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EXISTENCE OF EXTREMAL SOLUTIONS FOR FUZZY DIFFERENTIAL EQUATIONS DRIVEN BY LIU PROCESS†

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ABSTRACT. In this paper, we study existence of extremal solutions for fuzzy differential equations driven by Liu process. To show extremal solutions, we define partial ordering relative to fuzzy process. This is an extension of the results of Kwun et al. [5] and Rodríguez-López [13] to fuzzy differential equations in credibility space.

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1. Introduction

The theory on fuzzy random variables has be investigated by many authors. Zadeh [17] introduced the theory of possibility in order to measure a fuzzy event. Puri and Ralescu [12] studied fuzzy random variables as a generalization of random sets. Wang and Zhang [15] made fundamental properties for fuzzy stochastic processes.

In 2002, Liu and Liu [7] introduced the concept of credibility measure which is different from the above one. Credibility theory is deduced from the normality, monotonicity, self-duality, and maximality axioms. In order to deal with the evolution of fuzzy phenomena with time, the concept of fuzzy process presented by Liu [9]. The most important and useful fuzzy process is Liu process which has the same status as Brown motion in stochastic process. Based on this process, Liu integral and Liu formula were introduced by Liu [9], which are the counterparts of Ito integral and Ito formula. A new kind of fuzzy differential equation driven by Liu process was defined by Liu [9] as

$$dX_t = f(X_t, t)dt + g(X_t, t)dC_t$$

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where C_t is a standard Liu process, and f, g are some given functions. The solution of such equation is a fuzzy process.

Nieto and Rodríguez-López [11] studied existence of extremal solutions for quadratic fuzzy equations. Ezzati and Abbasbandy [2] proved existence of extremal solutions for fuzzy polynomials and their numerical solutions by the well-known fixed point theorem of Tarski. Recently, Kwun, Kim and Park [5] proved the existence of extremal solutions for impulsive fuzzy differential equations with periodic boundary value in n-dimensional fuzzy vector space.

In this paper, we study existence of extremal solutions for fuzzy differential equations driven by Liu process:

$$\begin{cases} dx(t,\theta) = f(t, x(t,\theta))dC(t), \ t \in [0,T], \\ x(0) = x_0 \in E_N, \end{cases} \tag{1}$$

where $T > 0$, in [10], E_N is the set of all upper semi-continuously convex fuzzy numbers on R, $(\Theta, \mathcal{P}, Cr)$ is a credibility space, $x : [0,T] \times (\Theta, \mathcal{P}, Cr) \to E_N$ is a fuzzy process, $f : [0,T] \times E_N \to E_N$ is a fuzzy function, $C(t)$ is a standard Liu process, $x_0 \in E_N$ is an initial value.

In section 2, we include some basic concepts relative to fuzzy sets and Liu process. In section 3, we define partial ordering relative to fuzzy process and prove the existence of extremal solution for the equation (1).

2. Preliminaries

In this section, we give basic definitions, terminologies, notations and Lemmas which are most relevant to our investigation and are needed in latter sections. All undefined concepts and notions used here are standard.

We consider E_N the space of one-dimensional fuzzy numbers $u : R \to [0,1]$, satisfying the following properties:

(1) u is normal, i.e., there exists an $u_0 \in R$ such that $u(t_o) = 1$;

(2) u is fuzzy convex, i.e., $u(\lambda t + (1-\lambda)s) \geq \min\{u(t), u(s)\}$ for any $t, s \in R$, $0 \leq \lambda \leq 1$;

(3) $u(t)$ is upper semi-continuous, i.e., $u(t_0) \geq \overline{\lim}_{k\to\infty} u(t_k)$ for any $t_k \in R$ $(k = 0, 1, 2, \cdots)$, $t_k \to t_0$;

(4) $[u]^0$ is compact.

The level sets of u, $[u]^\alpha = \{t \in R : u(t) \geq \alpha\}, \alpha \in (0,1]$, and $[u]^0$ are nonempty compact convex sets in R ([1]).

Definition 2.1 [14] We define a complete metric D_L on E_N by

$$\begin{aligned} D_L(u,v) &= \sup_{0\leq\alpha\leq 1} d_L([u]^\alpha, [v]^\alpha) \\ &= \sup_{0\leq\alpha\leq 1} \max\{|u_l^\alpha - v_l^\alpha|, |u_r^\alpha - v_r^\alpha|\}, \end{aligned}$$

for any $u, v \in E_N$, which satisfies $D_L(u+w, v+w) = D_L(u,v)$ for every $\alpha \in [0,1]$, $[u]^\alpha = [u_l^\alpha, u_r^\alpha]$, for every $\alpha \in [0,1]$ where $u_l^\alpha, u_r^\alpha \in R$ with $u_l^\alpha \leq u_r^\alpha$.

Definition 2.2 [6] For any $u, v \in C([0,T], E_N)$, the metric $H_1(u,v)$ on $C([0,T], E_N)$ is defined by

$$H_1(u,v) = \sup_{0 \le t \le T} D_L(u(t), v(t)).$$

Let Θ be a nonempty set, and let $\mathcal{P}$ the power set of Θ. Each element in $\mathcal{P}$ is called an event. In order to present an axiomatic definition of credibility, it is necessary to assign to each event A a number $Cr\{A\}$ which indicates the credibility that A will occur. In order to ensure that the number $Cr\{A\}$ has certain mathematical properties which we intuitively expect a credibility to have, we accept the following four axioms:

(1) $(Normality)$ $Cr\{\Theta\} = 1$.
(2) $(Monotonicity)$ $Cr\{A\} \le Cr\{B\}$ whenever $A \subset B$.
(3) $(Self - Duality)$ $Cr\{A\} + Cr\{A^c\} = 1$ for any event A.
(4) $(Maximality)$ $Cr\{\cup_i A_i\} = \sup_i Cr\{A_i\}$ for any events $\{A_i\}$ with $\sup_i Cr\{A_i\} < 0.5$.

The set function C_r is called a credibility measure if it satisfies the normality, monotonicity, self-duality and maximality axioms.

Definition 2.3 [8] Let Θ be a nonempty set, $\mathcal{P}$ the power set of Θ, and Cr a credibility measure. Then the triplet $(\Theta, \mathcal{P}, C_r)$ is called a credibility space.

Definition 2.4 [9] A fuzzy variable is defined as a function from a credibility space $(\Theta, \mathcal{P}, C_r)$ to the set of real numbers.

Definition 2.5 [9] Let T be an index set and let $(\Theta, \mathcal{P}, C_r)$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, C_r)$ to the set of real numbers.

That is, a fuzzy process $x(t,\theta)$ is a function of two variables such that the function $x(t^*, \theta)$ is a fuzzy variable for each t^*. For each fixed θ^*, the function $x(t, \theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $x(t,\theta)$ is said to be sample-continuous if the sample path is continuous for almost all $\theta \in \Theta$.

Definition 2.6 Let $(\Theta, \mathcal{P}, C_r)$ be a credibility space. For fuzzy process $x(t,\theta)$, the α-level set $[x(t,\theta)]^\alpha = [x_l^\alpha(t,\theta), x_r^\alpha(t,\theta)]$ is defined by

$$x_l^\alpha(t,\theta) = \inf x^\alpha(t,\theta) = \inf\{a \in R \mid x(t,\theta)(a) \ge \alpha\},$$
$$x_r^\alpha(t,\theta) = \sup x^\alpha(t,\theta) = \sup\{a \in R \mid x(t,\theta)(a) \ge \alpha\},$$

where $(x_t)_l^\alpha, (x_t)_r^\alpha \in R$ with $(x_t)_l^\alpha \le (x_t)_r^\alpha$ when $\alpha \in [0,1]$.

Definition 2.7 [7] Let ξ be a fuzzy variable and r be real number. Then the expected value of ξ is defined by

$$E\xi = \int_0^{+\infty} Cr\{\xi \ge r\} dr - \int_{-\infty}^0 Cr\{\xi \le r\} dr$$

provided that at least one of the integrals is finite.

