J. Appl. Math. & Informatics Vol. **39***(2021), No. 3 - 4, pp. 327 - 338*
https://doi.org/10.14317/jami.2021.327

RECURRENT STRUCTURE JACOBI OPERATOR OF REAL HYPERSURFACES IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIANS†

IMSOON JEONG AND CHANGHWA WOO*

Abstract. In this paper, we have introduced a new notion of recurrent structure Jacobi of real hypersurfaces in complex hyperbolic two-plane Grassmannians $G_2^*(\mathbb{C}^{m+2})$. Next, we show a non-existence property of real hypersurfaces in $G_2^*(\mathbb{C}^{m+2})$ satisfying such a curvature condition.

AMS Mathematics Subject Classification : 53C40, 53C15.
Key words and phrases : Hopf hypersurfaces, complex hyperbolic two-plane Grassmannians, Recurrent structure Jacobi operator.

Introduction

It is one of the main topics in submanifold geometry to investigate immersed real hypersurfaces of homogeneous type in Hermitian symmetric spaces of rank 2 (HSS2) with certain geometric conditions. Understanding and classifying real hypersurfaces in HSS2 is one of important problems in differential geometry. One of these spaces is the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2{\cdot}U_m)$ defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Another one is the complex hyperbolic two-plane Grassmannian $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2{\cdot}U_m)$ defined by the set of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$.

These are typical examples of HSS2. Characterizing typical model spaces of real hypersurfaces under certain geometric conditions is one of our main interests in the classification theory in $G_2(\mathbb{C}^{m+2})$ or $SU_{2,m}/S(U_2{\cdot}U_m)$.

Our recent interest is the study by applying geometric conditions used in submanifolds in $G_2(\mathbb{C}^{m+2})$ to submanifolds in $SU_{2,m}/S(U_2{\cdot}U_m)$.

Received February 23, 2021. Revised April 21, 2021. Accepted April 22, 2021. * Corresponding author.
†This work was supported by a Research Grant of Pukyong National University(2019).

$G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2{\cdot}U_m)$ has compact transitive group SU_{2+m}, however $SU_{2,m}/S(U_2{\cdot}U_m)$ has noncompact indefinite transitive group $SU_{2,m}$. This distinction gives various remarkable results.

The complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$ is the unique noncompact, irreducible, Kähler and quaternionic Kähler manifold which is not a hyperkähler manifold.

Let M be a real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$. Let N be a local unit normal vector field on M. Since the complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$ has the Kähler structure J, we may define a *Reeb vector field* $\xi = -JN$ and a 1-dimensional distribution $\mathcal{C}^\perp = \mathrm{Span}\{\,\xi\}$.

Let $\mathcal{C}$ be the orthogonal complement of distribution $\mathcal{C}^\perp$ in T_pM at $p \in M$. It is the complex maximal subbundle of T_pM. Thus the tangent space of M consists of the direct sum of $\mathcal{C}$ and $\mathcal{C}^\perp$ as follows: $T_pM = \mathcal{C} \oplus \mathcal{C}^\perp$. The real hypersurface M is said to be *Hopf* if $A\mathcal{C} \subset \mathcal{C}$, or equivalently, the Reeb vector field ξ is principal with principal curvature $\alpha = g(A\xi, \xi)$, where g denotes the metric. In this case, the principal curvature α is said to be a *Reeb curvature* of M.

From the quaternionic Kähler structure $\mathfrak{J} = \mathrm{span}\{J_1, J_2, J_3\}$ of the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, there naturally exist *almost contact 3-structure* vector fields $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Let $\mathcal{Q}^\perp = \mathrm{Span}\{\,\xi_1, \xi_2, \xi_3\}$. It is a 3-dimensional distribution in the tangent space T_pM of M at $p \in M$. In addition, $\mathcal{Q}$ stands for the orthogonal complement of $\mathcal{Q}^\perp$ in T_pM. It is the quaternionic maximal subbundle of T_pM. Thus the tangent space of M can be splitted into $\mathcal{Q}$ and $\mathcal{Q}^\perp$ as follows: $T_pM = \mathcal{Q} \oplus \mathcal{Q}^\perp$.

