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HIGHER ORDER STRONGLY EXPONENTIALLY PREINVEX FUNCTIONS

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ABSTRACT. In this paper, some new classes of the higher order strongly exponentially preinvex functions are introduced. New relationships among various concepts of higher order strongly exponentially preinvex functions are established. It is shown that the optimality conditions of differentiable higher order strongly exponentially preinvex functions can be characterized by exponentially variational-like inequalities. Parallelogram laws for Banach spaces are obtained as an application. As special cases, one can obtain various new and known results from our results. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

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1. Introduction

Recently, several extensions and generalizations have been considered for classical convexity. Strongly convex functions were introduced and studied by Polyak [51], which play an important part in the optimization theory and related areas. Karmardian [18] used the strongly convex functions to discuss the unique existence of a solution of the nonlinear complementarity problems. Strongly convex functions are used to investigate the convergence analysis for solving variational inequalities and equilibrium problems, see Zu and Marcotte [59]. See also Nikodem and Pales [19] and Qu and Li [49]. Awan et al[7, 8, 9] have derived Hermite-Hadamard type inequalities, which provide upper and lower estimate for the integrand. For more applications and properties of the strongly convex functions. See, for example, [1, 2], [4]-[59]. Hanson [16] introduced the concept of invex function for the differentiable functions, which played significant part

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in the mathematical programming. Ben-Israel and Mond [11] introduced the concept of invex set and preinvex functions. It is known that the differentiable preinvex function are invex functions. The converse also holds under certain conditions, see [20, 26]. Noor [24] proved that the minimum of the differentiable preinvex functions on the invex set can be characterized the variational-like inequality. Noor [28, 29, 30] proved that a function f is preinvex function, if and only if, it satisfies the Hermite-Hadamard type integral inequality. Noor at el. [31, 32, 33, 34, 35, 36, 37, 38, 39] investigated the applications of the strongly preinvex functions and their variant forms. It is known that more accurate and inequalities can be obtained using the logarithmically convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave) functions, the origin of exponentially convex functions can be traced back to Bernstein [12]. Avriel [6] introduced and studied the concept of $-$convex functions, where as the (r, p)-convex functions were studied by Antczak [5]. For further properties of the r-convex functions, see Zhao et al[58] and the references therein. Exponentially convex functions have important applications in information theory, big data analysis, machine learning and statistic. See [2, 3, 5, 6, 47, 50, 52] and the references therein. Noor and Noor [35, 36, 37, 38, 39, 41, 42] considered the concept of exponentially convex functions and discussed the basic properties. It is worth mentioning that these exponentially convex functions [35, 36, 37, 38] are distinctly different from the exponentially convex functions considered and studied by Bernstein [12], Awan et al.[7, 8, 9] and Pecaric et al.[49, 50]. We would like to point out that The definition of exponential convexity in Noor and Noor [35, 36, 3]] is quite different from Bernstein [12]. Noor and Noor [38, 39, 40] studied the properties of the exponentially preinvex functions and their variant forms. They have shown that the exponentially preinvex functions enjoy the same interesting properties which exponentially convex functions have.
Inspired by the research work going in this field, we introduce and consider some new classes of nonconvex functions with respect to an arbitrary non-negative arbitrary bifunction, which is called the higher order strongly exponentially generalized preinvex functions. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. Optimality conditions for differentiable strongly exponentially generalized preinvex functions are investigated. As special cases, one can obtain various new and refined versions of known results. This paper is continuation of our recent research in generalized convexity and variational inequalities. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminaries

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. Let $F : K_\eta \to R$ be a continuous function and let $\eta(.,.) : K_\eta \times K_\eta \to R$ be an arbitrary continuous bifunction.