Definition 2.8 [9] A fuzzy process $C(t)$ is said to be a Liu process if
(i) $C(0) = 0$,
(ii) $C(t)$ has stationary and independent increments,
(iii) every increment $C(t+s) - C(s)$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$, whose membership function is

$$\mu(x) = 2\Big(1 + \exp\Big(\frac{\pi|x - et|}{\sqrt{6}\sigma t}\Big)\Big)^{-1},\ x \in R.$$

The parameters e and σ are called the $drift$ and $diffusion$ coefficients, respectively. Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

Definition 2.9 [3] Let $x(t)$ be a fuzzy process and let $C(t)$ be a standard Liu process. For any partition of closed interval $[c, d]$ with $c = t_0 < \cdots < t_n = d$, the mesh is written as $\triangle = \max_{1 \le i \le n}(t_i - t_{i-1})$. Then the fuzzy integral of $x(t)$ with respect to $C(t)$ is

$$\int_c^d x(t)dC(t) = \lim_{\triangle \to 0} \sum_{i=1}^{n} x(t_{i-1})(C(t_i) - C(t_{i-1}))$$

provided that the limit exists almost surely and is a fuzzy variable.

Lemma 2.1 [3] Let $C(t)$ be a standard Liu process. For any given θ with $Cr\{\theta\} > 0$, the path $C(t,\theta)$ is Lipschitz continuous, that is, the following inequality holds

$$|C(t_1, \theta) - C(t_2, \theta)| < K(\theta)|t_1 - t_2|,$$

where K is a fuzzy variable called the Lipschitz constant of a Liu process with

$$K(\theta) = \begin{cases} \sup_{0 \le s < t} \frac{|C(t,\theta) - C(s,\theta)|}{t - s}, & Cr\{\theta\} > 0, \\ \infty, & \text{otherwise}, \end{cases}$$

and $E[K^p] < \infty,\ \forall p > 0$.

Lemma 2.2 [3] Let $C(t)$ be a standard Liu process, and let $h(t; c)$ be a continuously differentiable function. Define $x(t) = h(t; C(t))$. Then we have the following chain rule

$$dx(t) = \frac{\partial h(t; C(t))}{\partial t} dt + \frac{\partial h(t; C(t))}{\partial C} dC(t).$$

Lemma 2.3 [3] Let $f(t)$ be continuous fuzzy process, the following inequality of fuzzy integral holds

$$\Big| \int_c^d f(t)dC(t) \Big| \le K \int_c^d |f(t)|dt,$$

where $K = K(\theta)$ is defined in Lemma 2.1.

3. Existence of Extremal Solutions

In this section, we consider the existence and uniquencess of extremal solutions for the fuzzy differential equation (1).

For a positive constant M, $t \in [0,T]$, consider

$$\begin{cases} dx(t,\theta) = -Mx(t,\theta)dt + f(t,x(t,\theta))dC(t) + Mx(t,\theta)dt, \\ x(0) = x_0 \in E_N \end{cases} \tag{2}$$

and

$$\begin{cases} dx(t,\theta) = Mx(t,\theta)dt + f(t,x(t,\theta))dC(t) - Mx(t,\theta)dt, \\ x(0) = x_0 \in E_N. \end{cases} \tag{3}$$

Let's define the following an integral solution for equation (2)

$$\begin{cases} x(t,\theta) = U(t)x_0 + \int_0^t U(t-s)f(s,x(s,\theta))dC(s) \\ \qquad\qquad\qquad + M\int_0^t U(t-s)x(s,\theta)ds,\ t \in [0,T], \\ x(0) = x_0 \in E_N, \end{cases} \tag{4}$$

where $U(t) = e^{-Mt}$ is continuous with $U(0) = I$, $|U(t)| \le c,\ c > 0$, for all $t \in [0,T]$.

Also, let's define the following an integral solution for equation (3)

$$\begin{cases} x(t,\theta) = S(t)x_0 + \int_0^t S(t-s)f(s,x(s,\theta))dC(s) \\ \qquad\qquad\qquad - M\int_0^t S(t-s)x(s,\theta)ds,\ t \in [0,T], \\ x(0) = x_0 \in E_N, \end{cases} \tag{5}$$

where $S(t) = e^{Mt}$ is continuous with $S(0) = I$, $|S(t)| \le d,\ d > 0$, for all $t \in [0,T]$.

By Definition 2.11 and Lemma 2.1,

$$\begin{aligned} &\int_0^t U(t-s)x(s,\theta)ds \\ &= \lim_{\triangle\to 0}\sum_{i=1}^{n} U(t-t_{i-1})x(t_{i-1},\theta)(t_i - t_{i-1})(C(t_i) - C(t_{i-1}))\frac{1}{C(t_i)-C(t_{i-1})} \\ &\ge \lim_{\triangle\to 0}\sum_{i=1}^{n} U(t-t_{i-1})x(t_{i-1},\theta)(C(t_i) - C(t_{i-1}))\frac{1}{K} \\ &= \frac{1}{K}\int_0^t U(t-s)x(s,\theta)dC(s). \end{aligned}$$

Thus we have

$$\begin{aligned} &\int_0^t U(t-s)f(s,x(s,\theta))dC(s) + M\int_0^t U(t-s)x(s,\theta)ds \\ &\ge \int_0^t U(t-s)f(s,x(s,\theta))dC(s) + \frac{M}{K}\int_0^t U(t-s)x(s,\theta)dC(s) \end{aligned}$$

$$= \int_0^t U(t-s)G(s,x(s,\theta))dC(s).$$

Since

$$\int_0^t S(t-s)x(s,\theta)ds$$
$$= \lim_{\triangle\to 0}\sum_{i=1}^{n} S(t-t_{i-1})x(t_{i-1},\theta)(t_i-t_{i-1})(C(t_i)-C(t_{i-1}))\frac{1}{C(t_i)-C(t_{i-1})}$$
$$\geq \lim_{\triangle\to 0}\sum_{i=1}^{n} S(t-t_{i-1})x(t_{i-1},\theta)(C(t_i)-C(t_{i-1}))\frac{1}{K}$$
$$= \frac{1}{K}\int_0^t S(t-s)x(s,\theta)dC(s),$$

we get

$$\int_0^t S(t-s)f(s,x(s,\theta))dC(s) - M\int_0^t S(t-s)x(s,\theta)ds$$
$$\leq \int_0^t S(t-s)f(s,x(s,\theta))dC(s) - \frac{M}{K}\int_0^t S(t-s)x(s,\theta)dC(s)$$
$$= \int_0^t S(t-s)F(s,x(s,\theta))dC(s).$$

Therefore let us the following equations (6) and (7) instead of equations (4) and (5), respectively.

$$\begin{cases} x(t,\theta) = U(t)x_0 + \int_0^t U(t-s)G(s,x(s,\theta))dC(s), \\ x(0) = x_0 \in E_N, \end{cases} \tag{6}$$

where $U(t) = e^{-Mt}$ is continuous with $U(0) = I$, $|U(t)| \leq c$, $c > 0$, for all $t \in [0,T]$.

$$\begin{cases} x(t,\theta) = S(t)x_0 + \int_0^t S(t-s)F(s,x(s,\theta))dC(s), \\ x(0) = x_0 \in E_N, \end{cases} \tag{7}$$

where $S(t) = e^{Mt}$ is continuous with $S(0) = I$, $|S(t)| \leq d$, $d > 0$, for all $t \in [0,T]$.

Lemma 3.1 [4] For $n \in N$, let $\{x_n\}$ and x be integrably bounded fuzzy random variables. The following conditions are equivalent.
(i) $x_n \to^{p.D} x$ and $E||x_n|| \to E||x||$;
(ii) $ED(x_n,x) \to 0$;
(iii) $x_n \to^{p.D} x$ and $(||x_n||, n \in N)$ is uniformly integrable,
where $x_n \to^{p.D} x$ means that x_n converges in probability to x in D if $D(x_n,x) \to^p 0$.