Thus, we have considered two natural geometric conditions for real hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$ such that the subbundles $\mathcal{C}$ and $\mathcal{Q}$ of TM are both invariant under the shape operator. By using these geometric conditions, we will use the results in Berndt and Suh [1].

In this paper, we take the notion of *recurrent* structure Jacobi operator which is more general than *parallel* structure Jacobi operator. Actually, in [7], a nonzero tensor field K of type (r,s) on M is said to be *recurrent* if there exists a 1-form α such that $\nabla K = K \otimes \alpha$. Specifically, recurrent tensor fields can be applied in the problem of space classification of complex projective space $\mathbb{C}P^n$.

As mentioned above, since the structure Jacobi operator is (1,1)-type tensor on M. we consider the recurrent structure Jacobi operator given by Pérez and Santos [6] proved that there exist no real hypersurfaces with recurrent structure Jacobi operator in $(\nabla_X R_\xi)$(Y)=ω(X)$R_\xi(Y)$. for a certain 1-form ω on M. Using this notion, Pérez and Santos [6] proved that there exist no real hypersurfaces with recurrent structure Jacobi operator in complex projective space $\mathbb{C}P^n$, $n \geq 3$.

Now let us consider recurrent structure Jacobi operator defined by $(\nabla_X R_\xi)Y = \beta(X)R_\xi Y$ for any tangent vector fields X, Y to M and an 1-form β to TM. This notion is weaker than parallel structure Jacobi operator mentioned above. Then in this paper we give a non-existence theorem for Hopf hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$ with recurrent structure Jacobi operator as follows:

Main Theorem 1. There do not exist any connected Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$, $m\geq 3$, with recurrent structure Jacobi operator.

As mentioned above, the notion of recurrent is a kind of weaker condition of parallelism and can be regarded as the symmetric tensor of a Riemannian manifold. It means that if the symmetric tensor T is parallel, that is, $\nabla T = 0$, then T naturally becomes recurrent. If we apply such a relation to the structure Jacobi operator R_ξ for a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, we can give the following result from our Main Theorem.

Corollary. There do not exist any Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$, $m\geq 3$, with parallel structure Jacobi operator.

1. Key lemma

The structure Jacobi operator R_ξ of M is defined by $R_\xi X = R(X,\xi)\xi$ for any tangent vector $X \in T_pM$, $p \in M$ (see [4]).

Then for any tangent vector field X on M in $SU_{2,m}/S(U_2{\cdot}U_m)$, we calculate the structure Jacobi operator R_ξ

$$\begin{aligned} R_\xi(X) = \frac{1}{2}\Big[-X + \eta(X)\xi + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \\ + 3\eta_\nu(\phi X)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\big\}\Big] + \alpha AX - \eta(AX)A\xi, \end{aligned} \tag{3.1}$$

where α denotes the Reeb curvature defined by $g(A\xi,\xi)$.
Then the derivative of the structure Jacobi operator R_ξ is given by

$$\begin{aligned} &(\nabla_X R_\xi)Y \\ &= \frac{1}{2}g(\phi AX,Y)\xi + \frac{1}{2}\eta(Y)\phi AX \\ &\quad + \frac{1}{2}\sum_{\nu=1}^{3}\Bigg[g(\phi_\nu AX,Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\ &\quad + 3\Big\{g(\phi_\nu AX,\phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi \\ &\quad - \eta_\nu(\phi Y)\eta(AX)\xi_\nu + \eta_\nu(\phi Y)\phi_\nu\phi AX\Big\} \\ &\quad + 4\Big\{\eta_\nu(\xi)\eta_\nu(\phi Y)AX - \eta_\nu(\xi)g(AX,Y)\phi_\nu\xi\Big\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\Bigg] \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&+\eta((\nabla_X A)\xi)AY + 2\eta(A\phi AX)AY + \eta(A\xi)(\nabla_X A)Y \\
&-\eta((\nabla_X A)Y)A\xi - g(AY,\phi AX)A\xi \\
&-\eta(AY)(\nabla_X A)\xi - \eta(AY)A\phi AX\,.
\end{aligned}
$$

