR and modified AOR method. Considering the following preconditioner:

$$P = I + U_\beta = \begin{bmatrix} 1 & -\beta_1 a_{12\cdots 2} & -\beta_1 a_{13\cdots 3} & \cdots & -\beta_1 a_{1n\cdots n} \\ 0 & 1 & -\beta_2 a_{23\cdots 3} & \cdots & -\beta_2 a_{2n\cdots n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\beta_{n-1} a_{n-1,n\cdots n} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Just for convenience, let $\beta_1 = \beta_2 = \cdots = \beta_{n-1} = \beta$.

Example 1. Let $\mathcal{B} \in \mathbb{R}^{[3,n]}$ be a nonnegative tensor with $b_{i_1 i_2 i_3} = |\sin(i_1 + i_2 + i_3)|$. By [7], $\mathcal{A} = n^2\mathcal{I}_m - \mathcal{B}$ is a strong $\mathcal{M}$-tensor.

Let $\mathbf{b} = \mathbf{1}$ and initial vector $\mathbf{x}_0 = \mathbf{0}$. We take the parameters β from 0 to 4 and the interval is 0.3. The numerical results are shown in table 1.

TABLE 1. Numerical results of Example 1

	PAOR			modified PAOR		
β	IT	CPU	$(w,r)_{opt}$	IT	CPU	$(w,r)_{opt}$
0	27	2.055e-04	(1.4 , 1.4)	15	1.379e-04	(1.5 , 1.4)
0.3	26	1.422e-04	(1.4 , 1.3)	15	1.253e-04	(1.5 , 1.4)
0.6	23	1.226e-04	(1.5 , 1.5)	15	1.248e-04	(1.5 , 1.4)
0.9	23	1.153e-04	(1.4 , 1.4)	15	1.187e-04	(1.5 , 1.4)
1.2	22	1.113e-04	(1.4 , 1.3)	14	9.230e-05	(1.5 , 1.4)
1.5	21	1.132e-04	(1.4 , 1.3)	14	9.430e-05	(1.5 , 1.4)
1.8	20	1.131e-04	(1.4 , 1.2)	13	8.840e-05	(1.5 , 1.4)
2.1	18	9.450e-05	(1.4 , 1.4)	13	8.820e-05	(1.5 , 1.4)
2.4	17	9.140e-05	(1.4 , 1.3)	12	7.980e-05	(1.5 , 1.4)
2.7	16	8.090e-05	(1.4 , 1.4)	12	8.020e-05	(1.3 , 1.2)
3.0	16	7.540e-05	(1.4 , 1.4)	12	7.910e-05	(1.3 , 1.2)
3.3	16	8.990e-05	(1.3 , 1.3)	11	7.130e-05	(1.3 , 1.2)
3.6	15	1.343e-04	(1.3 , 1.3)	11	7.360e-05	(1.3 , 1.2)
3.9	16	8.540e-05	(1.3 , 1.2)	11	7.380e-05	(1.3 , 1.2)

Example 2 [24]. Let $\mathcal{A} \in \mathbb{R}^{[3,n]}$ and $\mathbf{b} \in \mathbb{R}^n$ with:

$$\begin{cases} a_{111} = (2+n)/2, & \\ a_{nnn} = 1, & \\ a_{iii} = 2, & i = 2,3,\cdots,n-1, \\ a_{1ii} = -1/2, & i = 2,3,\cdots,n-1, \\ a_{iii-1} = -1/2, & i = 2,3,\cdots,n-1, \\ a_{ii-1i-1} = -1/2, & i = 2,3,\cdots,n-1, \\ a_{ii+1i+1} = -1/2, & i = 2,3,\cdots,n-1. \end{cases}$$

and

$$\begin{cases} b_1 = c_0^2, & \\ b_i = a/(n-1)^2, & i = 2,3,\cdots,n-1, \\ b_n = c_1^2. & \end{cases}$$

Let initial vector $\mathbf{x}_0 = (1,1,\cdots,1)^T$, $c_0 = 1/2$, $c_1 = 1/3$ and $a = 2$. When n=10, we take the parameters β from 0 to 2 and the interval is 0.2. The numerical results are shown in table 2.

It is well known that we can get the Jacobi, the Gauss-Seidel and the successive overrelaxation (SOR) iteration methods by choosing certain values. The results of two numerical examples show that the AOR method is more effective than the GS and SOR method when we take the optimal parameter (w,r). We compare AOR with modified AOR iteration method in example 1 and example 2. It can be seen from Table 1 and table 2 that modified AOR iteration method requires less iterative steps and CPU time.

Table 2. Numerical results of Example 2

	PAOR			modified PAOR		
β	IT	CPU	$(w,r)_{opt}$	IT	CPU	$(w,r)_{opt}$
0	45	0.0034	(1.2 , 1.2)	24	0.0011	(1.3 , 1.2)
0.2	41	0.0404	(1.4 , 1.4)	22	0.0008	(1.3 , 1.2)
0.4	37	0.0365	(1.4 , 1.4)	21	0.0007	(1.3 , 1.0)
0.6	32	0.0303	(1.4 , 1.4)	21	0.0007	(1.3 , 1.0)
0.8	29	0.0014	(1.4 , 1.1)	21	0.0007	(1.3 , 0.8)
1.0	28	0.0279	(1.4 , 1.4)	21	0.0007	(1.1 , 1.0)
1.2	27	0.0009	(1.2 , 1.2)	20	0.0008	(1.1 , 1.0)
1.4	24	0.0009	(1.2 , 1.2)	19	0.0007	(1.1 , 1.0)
1.6	22	0.0026	(1.2 , 1.2)	21	0.0007	(1.3 , 1.0)
1.8	25	0.0232	(1.2 , 1.2)	33	0.0012	(1.3 , 1.2)
2.0	28	0.0007	(1.0 , 1.0)	69	0.0024	(1.5 , 1.4)

5. Conclusions

In this paper, we propose two preconditioned AOR iteration methods to solve multi-linear systems with $\mathcal{M}$-tensor. We prove that the two proposed methods are convergent when the preconditioner satisfies certain conditions. Moreover, the comparison theorem shows that the AOR iteration method with preconditioner converges faster than the method without preconditioner. A new preconditioner is given in the numerical examples. The results of the numerical examples illustrate the effectiveness of these methods. In the future, we will consider how to choose the best preconditioner.

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