Lemma 3.2 [16] Let $v(t) \in C[0,T]$ be continuous with $v(t) \geq 0$ and

$$v(t) \leq C \int_0^t v(s)ds, \quad t \in [0,T].$$

Then $v(t) \equiv 0$.

Assume the following:

(H1) For $L_1,\ L_2 > 0,\ x_0 \in E_N$,

$$d_L\Big([U(t)x_0]^\alpha, [x_0]^\alpha\Big) \leq L_1, \quad d_L\Big([S(t)x_0]^\alpha, [x_0]^\alpha\Big) \leq L_2.$$

(H2) For $x(\cdot), y(\cdot) \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$, $t \in [0,T]$, there exist positive numbers $m_1,\ m_2$ such that

$$d_L\Big([G(t,x)]^\alpha, [G(t,y)]^\alpha\Big) \leq m_1 d_L([x]^\alpha, [y]^\alpha),$$

$$d_L\Big([F(t,x)]^\alpha, [F(t,y)]^\alpha\Big) \leq m_2 d_L([x]^\alpha, [y]^\alpha)$$

and $F(0, \mathcal{X}_{\{0\}}(0)) \equiv 0,\ G(0, \mathcal{X}_{\{0\}}(0)) \equiv 0$.

(H3) For $L_3 > 0,\ x_0 \in E_N$, $d_L\Big([x_0]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big) \leq L_3$.

(H4) For $\varepsilon > 0$, $(L_1 + cm_1KL_3T)e^{cm_1KT} \leq \varepsilon$.

(H5) For $\varepsilon > 0$, $(L_2 + dm_2KL_3T)e^{dm_2KT} \leq \varepsilon$.

Theorem 3.1 If hypotheses (H1)- (H5) hold. Then the equations (6) and (7) have unique solutions, for $x_0 \in E_N$, respectively, $x(\cdot) \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$ and $y(\cdot) \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$.

Proof First, to show the existence of the solution, a successive approximation method will be introduced to construct a solution of the equation (6).

We define $x_0 = x_0(t,\theta)$ and for $n = 1, 2, \cdots$,

$$x_n(t,\theta) = U(t)x_0 + \int_0^t U(t-s)G(s, x_{n-1}(s,\theta))dC(s),\ t \in [0,T].$$

For any given θ with $Cr\{\theta\} > 0$, $t \in [0,T]$, by Lemma 2.3 and hypotheses (H1)-(H3), we have

$$\begin{aligned} & d_L\Big([x_1(t,\theta)]^\alpha, [x_0(t,\theta)]^\alpha\Big) \\ & = d_L\Big(\Big[U(t)x_0 + \int_0^t U(t-s)G(s,x_0)dC(s)\Big]^\alpha, [x_0]^\alpha\Big) \\ & \leq d_L\Big([U(t)x_0]^\alpha, [x_0]^\alpha\Big) \end{aligned}$$

$$
\begin{aligned}
&+d_L\Big(\Big[\int_0^t U(t-s)G(s,x_0)dC(s)\Big]^\alpha, \Big[\int_0^t U(t)G(s,\mathcal{X}_{\{0\}}(0))dC(s)\Big]^\alpha\Big)\\
&\le d_L\Big([U(t)x_0]^\alpha, [x_0]^\alpha\Big) + cK\int_0^t d_L\Big([G(s,x_0)]^\alpha, [G(s,\mathcal{X}_{\{0\}}(0))]^\alpha\Big)ds\\
&\le d_L\Big([U(t)x_0]^\alpha, [x_0]^\alpha\Big) + cm_1K\int_0^t d_L\Big([x_0]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big)ds\\
&\le L_1 + cm_1KL_3t.
\end{aligned}
$$

The inductive assumption is

$$
\begin{aligned}
&d_L\Big([x_n(t,\theta)]^\alpha, [x_{n-1}(t,\theta)]^\alpha\Big)\\
&= d_L\Big(\Big[U(t)x_0 + \int_0^t U(t-s)G(s,x_{n-1}(s,\theta))dC(s)\Big]^\alpha,\\
&\qquad\qquad \Big[U(t)x_0 + \int_0^t U(t-s)G(s,x_{n-2}(s,\theta))dC(s)\Big]^\alpha\Big)\\
&\le d_L\Big(\Big[\int_0^t U(t-s)G(s,x_{n-1}(t,\theta))dC(s)\Big]^\alpha,\\
&\qquad\qquad \Big[\int_0^t U(t-s)G(s,x_{n-2}(s,\theta))dC(s)\Big]^\alpha\Big)\\
&\le cK\int_0^t d_L\Big([G(s,x_{n-1}(s,\theta))]^\alpha, [G(s,x_{n-2}(s,\theta))]^\alpha\Big)ds\\
&\le cm_1K\int_0^t d_L\Big([x_{n-1}(s,\theta)]^\alpha, [x_{n-2}(s,\theta)]^\alpha\Big)ds\\
&\le (cm_1K)^{n-1}\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{n-1}} d_L\Big([x_1(s,\theta)]^\alpha, [x_0(s,\theta)]^\alpha\Big)ds\cdots dt_3dt_2\\
&\le (cm_1K)^{n-1}\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{n-1}} (L_1 + cm_1KL_3s)ds\cdots dt_3dt_2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&H_1\Big(x_n, x_{n-1}\Big)\\
&= \sup_{t\in[0,T]} D_L\Big(x_n(t,\theta), x_{n-1}(t,\theta)\Big)\\
&= \sup_{t\in[0,T]}\sup_{\alpha\in[0,1]} d_L\Big([x_n(t,\theta)]^\alpha, [x_{n-1}(t,\theta)]^\alpha\Big)\\
&\le \sup_{t\in[0,T]}\sup_{\alpha\in[0,1]} d_L\Big([x_n(t,\theta)]^\alpha, [x_{n-1}(t,\theta)]^\alpha\Big)\\
&\le \sup_{t\in[0,T]}\sup_{\alpha\in[0,1]} (cm_1K)^{n-1}\int_0^{t_1}\cdots\int_0^{t_{n-1}} d_L\Big([x_1(s,\theta)]^\alpha, [x_0(s,\theta)]^\alpha\Big)ds\cdots dt_2
\end{aligned}
$$

$$\leq \sup_{t\in[0,T]} \sup_{\alpha\in[0,1]} (cm_1K)^{n-1} \int_0^{t_1} \cdots \int_0^{t_{n-1}} (L_1 + cm_1KL_3s)ds \cdots dt_2$$
$$\leq (L_1 + cm_1KL_3T)\frac{(cm_1KT)^n}{n!}.$$

Therefore, for each θ with $Cr\{\theta\} > 0$, by hypothesis (H4) and Lemma 3.1, we get

$$\begin{aligned}\lim_{n\to\infty} E\Big(H_1(x_n, x)\Big) &\leq \sum_{n=1}^{\infty} E\Big(\sup_{t\in[0,T]} \sup_{\alpha\in[0,1]} d_L\Big([x_n(t,\theta)]^\alpha, [x_{n-1}(t,\theta)]^\alpha\Big)\Big)\\ &\leq \sum_{n=1}^{\infty} E\Big((L_1 + cm_1KL_3T)\frac{(cm_1KT)^n}{n!}\Big)\\ &= (L_1 + cm_1KL_3T)e^{cm_1KT}\\ &\leq \varepsilon.\end{aligned}$$

then $x_n(t,\theta)$ convergence uniformly to x in $C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$.