From this, together with the fact that M is Hopf, we have

$$
\begin{aligned}
&(\nabla_X R_\xi)Y \\
&\quad = \frac{1}{2}g(\phi AX,Y)\xi + \frac{1}{2}\eta(Y)\phi AX \\
&\qquad + \frac{1}{2}\sum_{\nu=1}^{3}\Big[g(\phi_\nu AX,Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\
&\qquad\quad + 3\Big\{g(\phi_\nu AX,\phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi \\
&\qquad\quad + \eta_\nu(\phi Y)(\phi_\nu\phi AX - \alpha\eta(X)\xi_\nu)\Big\} \\
&\qquad\quad + 4\eta_\nu(\xi)\Big\{\eta_\nu(\phi Y)AX - g(AX,Y)\phi_\nu\xi\Big\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\Big] \\
&\qquad + \eta((\nabla_X A)\xi)AY + \alpha(\nabla_X A)Y - \alpha\eta((\nabla_X A)Y)\xi - \alpha g(AY,\phi AX)\xi \\
&\qquad - \alpha\eta(Y)(\nabla_X A)\xi - \alpha\eta(Y)A\phi AX.
\end{aligned}
\tag{3.3}
$$

Let us assume that the structure Jacobi operator a Hopf hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$ is recurrent, that is, $(\nabla_X R_\xi)Y = \beta(X)R_\xi Y$ for any tangent vector fields X, Y to M.

By using above assumption our main purpose is to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the orthogonal complement $\mathcal{Q}^\perp$ such that $T_xM = \mathcal{Q}\oplus\mathcal{Q}^\perp$ for any point $x\in M$.

Hopf hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$ with recurrent structure Jacobi operator satisfies the following equation

$$
\begin{aligned}
&\frac{1}{2}g(\phi AX,Y)\xi + \frac{1}{2}\eta(Y)\phi AX \\
&\quad + \frac{1}{2}\sum_{\nu=1}^{3}\Big[g(\phi_\nu AX,Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\
&\qquad + 3\Big\{g(\phi_\nu AX,\phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi \\
&\qquad + \eta_\nu(\phi Y)(\phi_\nu\phi AX - \alpha\eta(X)\xi_\nu)\Big\} \\
&\qquad + 4\eta_\nu(\xi)\Big\{\eta_\nu(\phi Y)AX - g(AX,Y)\phi_\nu\xi\Big\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\Big] \\
&\quad + \eta((\nabla_X A)\xi)AY + \alpha(\nabla_X A)Y - \alpha\eta((\nabla_X A)Y)\xi - \alpha g(AY,\phi AX)\xi \\
&\quad - \alpha\eta(Y)(\nabla_X A)\xi - \alpha\eta(Y)A\phi AX
\end{aligned}
\tag{3.4}
$$

$$= \frac{1}{2}\beta(X)\Big[-Y+\eta(Y)\xi+\sum_{\nu=1}^{3}\big\{\eta_\nu(Y)\xi_\nu-\eta(Y)\eta_\nu(\xi)\xi_\nu$$
$$+3\eta_\nu(\phi Y)\phi_\nu\xi+\eta_\nu(\xi)\phi_\nu\phi Y\big\}\Big]+\alpha AY-\eta(AY)A\xi$$

for any tangent vector fields X, Y to M.

Lemma 1.1. Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ with recurrent structure Jacobi operator. If the distribution $\mathcal{Q}$ or $\mathcal{Q}^\perp$-component of the Reeb vector field ξ is invariant under the shape operator A of M, then the Reeb vector field ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.

Proof. In order to prove this lemma, we put

$$\xi=\eta(X_0)X_0+\eta(\xi_1)\xi_1 \quad \text{such that} \quad \eta(X_0)\eta(\xi_1)\neq 0 \tag{**}$$

for some unit vectors $X_0\in\mathcal{Q}$ and $\xi_1\in\mathcal{Q}^\perp$.