Definition 2.1. [11].*The set K_η in H is said to be invex set with respect to an arbitrary bifunction $\eta(\cdot,\cdot)$, if*

$$u + \lambda\eta(v,u) \in K_\eta, \qquad \forall u,v \in K_\eta, \lambda \in [0,1].$$

The invex set K_η is also called η-connected set. Note that the innvex set with $\eta(v,u) = v - u$ is a convex set K, but the converse is not true. For example, the set $K_\eta = R - (-\frac{1}{2}, \frac{1}{2})$ is an invex set with respect to η, where

$$\eta(v,u) = \begin{cases} v-u, & \text{for} \quad v>0, u>0 \quad \text{or} \quad v<0, u<0 \\ u-v, & \text{for} \quad v<0, u>0 \quad \text{or} \quad v<0, u<0. \end{cases}$$

It is clear that K is not a convex set.

From now onward K_η is a nonempty closed invex set in H with respect to the bifunction $\eta(\cdot,\cdot)$, unless otherwise specified.

We now introduce some new concepts of strongly exponentially preinvex functions and their variants forms, which is the main motivation of this paper.

Definition 2.2. *The function F on the invex set K_η is said to be higher order strongly exponentially preinvex with respect to the bifunction $\eta(\cdot,\cdot)$, if there exists a constant $\mu > 0$, such that*

$$e^{F(u+\lambda\eta(v,u))} \le (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p, \quad (1)$$
$$\forall u,v \in K_\eta, \lambda \in [0,1], p \ge 1.$$

Note that every higher order strongly convex function is a higher order strongly preinvex, but the converse is not true.

If $\lambda = \frac{1}{2}$, then

$$e^{F(\frac{2u+\eta(v,u)}{2})} \le \frac{1}{2}\{e^{F(u)} + e^{F(v)}\} - \mu\frac{1}{2^p}\|\eta(v,u)\|^p,$$
$$\forall u,v \in K_\eta, \lambda \in [0,1], p \ge 1.$$

The function F is said to be higher order strongly exponentially preconcave, if and only if, $-F$ is higher order strongly exponentially preinvex function. Consequently, we have a new concept.

Definition 2.3. *A function F is said to be higher order strongly affine exponentially preinvex involving an arbitrary bifunction $\eta(,)$, if there exists a constant $\mu > 0$, such that*

$$e^{F(u+\lambda\eta(v,u))} \le (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p, \quad (2)$$
$$\forall u,v \in K_\eta, \lambda \in [0,1], p \ge 1.$$

We now discuss some special cases of the higher order strongly exponentially preinvex functions.

(I).) If $\eta((v,u) = v-u$, then the higher order strongly generalized preinvex function becomes higher order strongly convex functions, that is,

Definition 2.4. *A function F is said to strongly exponentially convex functions, if*

$$e^{F(u+\lambda(v-u)))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\{\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|v-u\|^p, \quad \forall u,v \in K_\eta, \lambda \in [0,1].$$

For the properties of the higher order strongly exponentially convex functions in variational inequalities and equilibrium problems, see Noor et al. [37, 40, 41, 45].

(II). If $p=2$, then Definition 2.2 becomes:

Definition 2.5. *A function F is said to strongly exponential preinvex functions, if*

$$e^{F(u+\lambda\eta(v,u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\lambda(1-\lambda)\|\eta(v,u)\|^2, \quad \forall u,v \in K_\eta, \lambda \in [0,1]. \tag{3}$$

For applications in variational inequalities and equilibrium problems, see [24, 25, 32, 39, **?**] and the references therein.

Definition 2.6. *The function F on the invex set K_η is said to be higher order strongly exponentially quasi preinvex with respect to the bifunction $\eta(\cdot,\cdot)$, if there exists a constant $\mu > 0$, such that*

$$e^{F(u+\lambda\eta(v,u))} \leq \max\{e^{F(u)}, e^{F(v)}\} - \mu\{\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|\eta(v,u)\|^p, \quad \forall u,v \in K_\eta, \lambda \in [0,1], p \geq 1.$$