Next we show that the solution of the equation (6) is unique. Assume that both of ξ and η are solutions of (6). Then for any $\theta \in \Theta$ with $Cr\{\theta\} > 0$, by Lemma 2.3 and hypothesis (H2), we have

$$\begin{aligned}&d_L\Big([\xi(t,\theta)]^\alpha, [\eta(t,\theta)]^\alpha\Big)\\ &= d_L\Big(\Big[\int_0^t U(t-s)G(s,\xi(s,\theta))dC(s)\Big]^\alpha, \Big[\int_0^t U(t-s)G(s,\eta(s,\theta))dC(s)\Big]^\alpha\Big)\\ &\leq cK\int_0^t d_L([G(s,\xi(s,\theta))]^\alpha, [G(s,\eta(s,\theta))]^\alpha)ds\\ &\leq cm_1K\int_0^t d_L([\xi(s,\theta)]^\alpha, [\eta(s,\theta)]^\alpha)ds.\end{aligned}$$

It follows from Lemma 3.2 that $d_L([\xi(s,\theta)]^\alpha, [\eta(s,\theta)]^\alpha) \equiv 0$ for almost all θ. The uniqueness is proved. By hypotheses (H1)-(H3) and (H5), similarly, the equation (7) has a unique solution $y(\cdot) \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$.

The following Theorem 3.2 is an extension of Theorem 2 in [13] (Relative compactness criteria in $C(I, E^1)$ to stochastic fuzzy set in $C([0,T]\times(\Theta, \mathcal{P}, C_r), E_N)$.

Theorem 3.2 Let $I = [0,T]$ be a compact interval in R, $\Theta = (\Theta, \mathcal{P}, C_r)$ a compact set, and $B \subseteq C(I \times \Theta, E_N)$ such that for all $x \in B$ and $t \in I$, $\theta \in \Theta$, $x(t,\theta)$ be a continuous fuzzy process. Consider

$$\overline{B}_l = \{\overline{x}_l \mid x \in B\} \subseteq C([0,1] \times I \times \Theta, R),$$
$$\overline{B}_r = \{\overline{x}_r \mid x \in B\} \subseteq C([0,1] \times I \times \Theta, R),$$

where

$$\overline{x}_l : [0,1] \times I \times \Theta \to R,\ \text{by}\ \overline{x}_l(\alpha,t,\theta) = \overline{x}_l^\alpha(t,\theta) = x_l^\alpha(t,\theta)$$

and

$$\overline{x}_r : [0,1] \times I \times \Theta \to R, \text{ by } \overline{x}_r(\alpha, t, \theta) = \overline{x}_r^{\alpha}(t,\theta) = x_r^{\alpha}(t,\theta).$$

If $\overline{B}_l$ and $\overline{B}_r$ are relatively compact sets in $(C([0,1] \times I \times \Theta, R), ||\cdot||_\infty)$, then B is a relatively compact set in $C(I \times \Theta, E_N)$.

Proof We show that $\{x_n\} \subset B$ has a subsequence which converges to a point of $C(I \times \Theta, E_N)$.

Since $\overline{B}_l$ and $\overline{B}_r$ are a relatively compact set in $(C([0,1] \times I \times \Theta, R), ||\cdot||_\infty)$, $\{(\overline{x_n})_l\} \subset \overline{B}_l$ and $\{(\overline{x_n})_r\} \subset \overline{B}_r$ have subsequences which converge to points of $C([0,1] \times I \times \Theta, R)$, respectively. That is,

$$\{(\overline{x_{n_k}})_l\}(\subset \{(\overline{x_n})_l\}) \to \overline{x}_l = x_l,$$
$$\{(\overline{x_{n_k}})_r\}(\subset \{(\overline{x_n})_r\}) \to \overline{x}_r = x_r,$$

where $x_l,\ x_r \in C([0,1] \times I \times \Theta, R)$.

Consider $x : I \times \Theta \to E_N$ is fuzzy function defined by

$$[x(t,\theta)]^\alpha = [x_l^\alpha(t,\theta), x_r^\alpha(t,\theta)],\ \alpha \in [0,1].$$

For each $t \in I$, $\theta \in \Theta$ fixed, these intervals represent the family of level sets of some fuzzy number $x(t,\theta) \in E_N$. For all $\alpha \in [0,1]$, $[x_l^\alpha(t,\theta), x_r^\alpha(t,\theta)]$ are nonempty compact and convex sets in R, since

$$(\overline{x_{n_k}})_l^\alpha(t,\theta) = (x_{n_k})_l^\alpha(t,\theta) \le (x_{n_k})_r^\alpha(t,\theta) = (\overline{x_{n_k}})_r^\alpha(t,\theta),$$

in consequence, passing to the limit as k tends to $+\infty$, $x_l^\alpha(t,\theta) \le x_r^\alpha(t,\theta)$, for all $\alpha \in [0,1]$.

Using that $(\overline{x_{n_k}})_l^{\cdot}(t,\theta)$ is non-decreasing in α and $(\overline{x_{n_k}})_r^{\cdot}(t,\theta)$ is non-increasing in α, for every $k \in N$, and passing to the limit again, we deduce that $x_l^{\cdot}(t,\theta)$ is non-decreasing in α and $x_r^{\cdot}(t,\theta)$ is non-increasing in the variable α. Thus

$$[x_l^{\alpha_2}(t,\theta), x_r^{\alpha_2}(t,\theta)] \subseteq [x_l^{\alpha_1}(t,\theta), x_r^{\alpha_1}(t,\theta)], \quad 0 \le \alpha_1 \le \alpha_2 \le 1.$$

Finally, let $0 \le \alpha \le 1$ and $\{\alpha_k\} \subseteq [0,1]$, $\{\alpha_k\} \uparrow \alpha$, then $[x_l^{\alpha_k}(t,\theta), x_r^{\alpha_k}(t,\theta)]$ is a sequence of nested intervals, then by continuity of x_l and x_r, we get

$$\begin{aligned}\bigcap_{k\in N}[x_l^{\alpha_k}(t,\theta), x_r^{\alpha_k}(t,\theta)] &= \Big[\sup_{k\in N} x_l^{\alpha_k}(t,\theta), \inf_{k\in N} x_r^{\alpha_k}(t,\theta)\Big] \\ &= \Big[\lim_{k\to+\infty} x_l^{\alpha_k}(t,\theta), \lim_{k\to+\infty} x_r^{\alpha_k}(t,\theta)\Big] \\ &= [x_l^\alpha(t,\theta), x_r^\alpha(t,\theta)].\end{aligned}$$

By Theorem 1.5.1 of [10], for each $t \in I$, $\theta \in \Theta$, there exists $x(t,\theta) \in E_N$ such that

$$[x(t,\theta)]^\alpha = [x_l^\alpha(t,\theta), x_r^\alpha(t,\theta)], \quad \alpha \in (0,1]$$

and

$$[x(t,\theta)]^0 = \overline{\bigcup_{0<\alpha\le 1}[x_l^\alpha(t,\theta), x_r^\alpha(t,\theta)]} = [x_l^0(t,\theta), x_r^0(t,\theta)].$$

From the continuity of x_l and x_r, we have $x \in C(I \times \Theta, E_N)$. For $t_0 \in I$, a.s. $\theta \in \Theta$, we obtain

$$\begin{aligned}
& D_L\Big(x(t,\theta), x(t_0,\theta)\Big) \\
&= \sup_{\alpha\in[0,1]} d_L\Big([x(t,\theta)]^\alpha, [x(t_0,\theta)]^\alpha\Big) \\
&= \sup_{\alpha\in[0,1]} d_L\Big([x_l^\alpha(t,\theta), x_r^\alpha(t,\theta)], [x_l^\alpha(t_0,\theta), x_r^\alpha(t_0,\theta)]\Big) \\
&= \sup_{\alpha\in[0,1]} \max\Big\{|x_l^\alpha(t,\theta) - x_l^\alpha(t_0,\theta)|, |x_r^\alpha(t,\theta) - x_r^\alpha(t_0,\theta)|\Big\} \\
&\le \max\Big\{\sup_{\alpha\in[0,1]} |x_l^\alpha(t,\theta) - x_l^\alpha(t_0,\theta)|, \sup_{\alpha\in[0,1]} |x_r^\alpha(t,\theta) - x_r^\alpha(t_0,\theta)|\Big\},
\end{aligned}$$

where the last expression in the above inequality tends to zero as $t \to t_0$, since x_l and x_r are continuous on the compact set $[0,1] \times I \times \Theta$. Therefore, $x \in C(I \times \Theta, E_N)$.

