J. Appl. Math. & Informatics Vol. **39**(2021), No. 3 - 4, pp. 437 - 442
https://doi.org/10.14317/jami.2021.437

EXPANSION THEORY FOR THE TWO-SIDED BEST SIMULTANEOUS APPROXIMATIONS

HYANG JOO RHEE

Abstract. In this paper, we study the characterizations of two-sided best simultaneous approximations for $\ell-$tuple subset from a closed convex subset of $\mathbb{R}^m$ with $\ell_1^m(\mathbf{w})$-norm. *Main fact* is, k^* is a two-sided best simultaneous approximation to F from K if and only if there exist $f_1, \ldots, f_p$ in F, for any $k \in K$

$$|\sum_{i=1}^{m} sgn(f_{ji} - k_i^*)k_i w_i| \leq \sum_{i \in Z(f_j - k^*)} |k_i| w_i$$

for each $j = 1, \ldots, p$ and $\mathbf{w} \in W$.

AMS Mathematics Subject Classification : 41A28, 41A65, 54C60.
Key words and phrases : Best simultaneous approximation, two-sided best simultaneous approximation.

1. Introduction

Let X be a normed linear space and K be a closed convex subset of an n-dimensional subspace of X. For any subset F of X, we define

$$d(F, K) = \inf_{k \in K} \sup_{f \in F} ||f - k||$$

and the elements in K which attain the above infinimum are called the best simultaneous approximations for F from K.

We first fix some notation. B is a set, Σ a σ-field of subsets of B, and μ a positive measure defined on Σ. That is, $\mu(E) \geq 0$ for all $E \in \Sigma$. By $L^p(B, \mu)$, $1 \leq p < \infty$, we denote the set of all real valued μ-measurable functions f defined on B for which $|f|^p$ is μ-integrable over B. We consider two functions of $L^p(B, \mu)$ as equivalent if they are equal μ almost everywhere. Under this

Received April 9, 2021. Revised April 21, 2021. Accepted April 22, 2021.

convention $L^p(B,\mu)$ with norm

$$||f||_p = (\int_B |f(x)|^p d\mu(x))^{\frac{1}{p}}$$

is a normed linear space and is a Banach space.

$L^\infty(B,\mu)$ is defined analogously with norm

$$||f||_\infty = ess \sup_{x\in B} |f(x)|$$

where the essential supremum, $L^\infty(B,\mu)$ is also a Banach space, in which case we may identify the dual of $L^1(B,\mu)$ with $L^\infty(B,\mu)$,

$$L^1(B,\mu)^* = L^\infty(B,\mu).$$

2. Characterization

For each $f \in L^1(B,\mu)$, we define its zero set

$$Z(f) = \{x | f(x) = 0\}$$

and $N(f) = B \backslash Z(f)$. It is note that $Z(f)$ is μ-measurable, and for $f \in L^1(B,\mu)$, set

$$sgn(f(x)) = \begin{cases} 1, & f(x) > 0 \\ 0, & f(x) = 0 \\ -1, & f(x) < 0. \end{cases}$$

Let M be a subspace of $L^1(B,\mu)$. We first present a condition for characterizing best $L^1(B,\mu)$ approximations from K.

Theorem 2.1. [1]. *Let K be a subspace of $L^1(B,\mu)$ and $f \in L^1(B,\mu)\backslash\overline{K}$. Then k^* is a best $L^1(B,\mu)$ approximation to f from K if and only if*

$$|\int_B sgn(f-k^*)k d\mu| \le \int_{Z(f-k^*)} |k| d\mu$$

for all $k \in K$.

The above theorem is considered in the norm was given by

$$||f||_1 = \int_B |f| d\mu$$

with μ a non-atomic positive measure, i.e., if to each $E \in \Sigma$ with $\mu(E) > 0$, and each $\lambda \in (0,1)$, there exists an $E_\lambda \subset E$ ($E_\lambda \in \Sigma$) for which $\mu(E_\lambda) = \lambda\mu(E)$. Now we consider measures μ which are purely atomic. In fact we shall assume that μ has exactly m atoms. This corresponds to approximation in the vector space $\mathbb{R}^m$. The previous notation is now both cumbersome and inappropriate. We therefore redefine our problem in this new setting, introducing some new notation.