Together with (**) and a Hopf hypersurface condition, if $\alpha=g(A\xi,\xi)$ vanishes on M, then

$$Y\alpha=(\xi\alpha)\eta(Y)+2\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y).$$

implies $\eta(\xi_1)\phi\xi_1=0$ (see [4, (3.11)]). This gives ξ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$. So we may assume that α is non-vanishing.

Lee and Loo [2] show that if M is Hopf, then the Reeb function α is constant along the direction of structure vector field ξ, that is, $\xi\alpha=0$. Also in [4], we see that $\xi\alpha=0$ gives the distribution $\mathcal{Q}$ - and the $\mathcal{Q}^\perp$ -component of the Reeb vector field ξ is invariant by the shape operator A, that is,

$$AX_0=\alpha X_0, \quad \text{and} \quad A\xi_1=\alpha\xi_1. \tag{3.5}$$

In addition, from (**) and $\phi\xi=0$, we have

$$\begin{cases}\phi X_0=-\eta(\xi_1)\phi_1X_0,\\ \phi\xi_1=\phi_1\xi=\eta(X_0)\phi_1X_0,\\ \phi_1\phi X_0=\eta(\xi_1)X_0.\end{cases} \tag{3.6}$$

The equation

$$A\phi AY=\frac{\alpha}{2}(A\phi+\phi A)Y+\sum_{\nu=1}^{3}\big\{\eta(Y)\eta_\nu(\xi)\phi\xi_\nu+\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\big\}$$
$$-\frac{1}{2}\phi Y-\frac{1}{2}\sum_{\nu=1}^{3}\big\{\eta_\nu(Y)\phi\xi_\nu+\eta_\nu(\phi Y)\xi_\nu+\eta_\nu(\xi)\phi_\nu Y\big\}$$

yields $\alpha A\phi X_0 = (\alpha^2 - 2\eta^2(X_0))\phi X_0$ by substituting $X = X_0$ (see [4, (3.12)]). Since we assumed that the Reeb function α is non-vanishing, it becomes

$$A\phi X_0 = \sigma\phi X_0, \quad \text{where} \quad \sigma = \frac{\alpha^2 - 2\eta^2(X_0)}{\alpha}. \tag{3.7}$$

So we consider the case that the function α is non-vanishing. Putting $Y = \xi$ in (1), we have

$$\begin{aligned} 0 =& \phi AX - 2\alpha A\phi AX \\ &+ \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(\xi)\phi_\nu AX + 3\eta_\nu(AX)\phi_\nu\xi - 4\alpha\eta_\nu(\xi)\eta(X)\phi_\nu\xi\}. \end{aligned} \tag{3.8}$$

By applying (**) to (3.8), we get

$$\begin{aligned} 0 =& \phi AX - 2\alpha A\phi AX + \eta_1(\xi)\phi_1 AX - 4\alpha\eta_1(\xi)\eta(X)\phi_1\xi \\ &+ \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX)\xi_\nu + 3\eta_\nu(AX)\phi_\nu\xi\}. \end{aligned} \tag{3.9}$$

From this, by putting $X = X_0$ into (3.9) and $\phi_1\xi = \eta(X_0)\phi_1 X_0$, we have

$$\begin{aligned} 0 =& \phi AX_0 - 2\alpha A\phi AX_0 + \eta_1(\xi)\phi_1 AX_0 - 4\alpha\eta_1(\xi)\eta^2(X_0)\phi_1 X_0 \\ &+ \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX_0)\xi_\nu + 3\eta_\nu(AX_0)\phi_\nu\xi\}. \end{aligned} \tag{3.10}$$

By using $\eta_\nu(\phi X_0) = 0$, we have

$$0 = \alpha\phi X_0 - 2\alpha^2 A\phi X_0 + \alpha\eta_1(\xi)\phi_1 X_0 - 4\alpha\eta_1(\xi)\eta^2(X_0)\phi_1 X_0. \tag{3.11}$$