We need to show that $x_{n_k}(t,\theta)$ convergence to $x(t,\theta)$ in $C(I \times \Theta, E_N)$, for $t \in I$, a.s. $\theta \in \Theta$.

$$\begin{aligned}
& E\Big(H_1(x_{n_k}, x)\Big) \\
&= E\Big(\sup_{(t,\theta)\in I\times\Theta} D_L\Big(x_{n_k}(t,\theta), x(t,\theta)\Big)\Big) \\
&= E\Big(\sup_{(t,\theta)\in I\times\Theta} \sup_{\alpha\in[0,1]} d_L\Big([x_{n_k}(t,\theta)]^\alpha, [x(t,\theta)]^\alpha\Big)\Big) \\
&= E\Big(\sup_{(t,\theta)\in I\times\Theta} \sup_{\alpha\in[0,1]} d_L\Big([(x_{n_k})_l^\alpha(t,\theta), (x_{n_k})_r^\alpha(t,\theta)], [x_l^\alpha(t,\theta), x_r^\alpha(t,\theta)]\Big)\Big) \\
&= E\Big(\sup_{(t,\theta)\in I\times\Theta} \sup_{\alpha\in[0,1]} \max\Big\{|(x_{n_k})_l^\alpha(t,\theta) - x_l^\alpha(t,\theta)|, |(x_{n_k})_r^\alpha(t,\theta) - x_r^\alpha(t,\theta)|\Big\}\Big) \\
&\le E\Big(\max\Big\{\sup_{(t,\theta)\in I\times\Theta} \sup_{\alpha\in[0,1]} |(x_{n_k})_l^\alpha(t,\theta) - x_l^\alpha(t,\theta)|, \\
&\qquad\qquad \sup_{(t,\theta)\in I\times\Theta} \sup_{\alpha\in[0,1]} |(x_{n_k})_r^\alpha(t,\theta) - x_r^\alpha(t,\theta)|\Big\}\Big) \\
&= E\Big(\max\Big\{||(x_{n_k})_l(t,\theta) - x_l(t,\theta)||_\infty, ||(x_{n_k})_r(t,\theta) - x_r(t,\theta)||_\infty\Big\}\Big).
\end{aligned}$$

Hence, $\max\{||(x_{n_k})_l(t,\theta) - x_l(t,\theta)||_\infty, ||(x_{n_k})_r(t,\theta) - x_r(t,\theta)||_\infty\}$ tends to zero as $k \to \infty$, therefore by Lemma 3.1, $x_{n_k} \to x$ in $C(I \times \Theta, E_N)$. In consequence, B is a relatively compact set.

Definition 3.1 Let $x_0, y_0 \in E_N$ be fuzzy process and the partial ordering for fuzzy process be written as $\le_f$. Then we say that $x_0 \le_f y_0$ if and only if $x_0 \le y_0$.

Definition 3.2 Let $C(t)$ be standard Liu process and the partial ordering for stochastic fuzzy process be written as $\leq_{sf}$. Then we say that $gC(t) \leq_{sf} fC(t)$ if and only if a.s. $g \leq f$, where $g, f \in E_N$ are fuzzy function.

Definition 3.3 Let $x, y \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$. The partial ordering a.s. $x \leq_T y$ is defined by

$$x_0 \leq_f y_0 \text{ and } gC(t) \leq_{sf} fC(t),$$

for $x_0, y_0, g, f \in E_N$, and a standard Liu process $C(t)$, where $x = x_0 + gC(t),\ y = y_0 + fC(t)$.

Definition 3.4 For the partial ordering $\leq_T$, a function $a \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$ is a $\leq_T$-lower solution for equation (6) if

$$\begin{cases} a(t,\theta) \leq_T U(t)x_0 + \int_0^t U(t-s)G(s, a(s,\theta))dC(s),\ t \in [0,T], \\ a(0) \leq_T x_0 \in E_N \end{cases} \tag{8}$$

and a function $b \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$ is a $\leq_T$-upper solution for equation (7) if

$$\begin{cases} b(t,\theta) \geq_T S(t)x_0 + \int_0^t S(t-s)F(s, b(s,\theta))dC(s),\ t \in [0,T], \\ b(0) \geq_T x_0 \in E_N. \end{cases} \tag{9}$$

Assume the following:

(H6) For $x_0 \in E_N$, $\alpha, \beta \in [0,1]$, $\forall \varepsilon > 0$, $d_L([x_0]^\alpha, [x_0]^\beta) \leq \frac{\varepsilon}{2\min\{c,d\}}$.

(H7) For $x \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$, $\alpha,\ \beta \in [0,1]$, $\forall \varepsilon > 0$,

$$d_L([G(t,x)]^\alpha, [G(t,x)]^\beta) \leq \frac{\varepsilon}{2cKT}.$$

(H8) For $x \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$, $\alpha,\ \beta \in [0,1]$, $\forall \varepsilon > 0$,

$$d_L([F(t,x)]^\alpha, [F(t,x)]^\beta) \leq \frac{\varepsilon}{2dKT}.$$

Theorem 3.3 Let $a, b \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$ be, respectively, $\leq_T$-lower and $\leq_T$-upper solutions for equation (1) on $[0,T]$. Hypotheses (H1)-(H8) hold. Then, there exist monotone sequences $\{a_n\} \uparrow \rho,\ \{b_n\} \downarrow \gamma$ in $C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N)$, where ρ, γ are extremal solutions to equation (1) in the stochastic fuzzy functional interval $[a,b] := \{x \in C([0,T] \times (\Theta, \mathcal{P}, C_r), E_N) \mid a \leq_T x \leq_T b \text{ on } [0,T]\}$.

Proof We recall the following equations (10) and (11)

$$\begin{cases} x(t,\theta) = U(t)x_0 + \int_0^t U(t-s)G(s, x(s,\theta))dC(s), \\ x(0) = x_0 \in E_N \end{cases} \tag{10}$$

and

$$\begin{cases} x(t,\theta) = S(t)x_0 + \int_0^t S(t-s)F(s,x(s,\theta))dC(s), \\ x(0) = x_0 \in E_N. \end{cases} \tag{11}$$

By Theorem 3.1, the problems (10) and (11) have unique solutions. Let's $\eta,\ \xi \in [a,b]$ such that a.s. $\eta \leq_T \xi$. If $\eta,\ \xi \in [a,b]$ are solutions for the above problems such that

$$\begin{cases} x_\eta(t,\theta) = U(t)x_0 + \int_0^t U(t-s)G(s,\eta(s,\theta))dC(s), \\ x(0) = x_0 \in E_N, \end{cases} \tag{12}$$

or

$$\begin{cases} x_\eta(t,\theta) = S(t)x_0 + \int_0^t S(t-s)F(s,\eta(s,\theta))dC(s), \\ x(0) = x_0 \in E_N \end{cases} \tag{13}$$

and

$$\begin{cases} x_\xi(t,\theta) = U(t)x_0 + \int_0^t U(t-s)G(s,\xi(s,\theta))dC(s), \\ x(0) = x_0 \in E_N, \end{cases} \tag{14}$$

or

$$\begin{cases} x_\xi(t,\theta) = S(t)x_0 + \int_0^t S(t-s)F(s,\xi(s,\theta))dC(s), \\ x(0) = x_0 \in E_N. \end{cases} \tag{15}$$

Then we can consider the following three case :
(i) For η,ξ in (12) and (14),
(ii) For η,ξ in (13) and (15),
(iii) For η,ξ in (12) and (15).

Let's define

$$\mathcal{B} : [a,b] \to E_N$$
$$\upsilon \to \mathcal{B}\upsilon = x_\upsilon,$$

which satisfy:
(i) $\mathcal{B}([a,b]) \subseteq [a,b]$,
(ii) $\mathcal{B}$ is $\leq_T$-non-decreasing.