Let

$$W = \{\mathbf{w} : \mathbf{w} = (w_1,\ldots,w_m), w_i > 0, i = 1,\ldots,m\}.$$

We say that $\mathbf{w} \in W$ is a weight. On $\mathbb{R}^m$, we define the $\ell_1^m(\mathbf{w})$-norm given by

$$||\mathbf{x}||_{\mathbf{w}} = \sum_{i=1}^{m} |x_i| w_i$$

where $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$.

In this article, we study the theoretical problem of both two-sided $\ell_1^m(\mathbf{w})$-approximation and two-sided simultaneous $\ell_1^m(\mathbf{w})$-approximation from finite-dimensional subspace.

Let U be an n-dimensional subspace of $\mathbb{R}^m$. Given $\mathbf{w} \in W$ and $\mathbf{b} \in \mathbb{R}^m$, we consider

$$min\{||\mathbf{b} - \mathbf{u}||_{\mathbf{w}} : \mathbf{u} \in U\}$$

and the elements in U which attain the above minimum are called the two-sided $\ell_1^m(\mathbf{w})$-approximation for $\mathbf{b}$ from U. Since U is a finite-dimensional subspace of a normed linear space $\mathbb{R}^m$, there exists a $\mathbf{u}^* \in U$ for which

$$||\mathbf{b} - \mathbf{u}^*||_{\mathbf{w}} \leq ||\mathbf{b} - \mathbf{u}||_{\mathbf{w}}$$

for all $\mathbf{u} \in U$. The characterization of the best approximations follows from Theorem 2.1, which we now restate, translated to our present context.

Theorem 2.2. *Given $\mathbf{b} \in \mathbb{R}^m, \mathbf{w} \in W$, and a finite-dimensional subspace U of $\mathbb{R}^m$, then there exists a $\mathbf{u}^* \in U$ is a two-sided $\ell_1^m(\mathbf{w})$-approximation to $\mathbf{b}$ from U if and only if*

$$|\sum_{i=1}^{m} sgn(b_i - u_i^*) u_i w_i| \leq \sum_{i \in Z(\mathbf{b}-\mathbf{u}^*)} |u_i| w_i$$

for all $\mathbf{u} = (u_1, \ldots, u_m) \in U$.

We may assume that X be a normed linear space, let K is a closed convex subset of an n-dimensional subspace of X. Then the condition of Theorem 2.1 can be replaced by another form as follows.

Theorem 2.3. [3]. *Suppose that K is a closed convex subset of an n-dimensional subspace of X and let F be a compact subset of X. Then $k_0 \in K$ is a best simultaneous approximation for F from K if and only if there exist $f_1^*, \ldots, f_p^* \in F$ and positive real numbers $\lambda_1^*, \ldots, \lambda_p^*$ with $\sum_{i=1}^{p} \lambda_i^* = 1$ satisfying*

(1) $||f_i^* - k_0|| = \max_F ||f - k_0||, \ i = 1, \ldots, p,$

(2) $\sum_{i=1}^{p} \lambda_i^* ||f_i^* - k_0|| \leq \sum_{i=1}^{p} \lambda_i^* ||f_i^* - k||$ *for any* $k \in K$,

for some $1 \leq p \leq n+1$.

Since each finite set is compact, we obtain the following corollary.

Corollary 2.4. *Suppose that K is a closed convex subset of an n-dimensional subspace of X and let F be a singleton subset of X. Then $k_0 \in K$ is a best simultaneous approximation for F from K if and only if $k_0 \in K$ is satisfying*

(1) $||F - k_0|| = d(F, K)$

(2) $d(F, K) \leq ||F - k||$ *for any* $k \in K$.

In the above corollary, let the normed linear space X be replaced by $L^1(B,\mu)$. Consequently, we obtain the Theorem 2.1.

Corollary 2.5. *Suppose that K is a closed convex subset of an n-dimensional subspace of $L^1(B,\mu)$ and let F be a singleton subset of $L^1(B,\mu)$. Then $k_0 \in K$ is satisfying*

(1) $||F - k_0|| = d(F,K)$

(2) $d(F,K) \leq ||F-k||$ *for any* $k \in K$

if and only if

$$|\int_B sgn(F-k^*)kd\mu| \leq \int_{Z(F-k^*)} |k|d\mu$$

for all $k \in K$.