By applying (3.6) into (3.11), we obtain

$$0 = -\alpha\eta(\xi_1)\phi_1 X_0 - \alpha^2\eta(\xi_1)A\phi_1 X_0 + \alpha\eta(\xi_1)\phi_1 X_0 - 4\alpha\eta_1(\xi)\eta^2(X_0)\phi_1 X_0. \tag{3.12}$$

From (3.7), we get

$$\begin{aligned} 0 &= 2\alpha^2\eta(\xi_1)\frac{\alpha^2 - 2\eta^2(X_0)}{\alpha}\ \phi_1 X_0 - 4\alpha\eta_1(\xi)\eta^2(X_0)\phi_1 X_0 \\ &= -\alpha\eta(\xi_1)(\alpha^2 - 4\eta^2(X_0))\phi_1 X_0. \end{aligned}$$

Since $\alpha \neq 0$, $\eta(\xi_1) \neq 0$ and $\phi_1 X_0 \neq 0$, we get

$$\alpha^2 = 4\eta^2(X_0) \tag{3.13}$$

From this, by putting $X = \phi X_0$ into (3.9) and using (3.7), we have

$$\sigma(-1 + 2\alpha^2 - 2\eta^2(X_0))\phi X_0 = 0. \tag{3.14}$$

Since $\phi X_0 \neq 0$, we naturally two cases:

Case I. $\sigma = 0$**.**

$\sigma = 0$ means $\alpha^2 - 2\eta^2(X_0) = 0$. Using (3.13), we have $\eta^2(X_0) = 0$ which is a contradiction.

Case II. $1 + 2\alpha^2 - 2\eta^2(X_0) = 0$**.**

Also using (3.13), we have $1 + 6\eta^2(X_0) = 0$ which is a contradiction.

Both cases make a contradiction. Accordingly, we get a complete proof of our Lemma.

□

By virtue of Lemma 1.1, in next section, we consider the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$.

2. The Reeb vector field $\xi \in \mathcal{Q}^\perp$

Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ with recurrent structure Jacobi operator. Then by Lemma 1.1 we shall make an investigation into two cases depending on ξ belongs to either distribution $\mathcal{Q}^\perp$ or distribution $\mathcal{Q}$, respectively. So, in this section let us consider the case $\xi \in \mathcal{Q}^\perp$ (i.e., $JN \in \mathfrak{J}N$ where N is a unit normal vector field on M in $SU_{2,m}/S(U_2{\cdot}U_m)$). Since $\mathcal{Q}^\perp = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$, we may put $\xi = \xi_1$.

In [4, (3.6)], we have

$$\phi AX = 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX. \tag{4.1}$$

Lemma 2.1. Let M be a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$ with recurrent structure Jacobi operator. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$, then the shape operator A commutes with the structure operator ϕ, that is, $A\phi = \phi A$.

Proof. We may put $\xi = \xi_1$, because $\xi \in \mathcal{Q}^\perp$. By putting $Y = \xi$ into (1), we have $(\nabla_X R_\xi)\xi = 0$. So we obtain

$$\begin{aligned} 0 =& \phi AX - 2\alpha A\phi AX \\ &+ \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(\xi)\phi_\nu AX + 3\eta_\nu(AX)\phi_\nu\xi - 4\alpha\eta_\nu(\xi)\eta(X)\phi_\nu\xi\}. \end{aligned}$$

Since $\xi \in \mathcal{Q}^\perp$, without loss of generality, we may put $\xi = \xi_1$, thus we get

$$\begin{aligned} 0 =& \phi AX - 2\alpha A\phi AX - \eta_2(\phi AX)\xi_2 - \eta_3(\phi AX)\xi_3 + \phi_1 AX \\ &+ 3\eta_2(AX)\phi_2\xi + 3\eta_3(AX)\phi_3\xi - 4\alpha\eta(X)\phi_1\xi. \end{aligned}$$

And we know that $\eta_2(\phi AX) = \eta_3(AX)$ and $\eta_3(\phi AX) = -\eta_2(AX)$.

From these, we obtain

$$0 = \phi AX - 2\alpha A\phi AX + 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX \tag{4.2}$$

for any vector field $X \in TM$.