We show that $\mathcal{B}$ is $\leq_T$-non-decreasing. The cases (i) and (ii) are clear since definition of lower and upper solutions. So we consider the case (iii). Indeed, for $\eta,\ \xi \in [a,b]$ with a.s. $\eta \leq_T \xi$, we have

$$E(U(t)\eta) \leq E(S(t)\xi),$$
$$E\Big(\int_0^t U(t-s)G(s,\eta(s,\theta))dC(s)\Big) \leq E\Big(\int_0^t S(t-s)F(s,\xi(s,\theta))dC(s)\Big).$$

Then $\mathcal{B}\eta$ and $\mathcal{B}\xi$ are functions in E_N and

$$\begin{aligned} \mathcal{B}\eta(t,\theta) &= U(t)\eta_0 + \int_0^t U(t-s)G(s,\eta(s,\theta))dC(s) \\ &\leq_T S(t)\xi_0 + \int_0^t S(t-s)F(s,\xi(s,\theta))dC(s) = \mathcal{B}\xi(t,\theta),\ t \in [0,T], \\ \mathcal{B}\eta(0) &= x_0 \leq_T \mathcal{B}\xi(0) = x_0, \end{aligned}$$

which implies a.s. $\mathcal{B}\eta \leq_T \mathcal{B}\xi$ on $[0,T]$. Moreover, let $\mathcal{B}(\cdot) \in C([a,b],E_N)$, which satisfies, by the properties of the $\leq_T$-lower solution, upper solution and the partial ordering,

$$\begin{cases} a(t,\theta) \leq_T U(t)x_0 + \int_0^t U(t-s)G(s,a(s,\theta))dC(s),\ t \in [0,T], \\ a(0) \leq_T \mathcal{B}a(0), \end{cases}$$

$$\begin{cases} b(t,\theta) \geq_T S(t)x_0 + \int_0^t S(t-s)F(s,a(s,\theta))dC(s),\ t \in [0,T], \\ b(0) \geq_T \mathcal{B}b(0), \end{cases}$$

then $a \leq_T \mathcal{B}a \leq_T \mathcal{B}b \leq_T b$ on $[0,T]$.

This prove that $\mathcal{B} : [a,b] \to [a,b]$ and $\mathcal{B}$ is non-decreasing. Define the sequences $\{a_n\}, \{b_n\}$ such that $a = a_0,\ b = b_0,\ a_{n+1} = \mathcal{B}a_n,$ and $b_{n+1} = \mathcal{B}b_n$ for $n \in N$. It can be shown that $\{a_n\}$ is non-decreasing, $\{b_n\}$ is non-increasing, and

$$a = a_0 \leq_T a_1 \leq_T \cdots \leq_T a_n \leq_T b_n \leq_T \cdots \leq_T b_1 \leq_T b_0 = b.$$

Note that a_n $(n \in N)$ is the solution to

$$\begin{cases} x(t,\theta) = U(t)x_0 + \int_0^t U(t-s)G(s,a_{n-1}(s,\theta))dC(s),\ t \in [0,T], \\ x(0) = x_0. \end{cases}$$

and b_n $(n \in N)$ is the solution to

$$\begin{cases} x(t,\theta) = S(t)x_0 + \int_0^t S(t-s)F(s,b_{n-1}(s,\theta))dC(s),\ t \in [0,T], \\ x(0) = x_0, \end{cases}$$

We need to check that for $n \in N$, $\{a_n\}$ and $\{b_n\}$ are relatively compact sets in $C([0,T] \times (\Theta,\mathcal{P},C_r),E_N)$, by using Theorem 3.2, the relative compactness of $\{\overline{a_n}\}_l, \{\overline{a_n}\}_r, \{\overline{b_n}\}_l, \{\overline{b_n}\}_r$ in the space $(C([0,T] \times (\Theta,\mathcal{P},C_r),R), ||\cdot||_\infty)$.

For each $n \in N$, we obtain

$$\begin{cases} a_n(t,\theta) = U(t)x_0 + \int_0^t U(t-s)G(s,a_{n-1}(s,\theta))dC(s),\ t \in [0,T], \\ a_n(0) = x_0, \end{cases}$$

$$\begin{cases} b_n(t,\theta) = S(t)x_0 + \int_0^t S(t-s)F(s,b_{n-1}(s,\theta))dC(s),\ t \in [0,T], \\ b_n(0) = x_0. \end{cases}$$

and $a_n(t,\theta),\ b_n(t,\theta)$ are continuous fuzzy numbers, for every $t \in I$, a.s. $\theta \in \Theta$, and $n \in N$. Since function a_n belongs to $[a,b]$, for every $n \in N$, $\{a_n\}$ is bounded. Hence, the set $\{\overline{a_n}\}_l, \{\overline{a_n}\}_r$ are uniformly bounded, where $(\overline{a_n})_l^\alpha(t,\theta) = (a_n)_l^\alpha(t,\theta)$, $(\overline{a_n})_r^\alpha(t,\theta) = (a_n)_r^\alpha(t,\theta)$.

We show the set $\{\{a_n\} \mid n \in N\}$ is uniformly equicontinuous at the variable $\alpha \in [0,1]$. For $\alpha,\ \beta \in [0,1]$ and $n \in N$, take $t \in I$ fixed, a.s. $\theta \in \Theta$, then, by Lemma 2.3 and hypotheses (H6), (H7)

$$\begin{aligned} & d_L\Big([a_n(t,\theta)]^\alpha, [a_n(t,\theta)]^\beta\Big) \\ & \leq d_L\Big(\Big[U(t)x_0 + \int_0^t U(t-s)G(s,a_{n-1}(s,\theta))dC(s)\Big]^\alpha, \end{aligned}$$

$$\Big[U(t)x_0+\int_0^t U(t-s)G(s,a_{n-1}(s,\theta))dC(s)\Big]^{\beta}\Big)$$

$$\le cd_L\Big([x_0]^{\alpha},[x_0]^{\beta}\Big)$$

$$+cd_L\Big(\Big[\int_0^t G(s,a_{n-1}(s,\theta))dC(s)\Big]^{\alpha},\Big[\int_0^t G(s,a_{n-1}(s,\theta))dC(s)\Big]^{\beta}\Big)$$

$$\le cd_L\Big([x_0]^{\alpha},[x_0]^{\beta}\Big)$$

$$+cK\int_0^t d_L\Big(\Big[G(s,a_{n-1}(s,\theta))\Big]^{\alpha},\Big[G(s,a_{n-1}(s,\theta))\Big]^{\beta}\Big)ds$$

$$\le c\Big(\frac{\varepsilon}{2c}\Big)+cK\Big(\int_0^t\frac{\varepsilon}{2cKT}ds\Big).$$

Hence

$$E\Big(H_1(a_n,a_n)\Big)$$

$$=E\Big(\sup_{t\in[0,T]}D_L(a_n(t,\theta),a_n(t,\theta))\Big)$$

$$=E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha,\beta\le 1}d_L\Big([a_n(t,\theta)]^{\alpha},[a_n(t,\theta)]^{\beta}\Big)\Big)$$

$$\le E\Big(c\sup_{t\in[0,T]}\sup_{0<\alpha,\beta\le 1}\Big(\frac{\varepsilon}{2c}\Big)+cK\sup_{t\in[0,T]}\sup_{0<\alpha,\beta\le 1}\Big(\int_0^t\frac{\varepsilon}{2cKT}ds\Big)\Big)$$

$$\le \varepsilon.$$

Thus the set $\{\{a_n\}\mid n\in N\}$ is uniformly equicontinuous at the variable $\alpha\in[0,1]$. And we show the set $\{\{a_n\}\mid n\in N\}$ is uniformly equicontinuous in the variable $t\in[0,T]$. We take $0<t'<t<T$, then by Lemma 2.3 and hypothesis (H2), we get

$$d_L\Big([a_n(t,\theta)]^{\alpha},[a_n(t-t',\theta)]^{\alpha}\Big)$$

$$\le d_L\Big(\Big[\int_0^t U(t-s)G(s,a_{n-1}(s,\theta))dC(s)\Big]^{\alpha},$$

$$\Big[\int_0^{t-t'}U(t-s+s')G(s,a_{n-1}(s-s',\theta))dC(s)\Big]^{\alpha}\Big)$$

$$\le cKd_L\Big(\int_0^t[G(s,a_{n-1}(s,\theta))]^{\alpha}ds,\int_0^{t-t'}[G(s,a_{n-1}(s-s',\theta))]^{\alpha}ds\Big).$$