Definition 2.7. *The function F on the invex set K is said to be higher order strongly exponentially log-preinvex with respect to the bifunction $\eta(\cdot,\cdot)$, if there exists a constant $\mu > 0$, such that*

$$e^{F(u+\lambda\eta(v,u))} \leq (F(u)^{1-\lambda})(e^{F(v)})^\lambda - \mu\{\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|\eta(v,u)\|^p, \quad \forall u,v \in K_\eta, \lambda \in [0,1].p \geq 1,$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} e^{F(u+\lambda\eta(v,u))} &\leq (e^{F(u)})^{1-\lambda}(e^{F(v)})^\lambda - \mu\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|\eta(v,u)\|^p \\ &\leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|\eta(v,u)\|^p \\ &\leq \max\{e^{F(u)}, e^{F(v)}\} - \mu\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|\eta(v,u)\|^p. \end{aligned}$$

This shows that every higher order strongly exponentially log-preinvex function is higher order strongly exponentially preinvex function and every higher order strongly exponentially preinvex function is a higher order strongly exponentially quasi-preinvex function. However, the converse is not true.

For $t = 1$, Definition 2.2 and 2.7 reduce to the following condition, which is mainly due to Noor and Noor [5].

Condition A.

$$e^{F(u+\eta(v,u))} \quad \leq \quad e^{F(v)}, \quad \forall v \in K_\eta.$$

Definition 2.8. *The differentiable function F on the invex set K_η is said to be higher order strongly exponentially invex function with respect to the bifunction $\eta(\cdot,\cdot)$, if there exists a constant $\mu > 0$ such that*

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u), \eta(v,u)\rangle + \mu\|\eta(v,u)||^p, \quad \forall u, v \in K_\eta, p \geq 1,$$

where $F'(u))$ is the differential of F at u.

It is noted that, if $\mu = 0$, then the Definition 2.8 reduces to the definition of the exponentially invex function as introduced by Noor and Noor [38, 39]. It is well known that the concepts of preinvex and invex functions play a significant role in the mathematical programming and optimization theory.

Definition 2.9. *A differentiable function F on the invex set K_η is said to be higher order strongly exponentially pseudo η-invex function, iff, if there exists a constant $\mu > 0$ such that*

$$\langle e^{F(u)}F'(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^p \geq 0 \Rightarrow F(v) - F(u) \geq 0, \qquad \forall u, v \in K_\eta, p \geq 1.$$

Definition 2.10. *A differentiable function F on K_η is said to be higher order strongly exponentially quasi-invex function, iff, if there exists a constant $\mu > 0$ such that*

$$\begin{aligned} & e^{F(v)} \leq e^{F(u)} \\ & \Rightarrow \\ & \langle e^{F(u)}F'(u), \eta(v,u)\rangle + \mu\|\eta(u,v)\|^p \leq 0, \quad \forall u, v \in K_\eta, p \geq 1. \end{aligned}$$

Definition 2.11. *The function F on the set K_η is said to be exponentially pseudo-invex, iff,*

$$\langle e^{F(u)}F'(u), \eta(v,u)\rangle \geq 0 \Rightarrow e^{F(v)} \geq e^{F(u)}, \qquad \forall u, v \in K_\eta.$$

Definition 2.12. *The differentiable function F on the K_η is said to be exponentially quasi-invex function, iff,*

$$e^{F(v)} \leq e^{F(u)} \Rightarrow \langle e^{F(u)}F'(u), \eta(v,u)\rangle \leq 0, \qquad \forall u, v \in K_\eta.$$

If $\eta(v,u) = -\eta(v,u), \forall u, v \in K_\eta$, that is, the function $\eta(\cdot,\cdot)$ is skew-symmetric, then Definitions **??**-2.12 reduce to the known ones. This shows that the concepts introduced in this paper represent significant improvement of the previously known ones. All these new concepts may play important and fundamental part in the mathematical programming and optimization.