By using (4.1), (4.2) becomes

$$0 = \phi AX - \alpha A\phi AX. \tag{4.3}$$

Taking symmetric part of (4.3), we have

$$0 = A\phi X - \alpha A\phi AX. \tag{4.4}$$

By virtue of (4.3) and (4.4), we have $A\phi = \phi A$.

Summing up these observations, it is natural that the shape operator A commutes with the structure tensor field ϕ under our assumption. □

Let us check whether the structure Jacobi operator of real hypersurfaces of Type (A) is recurrent or not. In order to do this, we recall a proposition due to Berndt and Suh [1] as follows: In [8], Suh gave some information related to the shape operator A of $\mathcal{T}_A$ and $\mathcal{H}_A$ as follows:

Proposition A. Let M be a connected real hypersurface in complex hyperbolic two-plane Grassamannian $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \in \mathfrak{J}N$, then one of the following statements holds:

($\mathcal{T}_A$) M has exactly four distinct constant principal curvatures

$$\alpha = 2\coth(2r),\ \beta = \coth(r),\ \lambda_1 = \tanh(r),\ \lambda_2 = 0,$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = \mathcal{C} \ominus \mathcal{Q},\ T_{\lambda_1} = E_{-1},\ T_{\lambda_2} = E_{+1}.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are complex (with respect to J) and totally complex (with respect to $\mathfrak{J}$).

($\mathcal{H}_A$) M has exactly three distinct constant principal curvatures

$$\alpha = 2,\ \beta = 1,\ \lambda = 0$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = (\mathcal{C} \ominus \mathcal{Q}) \oplus E_{-1},\ T_\lambda = E_{+1}.$$

Here, E_{+1} and E_{-1} are the eigenbundles of $\phi\phi_1|_{\mathcal{Q}}$ with respect to the eigenvalues $+1$ and -1, respectively.

By putting $Y = \xi$ and applying $\xi = \xi_1$ into (1), we obtain

$$0 = \phi AX + \alpha A\phi AX + \phi_1 AX + \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX)\xi_\nu + 3\eta_\nu(AX)\phi_\nu\xi\} \tag{4.5}$$

for any tangent vector field X to M.

Case I: Tube ($\mathcal{T}_A$).

From (4.5), let us consider a unit eigenvector $X \in T_{\lambda_1}$. Then we have

$$0 = \lambda_1\phi X + 2\alpha\lambda_1 A\phi X + \lambda_1\phi_1 X + \sum_{\nu=1}^{3}\{-\lambda_1\eta_\nu(\phi X)\xi_\nu + 3\lambda_1\eta_\nu(X)\phi_\nu\xi\}.$$

Since $\phi X = \phi_1 X$, we have

$$0 = \lambda_1\phi X + \alpha\lambda_1 A\phi X.$$

Since $\phi T_{\lambda_1} \subset T_{\lambda_1}$ and $AX = \lambda_1 X$, we have $A\phi X = \lambda_1\phi X$. Thus we get $\lambda_1(1+\alpha\lambda_1)\phi X = 0$.

As $\alpha = 2\coth(2r)$, $\lambda_1 = \tanh(r)$, we have

$$2 - \tanh^2(r) = 0 \tag{4.6}$$

From (4.5), putting $X = \xi_2$, we have

$$\begin{aligned} 0 =& (\alpha\beta - 1)\beta\xi_3 \\ =& (\coth^2(r) - 1)\xi_3 \end{aligned} \tag{4.7}$$

By using (4.6) and (4.7), we have $\xi_3 = 0$ which gives a contradiction.

Case II: horosphere ($\mathcal{H}_A$).

From the given condition, we may have

$$0 = \phi AX - \alpha A\phi AX. \tag{4.8}$$

Let us consider a unit eigenvector $X = \xi_2 \in T_\beta$, then we have

$$\begin{aligned} 0 =& \beta(-1 + \alpha\beta)\xi_3 \\ =& \xi_3. \end{aligned} \tag{4.9}$$

This also gives a contradiction.