Thus

$$E\Big(H_1(a_n,a_n)\Big)$$

$$=E\Big(\sup_{t\in[0,T]}D_L(a_n(t,\theta),a_n(t-t',\theta))\Big)$$

$$
\begin{aligned}
&= E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\le 1} d_L\Big([a_n(t,\theta)]^\alpha,[a_n(t-t',\theta)]^\alpha\Big)\Big)\\
&\le E\Big(cK\sup_{t\in[0,T]}\sup_{0<\alpha\le 1} d_L\Big(\int_0^t [G(s,a_{n-1}(s,\theta))]^\alpha ds,\\
&\qquad\qquad \int_0^{t-t'}[G(s,a_{n-1}(s-s',\theta))]^\alpha ds\Big)\Big)\\
&\le E\Big(cm_1K\sup_{t\in[0,T]}\sup_{0<\alpha\le 1} d_L\Big(\int_0^t [a_{n-1}(s,\theta)]^\alpha ds,\int_0^{t-t'}[a_{n-1}(s-s',\theta)]^\alpha ds\Big)\Big)\\
&\le E\Big(cm_1K\sup_{t\in[0,T]} D_L\Big(\int_0^t a_{n-1}(s,\theta)ds,\int_0^{t-t'}a_{n-1}(s-s',\theta)ds\Big)\Big)\\
&\le E\Big(cm_1KTH_1(a_{n-1}(t,\theta),a_{n-1}(t-t',\theta))\Big)\\
&= cm_1KTE\Big(H_1(a_{n-1}(t,\theta),a_{n-1}(t-t',\theta))\Big).
\end{aligned}
$$

Hence, a.s. $\theta\in\Theta$, $H_1(a_{n-1}(t,\theta),a_{n-1}(t-t',\theta))\to 0$ as $t'\to 0$ the set $\{\{a_n\}\mid n\in N\}$ is uniformly equicontinuous in the variable $t\in[0,T]$. The case of equicontinuity from the right is similar.

In consequence, $\{\{a_n\}\mid n\in N\}$ is uniformly equicontinuous at the variable $\alpha\in[0,1]$, $t\in[0,T]$, a.s. $\theta\in\Theta$. This proves that $\{\{a_n\}\mid n\in N\}$ is uniformly equicontinuous in $C([0,T]\times(\Theta,\mathcal{P},C_r),E_N)$. And similarly, it can be shown for $\{b_n\}$. Hence $\{a_n\},\{b_n\}\in B$. Next, we have to prove that the following ρ, γ are extremal solutions to equation (1), that is,

$$
\begin{cases}
\rho(t,\theta)=U(t)\rho_0+\int_0^t U(t-s)G(s,\rho(s,\theta))dC(s),\ t\in[0,T],\\
\rho(0)=\rho_0,
\end{cases}
$$

and

$$
\begin{cases}
\gamma(t,\theta)=S(t)\gamma_0+\int_0^t S(t-s)F(s,\gamma(s,\theta))dC(s),\ t\in[0,T],\\
\gamma(0)=\gamma_0.
\end{cases}
$$

The above problem has a solution $\rho\in C([0,T]\times(\Theta,\mathcal{P},C_r),E_N)$. Since $a\le_T a_n\le_T b$ on $C([0,T]\times(\Theta,\mathcal{P},C_r),E_N)$, $a\le_T\rho\le_T b$ on $C([0,T]\times(\Theta,\mathcal{P},C_r),E_N)$. By Lemma 2.3 and hypothesis (H2), we get

$$
\begin{aligned}
&d_L\Big([a_n(t,\theta)]^\alpha,[\rho(t,\theta)]^\alpha\Big)\\
&\le d_L\Big(\Big[\int_0^t U(t-s)G(s,a_{n-1}(s,\theta))dC(s)\Big]^\alpha,\\
&\qquad\qquad \Big[\int_0^t U(t-s)G(s,\rho(s,\theta))dC(s)\Big]^\alpha\Big)\\
&\le cK\int_0^t d_L\Big([G(s,a_{n-1}(s,\theta))]^\alpha,[G(s,\rho(s,\theta))]^\alpha\Big)ds
\end{aligned}
$$

$$\leq cm_1K\int_0^t d_L\Big([a_{n-1}(s,\theta)]^\alpha,[\rho(s,\theta)]^\alpha\Big)ds.$$

Hence

$$\begin{aligned}
&E\Big(H_1(a_n,\rho)\Big)\\
&=E\Big(\sup_{t\in[0,T]}D_L\Big(a_n(t,\theta),\rho(t,\theta)\Big)\Big)\\
&=E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\leq1}d_L\Big([a_n(t,\theta)]^\alpha,[\rho(t,\theta)]^\alpha\Big)\Big)\\
&\leq E\Big(cm_1K\sup_{t\in[0,T]}\sup_{0<\alpha\leq1}\int_0^t d_L\Big([a_{n-1}(s,\theta)]^\alpha,[\rho(s,\theta)]^\alpha\Big)ds\Big)\\
&\leq E\Big(cm_1KH_1(a_{n-1}(t,\theta),\rho(t,\theta))\Big)\\
&=cm_1KE\Big(H_1(a_{n-1}(t,\theta),\rho(t,\theta))\Big).
\end{aligned}$$

Therefore $H_1(a_{n-1}(t,\theta),\rho(t,\theta))\to 0$ as $n\to+\infty$, a.s. $\theta\in\Theta$. Thus we obtain that ρ is a extremal solution to equation (1). Also, the above problem has a solution $\gamma\in C([0,T]\times(\Theta,\mathcal{P},C_r),E_N)$. Since $a\leq_T b_n\leq_T b$ on $C([0,T]\times(\Theta,\mathcal{P},C_r),E_N)$, $a\leq_T\gamma\leq_T b$ on $C([0,T]\times(\Theta,\mathcal{P},C_r),E_N)$. By Lemma 2.3 and hypothesis (H2), we have

$$\begin{aligned}
&d_L\Big([b_n(t,\theta)]^\alpha,[\gamma(t,\theta)]^\alpha\Big)\\
&\leq d_L\Big(\Big[\int_0^t S(t-s)F(s,b_{n-1}(s,\theta))dC(s)\Big]^\alpha,\\
&\qquad\qquad\Big[\int_0^t S(t-s)F(s,\gamma(s,\theta))dC(s)\Big]^\alpha\Big)\\
&\leq dK\int_0^t d_L\Big([F(s,b_{n-1}(s,\theta))]^\alpha,[F(s,\gamma(s,\theta))]^\alpha\Big)ds\\
&\leq dm_2K\int_0^t d_L\Big([b_{n-1}(s,\theta)]^\alpha,[\gamma(s,\theta)]^\alpha\Big)ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&E\Big(H_1(b_n,\gamma)\Big)\\
&=E\Big(\sup_{t\in[0,T]}D_L\Big(b_n(t,\theta),\gamma(t,\theta)\Big)\Big)\\
&=E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\leq1}d_L\Big([b_n(t,\theta)]^\alpha,[\gamma(t,\theta)]^\alpha\Big)\Big)\\
&\leq E\Big(dm_2K\sup_{t\in[0,T]}\sup_{0<\alpha\leq1}\int_0^t d_L\Big([b_{n-1}(s,\theta)]^\alpha,[\gamma(s,\theta)]^\alpha\Big)ds\Big)
\end{aligned}$$

$$\leq E\Big(dm_2KH_1(b_{n-1}(t,\theta),\gamma(t,\theta))\Big)$$
$$= dm_2KE\Big(H_1(b_{n-1}(t,\theta),\gamma(t,\theta))\Big).$$

Therefore $H_1(b_{n-1}(t,\theta),\gamma(t,\theta)) \to 0$ as $n \to +\infty$, a.s. $\theta \in \Theta$. Thus we obtain that γ is an extremal solution to equation (1). Finally, if x is a solution to equation (1) such that $a \leq_T x \leq_T b$, since $\mathcal{B}$ is nondecreasing, we obtain

$$a_n = \mathcal{B}^n a \leq_T \mathcal{B}^n x = x \leq_T \mathcal{B}^n b = b_n$$

and then

$$\rho \leq_T x \leq_T \gamma.$$

In conclusion, x exists between ρ and γ in credibility space.