We also need the following assumption regarding the bifunction $\eta(\cdot,\cdot)$. which is mainly due to Mohen and Neogy [20]/

Condition C[20]**.** Let $\eta(\cdot,\cdot) : K_\eta \times K_\eta \to H$ satisfy assumptions

$$\eta(u, u + \lambda\eta(v,u)) = -\lambda\eta(v,u)$$
$$\eta(v, u + \lambda\eta(v,u)) = (1-\lambda)\eta(v,u), \quad \forall u, v \in K_\eta, \lambda \in [0,1].$$

Clearly for $\lambda = 0$, we have $\eta(u,v) = 0$, if and only if $u = v, \forall u, v \in K_\eta$. One can easily show [25] that $\eta(u + \lambda\eta(v,u), u) = \lambda\eta(v,u), \forall u, v \in K_\eta$.

3. Main Results

In this section, we consider some basic properties of higher order strongly preinvex functions on the invex set K_η and their applications.

Theorem 3.1. *Let F be a differentiable function on the invex set K_η in H and let the condition C hold. Then the function F is higher order strongly exponentially preinvex function, if and only if, F is a higher order strongly exponentially invex function.*

Proof. Let F be a higher strongly exponentially preinvex function on the invex set K_η. Then

$$e^{F(u+\lambda\eta(v,u))} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p, \quad \forall u, v \in K_\eta, \lambda \in [0,1], p \geq 1.$$

which can be written as

$$e^{F(v)} - e^{F(u)} \geq \{\frac{e^{F(u+\lambda\eta(v,u))} - e^{F(u)}}{\lambda}\} + \mu\{\lambda^{p-1}(1-\lambda) + (1-\lambda)^p\}\|\eta(v,u)\|^p.$$

Taking the limit in the above inequality as $t \to 0$, we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u), \eta(v,u))\rangle + \mu\|\eta(v,u)\|^p.$$

This shows that F is a higher order strongly exponentially invex function.

Conversely, let F be a higher order strongly exponentially invex function on the invex set K_η. Then, $\forall u, v \in K_\eta, \lambda \in [0,1]$, $v_t = u + \lambda\eta(v,u) \in K_\eta$ and using the condition C, we have

$$\begin{aligned} & e^{F(v)} - e^{F(u+\lambda\eta(v,u))} \\ \geq \quad & \langle e^{F(u+\lambda\eta(v,u)}F'(u+\lambda\eta(v,u)), \eta(v, u+\lambda\eta(v,u))\rangle + \mu\|\eta(v, u+\lambda\eta(v,u))\|^p \end{aligned}$$

$$= \quad (1-\lambda)\langle e^{F(u+\lambda\eta(v,u)}F'(u+\lambda\eta(v,u)),\eta(v,u)\rangle + \mu(1-\lambda)^p\|\eta(v,u)\|^p. \quad (4)$$

In a similar way, we have

$$\begin{aligned} & e^{F(u)} - e^{F(u+\lambda\eta(v,u))} \\ \geq \quad & \langle e^{F(u+\lambda\eta(v,u)}F'(u+\lambda\eta(v,u)),\eta(u,u+\lambda\eta(v,u))\rangle + \mu\|\eta(u,u+\lambda\eta(v,u))\|^p \\ = \quad & -\lambda\langle e^{F(u+\lambda\eta(v,u)}F'(u+t\eta(v,u)),\eta(v,u)\rangle + \mu\lambda^p\|\eta(v,u)\|^p. \quad (5) \end{aligned}$$

Multiplying (4) by λ and (5) by $(1-\lambda)$ and adding the resultant, we have

$$e^{F(u+\lambda\eta(v,u)} \leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - \{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p,$$

showing that F is a higher order strongly exponentially preinvex function. □

Theorem 3.2. *Let F be differentiable higher order strongly exponentially preinvex function on the invex set K_η. If F is a higher order strongly exponentially invex function, then*

$$\begin{aligned} & \langle e^{F(u)}F'(u),\eta(v,u))\rangle + \langle e^{F(v)}F'(v),\eta(u,v)\rangle \\ & \leq -\mu\{\|\eta(v,u)\|^p + \|\eta(u,v)\|^p\}, \forall u,v \in K_\eta. \end{aligned} \quad (6)$$

Proof. Let F be a higher order strongly exponentially invex function on the invex set K_η. Then