Specially, let the normed linear space X be replaced by $\mathbb{R}^m$ with $\ell_1^m(\mathbf{w})$. Moreover, we obtain the Theorem 2.2.

Corollary 2.6. *Suppose that K is a closed convex subset of $\mathbb{R}^m$ with $\ell_1^m(\mathbf{w})$-norm and f in $\mathbb{R}^m$. Then $k^* \in K$ is satisfying $||f-k^*|| = d(f,K)$ and $d(f,K) \leq ||f-k||$ for any $k \in K$ if and only if*

$$|\sum_{i=1}^{m} sgn(f_i - k_i^*)k_i w_i| \leq \sum_{i \in Z(f-k^*)} |k_i| w_i$$

for all $k = (k_1, \ldots, k_m) \in K$ and $\mathbf{w} \in W$.

Proof. Suppose that $k \in K$ is satisfying $||f-k^*||_1 = d(f,K)$ and $d(f,K) \leq ||f-k||_1$ for all $k \in K$. Then

$$\begin{aligned} ||f-k^*||_1 &= \textstyle\sum_{i=1}^m |f_i - k_i^*| w_i \\ &= \textstyle\sum_{i=1}^m sgn(f_i-k_i^*)(f_i-k_i^*)w_i \\ &= \textstyle\sum_{i=1}^m sgn(f_i-k_i^*)(f_i-k_i)w_i + \sum_{i=1}^m sgn(f_i-k_i^*)(k_i-k_i^*)w_i \\ &\leq \textstyle\sum_{N(f-k^*)} |f_i-k_i|w_i + \sum_{Z(f-k^*)} |k_i-k_i^*|w_i \\ &= \textstyle\sum_{N(f-k^*)} |f_i-k_i|w_i + \sum_{Z(f-k^*)} |f_i-k_i|w_i \\ &= ||f-k||_1 \end{aligned}$$

for any $k \in K$ and $\mathbf{w} \in W$. □

In corroboration of this theory, let F be a ℓ-tuple subset of X. The main theorem of this article is the following.

Theorem 2.7. *Suppose that F be a ℓ-tuple subset of $\mathbb{R}^m$ and K is a closed convex subset of $\mathbb{R}^m$ with $\ell_1^m(\mathbf{w})$-norm. Then k^* is a two-sided best simultaneous approximation to F from K if and only if there exist $f_1, \ldots, f_p$ in F, for any $k \in K$*

$$|\sum_{i=1}^{m} sgn(f_{ji} - k_i^*)k_i w_i| \leq \sum_{i \in Z(f_j-k^*)} |k_i| w_i$$

for each $j = 1, \ldots, p$ and $\mathbf{w} \in W$.

Proof. By Theorem 2.3, since $k^* \in K$ is a best simultaneous approximation for F from K if and only if there exist $f_1, \ldots, f_p \in F$ and positive real numbers $\lambda_1^*, \ldots, \lambda_p^*$ with $\sum_{i=1}^p \lambda_i^* = 1$ satisfying $||f_i - k^*||_1 = d(F, K)$ and $d(F, K) \leq \sum_{i=1}^p \lambda_i^* ||f_i - k||_1$ for any $k \in K$. By the hypothesis, for each $j = 1, \ldots, p$, $||f_j - k^*||_1 = d(F, K)$,

$$\begin{aligned} ||f_j - k^*||_1 &= \textstyle\sum_{i=1}^m |f_{ji} - k_i^*| w_i \\ &= \textstyle\sum_{i=1}^m sgn(f_{ji} - k_i^*)(f_{ji} - k_i^*) w_i \\ &= \textstyle\sum_{i=1}^m sgn(f_{ji} - k_i^*)(f_{ji} - k_i) w_i + \sum_{i=1}^m sgn(f_{ji} - k_i^*)(k_i - k_i^*) w_i \\ &\leq \textstyle\sum_{N(f_j - k^*)} |f_{ji} - k_i| w_i + \sum_{Z(f_j - k^*)} |k_i - k_i^*| w_i \\ &= \textstyle\sum_{N(f_j - k^*)} |f_{ji} - k_i| w_i + \sum_{Z(f_j - k^*)} |f_{ji} - k_i| w_i \\ &= ||f_j - k||_1 \end{aligned}$$

for any $k \in K$ and $\mathbf{w} \in W$. Hence

$$|\sum_{i=1}^m sgn(f_{ji} - k_i^*) k_i w_i| \leq \sum_{i \in Z(f_j - k^*)} |k_i| w_i$$

for each $j = 1, \ldots, p$

Conversely if it satisfied the necessary conditions, then for each $j = 1, \ldots, p$, $||f_j - k^*||_1 \leq ||f_j - k||_1$ for any $k \in K$ from the above inequality. Then

$$\sup_{f \in F} ||f - k^*||_1 \leq \inf_{k \in K} \sup_{f \in F} ||f - k||_1 = d(F, K).$$