4. Example

We consider the following fuzzy differential equation driven by Liu process

$$\begin{cases} dx(t,\theta) = f(t,x(t,\theta))dC(t), \ t \in [0,T], \\ x(0) = x_0 \in E_N, \end{cases} \tag{16}$$

where $T > 0$, E_N is the set of all upper semi-continuously convex fuzzy numbers on R, $(\Theta,\mathcal{P},Cr)$ is a credibility space, $x : [0,T] \times (\Theta,\mathcal{P},Cr) \to E_N$ is a fuzzy process, $f : [0,T] \times E_N \to E_N$ is a regular fuzzy function, $C(t)$ is a standard Liu process, $x_0 \in E_N$ is an initial value.

Let $f(t,x(t,\theta)) = \widetilde{3}tx^2(t,\theta), \ t \in [0,T]$.

For a positive constant M, let

$$\begin{cases} dx(t,\theta) = Mx(t,\theta)dt + \widetilde{3}tx^2(t,\theta)dC(t) - Mx(t,\theta)dt, \ t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{17}$$

Then let us define the following integral equation for equation (16)

$$\begin{cases} x(t,\theta) = S(t)x_0 + \int_0^t S(t-s)\widetilde{3}sx^2(s,\theta)dC(s) \\ \qquad\qquad -M\int_0^t S(t-s)x(s,\theta)ds, t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{18}$$

Put

$$\int_0^t S(t-s)\widetilde{3}sx^2(s,\theta)dC(s) - M\int_0^t S(t-s)x(s,\theta)ds$$
$$\leq \int_0^t S(t-s)\widetilde{3}sx^2(s,\theta)dC(s) - \frac{M}{K}\int_0^t S(t-s)x(s,\theta)dC(s)$$
$$= \int_0^t S(t-s)F(s,x(s,\theta))dC(s)$$

and $S(t) = e^{Mt}$.

Then the balance equation becomes

$$\begin{cases} x(t,\theta) = S(t)x_0 + \int_0^t S(t-s)F(s,x(s,\theta))dC(s), \ t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{19}$$

The α-level sets of fuzzy numbers are the following:

$[\widetilde{0}]^\alpha = [\alpha-1, 1-\alpha]$, $[\widetilde{2}]^\alpha = [\alpha+1, 3-\alpha]$ for all $\alpha \in [0,1], M = 1$. Then α-level sets of $F(t,x(t,\theta))$ is

$$[F(t,x(t,\theta))]^\alpha = t[(\alpha+1)(x_l^{2\alpha}(t,\theta) - x_l^\alpha(t,\theta)), (3-\alpha)(x_r^{2\alpha}(t,\theta) - x_r^\alpha(t,\theta))].$$

Further, we have

$$\begin{aligned}
&d_L\Big([F(t,x(t,\theta))]^\alpha, [F(t,y(t,\theta))]^\alpha\Big) \\
&= d_L\Big(t[(\alpha+1)(x_l^{2\alpha}(t,\theta) - x_l^\alpha(t,\theta)), (3-\alpha)(x_r^{2\alpha}(t,\theta) - x_r^\alpha(t,\theta))], \\
&\qquad t[(\alpha+1)(y_l^{2\alpha}(t,\theta) - y_l^\alpha(t,\theta)), (3-\alpha)(y_r^{2\alpha}(t,\theta) - y_r^\alpha(t,\theta))]\Big) \\
&= t\max\Big\{(\alpha+1)|(x_l^{2\alpha}(t,\theta) - x_l^\alpha(t,\theta)) - (y_l^{2\alpha}(t,\theta) - y_l^\alpha(t,\theta))|, \\
&\qquad (3-\alpha)|(x_r^{2\alpha}(t,\theta) - x_r^\alpha(t,\theta)) - (y_r^{2\alpha}(t,\theta) - y_r^\alpha(t,\theta))|\Big\} \\
&\le 3T\max\Big\{|x_l^\alpha(t,\theta) + y_l^\alpha(t,\theta) - 1|, |x_r^\alpha(t,\theta) + y_r^\alpha(t,\theta) - 1|\Big\} \\
&\qquad \times\Big\{|x_l^\alpha(t,\theta) - y_l^\alpha(t,\theta))|, |x_r^\alpha(t,\theta) - y_r^\alpha(t,\theta)|\Big\} \\
&= m_2 d_L([x(t,\theta)]^\alpha, [y(t,\theta)]^\alpha),
\end{aligned}$$

where $m_2 = 3T\max\Big\{|x_l^\alpha(t,\theta) + y_l^\alpha(t,\theta) - 1|, |x_r^\alpha(t,\theta) + y_r^\alpha(t,\theta) - 1|\Big\}$ satisfies the inequality in hypothesis (H2). Since $S(t)$ is continuous we know that hypothesis (H1) holds. Thus all the conditions stated in Theorem 3.1 are satisfied. Next, we show that hypothesis (H8) satisfies. For $\alpha, \beta \in [0,1]$ and given $\varepsilon = 6dKT^2\max\Big\{|(x_l^\alpha(t,\theta)+x_l^\beta(t,\theta)-1)(x_l^\alpha(t,\theta)-x_l^\beta(t,\theta))|, |(x_r^\alpha(t,\theta)+x_r^\beta(t,\theta)-1)(x_r^\alpha(t,\theta) - x_r^\beta(t,\theta))|\Big\} > 0$, we have

$$\begin{aligned}
&d_L\Big([F(t,x(t,\theta))]^\alpha, [F(t,x(t,\theta))]^\beta\Big) \\
&= d_L\Big(t[(\alpha+1)(x_l^{2\alpha}(t,\theta) - x_l^\alpha(t,\theta)), (3-\alpha)(x_r^{2\alpha}(t,\theta) - x_r^\alpha(t,\theta))], \\
&\qquad t[(\beta+1)(x_l^{2\beta}(t,\theta) - x_l^\beta(t,\theta)), (3-\beta)(x_r^{2\beta}(t,\theta) - x_r^\beta(t,\theta))]\Big) \\
&= t\max\Big\{|(\alpha+1)(x_l^{2\alpha}(t,\theta) - x_l^\alpha(t,\theta)) - (\beta+1)(x_l^{2\beta}(t,\theta) - x_l^\beta(t,\theta))|, \\
&\qquad |(3-\alpha)(x_r^{2\alpha}(t,\theta) - x_r^\alpha(t,\theta)) - (3-\beta)(x_r^{2\beta}(t,\theta) - x_r^\beta(t,\theta))|\Big\} \\
&\le 3T\max\Big\{|(x_l^\alpha(t,\theta) + x_l^\beta(t,\theta) - 1)(x_l^\alpha(t,\theta) - x_l^\beta(t,\theta))|,
\end{aligned}$$

$$|(x_r^\alpha(t,\theta) + x_r^\beta(t,\theta) - 1)(x_r^\alpha(t,\theta) - x_r^\beta(t,\theta))|\Big\}$$
$$= \frac{\varepsilon}{2dKT}.$$

Hence all the conditions stated Theorem 3.3 are satisfied. Thus the problem (16) has extremal solutions.

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