Thus we know that the structure Jacobi operator R_ξ of real hypersurface of Type (A) in $SU_{2,m}/S(U_2{\cdot}U_m)$ is not recurrent if ξ belongs to the distribution $\mathcal{Q}^\perp$. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$, then there exist no hypersurface of Type (A) in $SU_{2,m}/S(U_2{\cdot}U_m)$ with recurrent structure Jacobi operator.

3. The Reeb vector field $\xi \in \mathcal{Q}$

Next, we check for the case $\xi \in \mathcal{Q}$ whether the structure Jacobi operator of real hypersurfaces of Type (B) is recurrent or not. In order to do this we introduce a proposition due to Berndt and Suh [1] as follows:
By virtue of the result in [9], we assert that a Hopf hypersurface M in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$ satisfying the hypotheses is locally congruent to one of the following real hypersurfaces

($\mathcal{T}_B$) An open part of a tube around a totally geodesic quaternionic hyperbolic space $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$,

($\mathcal{H}_B$) An open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \perp \mathfrak{J}N$, or

($\mathcal{E}$) The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$,

when $\xi \in \mathcal{Q}$. Hereafter, the model spaces of $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ is denoted by M_B. Let us check whether the shape operator A of model spaces are M_B satisfy our conditions, conversely. In order to do this, let us introduce the following proposition given by Suh [9].

Proposition B. Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \perp \mathfrak{J}N$, then one of the following statements holds:

($\mathcal{T}_B$) M has five (four for $r = \sqrt{2}\tanh^{-1}(1/\sqrt{3})$ in which case $\alpha = \lambda_2$) distinct constant principal curvatures

$$\alpha = \sqrt{2}\tanh(\sqrt{2}r),\ \beta = \sqrt{2}\coth(\sqrt{2}r),\ \gamma = 0,$$
$$\lambda_1 = \frac{1}{\sqrt{2}}\tanh(\frac{1}{\sqrt{2}}r),\ \lambda_2 = \frac{1}{\sqrt{2}}\coth(\frac{1}{\sqrt{2}}r),$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = TM \ominus \mathcal{Q},\ T_\gamma = J(TM \ominus \mathcal{Q}) = JT_\beta.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are invariant under $\mathfrak{J}$ and are mapped onto each other by J. In particular, the quaternionic dimension of $SU_{2,m}/S(U_2U_m)$ must be even.

($\mathcal{H}_B$) M has exactly three distinct constant principal curvatures

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}),\ T_\gamma = J(TM \ominus \mathcal{Q}),\ T_\lambda = \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

($\mathcal{E}$) M has at least four distinct principal curvatures, three of which are given by

$$\alpha = \beta = \sqrt{2},\ \gamma = 0,\ \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}),\ T_\gamma = J(TM \ominus \mathcal{Q}),\ T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$. Thus, the corresponding multiplicities are

$$m(\alpha) = 4, \quad m(\gamma) = 3, \quad m(\lambda), \quad m(\mu).$$

By putting $Y = \xi$ and applying $\xi \in \mathcal{Q}$ into ((1)), we have

$$0 = \phi AX + 2\alpha A\phi AX + \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX)\xi_\nu + 3\eta_\nu(AX)\phi_\nu\xi\}. \tag{5.1}$$

Case I: tube ($\mathcal{T}_B$).

From (5.1), we consider a unit eigenvector $X \in T_\lambda$. Then it follows that

$$\begin{aligned} 0 =& \lambda\phi X + 2\alpha\lambda A\phi X \\ &+ \sum_{\nu=1}^{3}\{-\lambda\eta_\nu(\phi X)\xi_\nu + 3\lambda\eta_\nu(X)\phi_\nu\xi\} \\ =& \lambda\phi X + 2\alpha\lambda A\phi X. \end{aligned}$$

Since $JT_\lambda = T_\mu$ and $X \in T_\lambda$, we know that $\phi X \in T_\mu$. This means that $A\phi X = \mu\phi X$. Naturally we also have $0 = \lambda\phi X + 2\alpha\lambda\mu\phi X = \lambda(1 + 2\alpha\mu)\phi X$. And we have

$$\begin{aligned} 1 + 2\alpha\mu =& 1 + (\sqrt{2}\tanh(2\theta))(\frac{1}{\sqrt{2}}\coth(\theta)) \\ =& 1 + 2\frac{\coth^2(\theta)}{\coth^2(\theta) + \tanh^2(\theta)} \\ &> 0. \end{aligned}$$

where $\theta = \sqrt{2}r$.