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u),\eta(v,u))\rangle + \mu\|\eta(v,u)\|^p, \quad \forall u,v \in K_\eta. \quad (7)$$

Changing the role of u and v in (7), we have

$$e^{F(u)} - e^{F(v)} \geq \langle e^{F(v)}F'(v),\eta(u,v)\rangle + \mu\|\eta(u,v)\|^p, \quad \forall u,v \in K_\eta. \quad (8)$$

Adding (7) and (8), we have

$$\begin{aligned} & \langle e^{F(u)}F'(u),\eta(v,u))\rangle + \langle e^{F(v)}F'(v),\eta(u,v)\rangle \\ & \leq -\mu\{\|\eta(v,u)\|^p + \|\eta(u,v)\|^p\}, \forall u,v \in K_\eta, \end{aligned} \quad (9)$$

which shows that $F'(.)$ is higher order strongly η-monotone operator. □

We note that the converse of Theorem 3.2 is true only for $p = 2$. However, we have:

Theorem 3.3. *If the differential $F'(.)$ is a higher order strongly η-monotone, then*

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u),\eta(v,u)\rangle + \frac{2}{p}\mu\|\eta(v,u)\|^p.$$

Proof. Let $F'(.)$ be higher order strongly η-monotone. From (9), we have

$$\langle e^{F(v)}F'(v),\eta(u,v)\rangle \geq \langle e^{F(v)}F'(u),\eta(v,u))\rangle - \mu\{\|\eta(v,u)\|^p + \|\eta(u,v)\|^p\}. \quad (10)$$

Since K is an invex set, $\forall u,v \in K_\eta$, $\lambda \in [0,1]$ $v_t = u + \lambda\eta(v,u) \in K_\eta$. Taking $v = v_\lambda$ in (10) and using Condition C, we have

$$\begin{aligned} & \langle e^{F(v_\lambda)}F'(v_\lambda),\eta(u,u+\lambda\eta(v,u))\rangle \\ \leq \quad & \langle e^{F(u)}F'(u),\eta(u+\lambda\eta(v,u),u))\rangle \end{aligned}$$

$$\begin{aligned} & -\mu\{\|\eta(u+\lambda\eta(v,u),u)\|^p + \|\eta(u,u+\lambda\eta(v,u)\|^p\} \\ = \quad & -\lambda\langle e^{F(u)}F'(u),\eta(v,u)\rangle - 2\lambda^p\mu\|\eta(v,u)\|^p, \end{aligned}$$

which implies that

$$\langle e^{F(v_\lambda)}F'(v_\lambda),\eta(v,u)\rangle \geq \langle e^{F(u)}F'(u),\eta(v,u)\rangle + 2\mu\lambda^{p-1}\|\eta(v,u)\|^p. \tag{11}$$

Let $g(\lambda) = F(u+\lambda\eta(v,u))$. Then, from (11), we have

$$\begin{aligned} \xi'(\lambda) &= \langle e^{F(u+\lambda\eta(v,u)}F'(u+\lambda\eta(v,u)),\eta(v,u)\rangle \\ &\geq \langle e^{F(u)}F'(u),\eta(v,u)\rangle + 2\mu\lambda^{p-1}\|\eta(v,u)\|^p. \end{aligned} \tag{12}$$

Integrating (12) between 0 and 1, we have

$$\xi(1) - \xi(0) \geq \langle e^{F(u)}F'(u),\eta(v,u)\rangle + \frac{2}{p}\mu\|\eta(v,u)\|^p.$$

that is,

$$e^{F(u+\eta(v,u))} - e^{F(u)} \geq \langle e^{F(u)}F'(u),\eta(v,u)\rangle + \frac{2}{p}\mu\|\eta(v,u\|^p).$$

By using Condition A, we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u),\eta(v,u)\rangle + \frac{2}{p}\mu\|\eta(v,u)\|^p.$$

the required result. □

We now give a necessary condition for higher order strongly exponentially η-pseudo-invex function.