Hence k^* is a two-sided best simultaneous approximation to F from K. □

The case $C_1(X)$ is not a Banach space. It is however a dense linear subspace of $L_1(X, \mu)$. Suppose that F is a compact subset of $C_1(X)$ and S is a finite-dimensional subspace of $C_1(X)$, throughout this chapter. Let

$$S(F) = \bigcap_{f \in F} S(F) = \bigcap_{f \in F} \{s^* \in S | s^* \leq f\}.$$

If there exists a function $\tilde{s} \in S(F)$ such that

$$\max_{f \in F} ||f - \tilde{s}||_1 = d(F, S(F)) = \inf_{s \in S(F)} \sup_{f \in F} ||f - s||,$$

it is called a one-sided best simultaneous L_1-approximation for F. Moreover, if F consists of $\ell-$elements only, then $\tilde{s}$ is called a one-sided best $\ell-$simultaneous L_1-approximation for F[4]. Particularly, if ℓ=1, then $\tilde{s}$ is called a one-sided best L_1-approximation for F[1].

We will study exact conditions on a finite-dimensional subspace S of $C_1(X)$ which imply that there exists a one-sided best simultaneous L_1-approximation for each compact subset $F \subset C_1(X)$ from $S(F)$ and we find the characterizations of the one-sided best simultaneous L_1-approximation. Moreover, we have a necessary and sufficient conditions on a subspace S of $C_1(X)$ in order that for each compact set F, there exists a unique one-sided best simultaneous L_1-approximation from $S(F)$.

The motivation is the one-sided best approximation of an element, which has been studied by R. Bojanic, R. DeVore(1966), H. Strauss(1982), G. Nürnberger (1985), A. Pinkus and V. Totik(1986). They proved, by looking at Gaussian-type quadrature formula, the existence of $f \in C[a,b]$ with more than the best one-sided L_1-approximation from π_n. They also showed, for f continuous on $[a,b]$ and differentiable on (a,b), the uniqueness of the best one-sided L_1-approximation from π_n. DeVore essentially generalized these results to T- and ET_2-systems [1].

Very little else was then done on the general theory except as it pertained to Gaussian-type quadrature formula and f in the convexity cone of a WT-system [1]. In Pinkus are to be found extensions of the first two above-mentioned results of Bojanic, DeVore to splines with simple fixed knots. It is only in this decade that a more systematic study of this theory has begun.

References

1. Allan M. Pinkus, *On L^1-approximation*, Cambridge University Press, 1988.
2. K. Fan, *Minimax theorem*, Proc. Nat. Acad. Soc. U.S.A. **39** (1953), 42-47.
3. S.H. Park and H.J. Rhee, *Best simultaneous approximations from a convex subset*, Bull. Korean Math. Soc. **33** (1996), 193-204.
4. S.H. Park and H.J. Rhee, *One-sided best simultaneous L_1-approximation*, Jour. Korean Math. Soc. **33** (1996), 155-167.
5. S.H. Park and H.J. Rhee, *One-sided best simultaneous L_1-approximation for a compact set*, Bull. Korean Math. Soc. **35** (1998), 127-140.
6. S. Tanimoto, *Uniform approximation and a generalized minimax theorem*, J. Approx. Theory **45** (1985), 1-10.
7. S. Tanimoto, *A characterization of best simultaneous approximations*, J. Approx. Theory **59** (1989), 359-361.

Hyang Joo Rhee received M.Sc. and Ph.D. at Sogang University at Seoul, Korea. Since 1996 she has been at Duksung Women's University. Her research interests include approximation and optimization.

Department of Mathematics, College of Natural Sciences, Duksung Women's University, Seoul 01369, Korea.

e-mail: rhj@duksung.ac.kr