So we get $\phi X = 0$. This gives a contradiction.

Case II: horosphere or exceptional case ($\mathcal{H}_B$) or ($\mathcal{E}$)

From (5.1), we consider a unit eigenvector $X = \xi_1 \in T_\beta$. Then it follows that

$$\begin{aligned} 0 =& \phi A\xi_1 + 2\alpha A\phi A\xi_1 + \sum_{\nu=1}^{3}\{-\eta_\nu(\phi A\xi_1)\xi_\nu + 3\eta_\nu(A\xi_1)\phi_\nu\xi\} \\ =& 4\beta\phi_1\xi \\ =& 4\sqrt{2}\phi_1\xi. \end{aligned}$$

So we get $\phi_1\xi = 0$. This gives a contradiction.

Thus we know that the structure Jacobi operator R_ξ of real hypersurface of Type (B) in $SU_{2,m}/S(U_2{\cdot}U_m)$ is not recurrent if the Reeb vector field ξ belongs to the distribution $\mathcal{Q}$.

Summing up Lemmas and using Theorems in [1], [9], we know that any connected Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ with recurrent structure Jacobi operator is locally congruent to one either of Type (A) or of Type (B). But, by using Propositions in [1], we checked that the structure Jacobi operator R_ξ of any real hypersurfaces of Type (A) or of Type (B) is not recurrent. So we complete the proof of our Main Theorem in the introduction.

References

1. J. Berndt and Y.J. Suh, *Hypersurfaces in noncompact complex Grassmannians of rank two*, Internat. J. Math. **23** (2012), 1250103(35 pages).
2. R.-H. Lee and T.-H. Loo, *Hopf hypersurfaces in complex Grassmannians of rank two*, Results Math. **71** (2017), 1083-1107.
3. H. Lee, Y.J. Suh and C. Woo, *Reeb parallel Ricci tensor for homogeneous real hypersurfaces in complex hyperbolic two-plane Grassmannians*, Math. Nachr. **288** (2015), 1-12.
4. H. Lee, Y.J. Suh and C. Woo, *Real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting structure Jacobi operators*, Mediterr. J. Math. **13** (2016), 3389-3407.
5. J.D. Pérez, Y.J. Suh and C. Woo, *Real Hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting shape operator*, Open Math. **13** (2015), 493-501.
6. J.D. Pérez and F.G. Santos, *Real hypersurfaces in complex projective space recurrent structure Jacobi operator*, Diffential geometry and its Applications **26** (2008), 218-223.
7. S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry Vol. 1*, Wiley, New York, 1996.
8. Y.J. Suh, *Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians*, Adv. Appl. Math. **50** (2013), 645-659.
9. Y.J. Suh, *Real hypersurfaces in complex hyperbolic two-plane Grassmannians with Reeb vector field*, Adv. Appl. Math. **55** (2014), 131-145.
10. Y.J. Suh and C. Woo, *Real Hypersurfaces in complex hyperbolic two-plane Grassmannians with parallel Ricci tensor*, Math. Nachr. **287** (2014), 1524-1529.

Imsoon Jeong received M.Sc. and Ph.D. from Daegu Catholic University. Since 2020, she has been at Cheongju University as an assistant professor. Her research interest is differential geometry.

Department of Mathematics Education, Cheongju University, Cheongju 28503, Korea.
e-mail: isjeong@cju.ac.kr

Changhwa Woo received M.Sc. and Ph.D. from Kyungpook National University. Since 2019, he is currently an assistant professor at Pukyong National University. His research interest is differential geometry.

Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Busan 48513, Korea.
e-mail: legalgwch@pknu.ac.kr