Theorem 3.4. *Let $F'(.)$ be a higher order strongly exponentially relaxed η-pseudomonotone operator and Condition A and C hold. Then F is a higher order strongly exponentially η-pseudo-invex function.*

Proof. Let F' be higher order strongly exponentially relaxed η-pseudomonotone. Then, $\forall u,v \in K_\eta$,

$$\langle e^{F(u)}F'(u),\eta(v,u)\rangle \geq 0,$$

implies that

$$-\langle e^{F(v)}F'(v),\eta(u,v)\rangle \geq \alpha\|\eta(u,v)\|^p. \tag{13}$$

Since K is an invex set, $\forall u,v \in K_\eta$, $\lambda \in [0,1]$, $v_\lambda = u+\lambda\eta(v,u) \in K_\eta$. Taking $v = v_\lambda$ in (13) and using condition Condition C, we have

$$-\langle e^{F(u+\lambda\eta(v,u)}F'(u+\lambda\eta(v,u)),\eta(u,v)\rangle \geq \lambda\alpha\|\eta(v,u)\|^p. \tag{14}$$

Let

$$\xi(\lambda) = e^{F(u+\lambda\eta(v,u))}, \quad \forall u,v \in K_\eta, \lambda \in [0,1].$$

Then, using (14), we have

$$\xi'(\lambda) = \langle e^{F(u+\lambda\eta(v,u)}F'(u+\lambda\eta(v,u)),\eta(u,v)\rangle \geq \lambda\alpha\|\eta(v,u)\|^p.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) \geq \frac{\alpha}{2}\|\eta(v,u)\|^p,$$

that is,

$$e^{F(u+\lambda\eta(v,u))} - e^{F(u)} \geq \frac{\alpha}{2}\|\eta(v,u)\|^p,$$

which implies, using Condition A,

$$e^{F(v)} - e^{F(u)} \geq \frac{\alpha}{2}\|\eta(v,u)\|^p,$$

showing that F is a higher order strongly exponentially η-pseudo-invex function. □

As special cases of Theorem 3.4, we have the following:

Theorem 3.5. *Let the differentiable $F'(u)$ of a function $F(u)$ on the invex set K_η be higher order strongly exponentially η-pseudomonotone operator. If Conditions A and C hold, then F is higher order strongly exponentially pseudo η-invex function.*

Theorem 3.6. *Let the differential $F'(u)$ of a function $F(u)$ on the invex set K_η be higher order strongly exponentially η-pseudomonotone. If Conditions A and C hold, then F is higher order strongly exponentially pseudo η-invex function.*

Theorem 3.7. *Let the differential $F'(u)$ of a function $F(u)$ on the invex set K_η be higher order strongly exponentially η-pseudomonotone. If Conditions A and C hold, then F is higher order strongly exponentially pseudo η-invex function.*

Theorem 3.8. *Let the differential $F'(u)$ of a function $F(u)$ on the invex set K_η be η-pseudomonotone. If Conditions A and C hold, then F is higher order strongly exponentially pseudo invex function.*

Theorem 3.9. *Let the differential $F'(u)$ of a differentiable higher order strongly exponentially preinvex function $F(u)$ be Lipschitz continuous on the invex set K_η with a constant $\beta > 0$. Then*

$$e^{F(u+\eta(v,u))} - e^{F(u)} \leq \langle e^{F(u)}F'(u), \eta(v,u)\rangle + \frac{\beta}{2}\|\eta(v,u)\|^2, \quad u, v \in K_\eta.$$

Proof. Its proof follow from Noor and Noor [24]. □

Definition 3.1. *The function F is said to be sharply higher order strongly exponentially pseudo preinvex, if there exists a constant $\mu > 0$ such that*

$$\begin{aligned}
&\langle e^{F(u)}F'(u), \eta(v,u)\rangle \geq 0 \\
&\Rightarrow \\
&e^{F(v)} \geq e^{F(v+\lambda\eta(v,u))} + \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p, \forall u, v \in K_\eta, \lambda \in [0,1].
\end{aligned}$$

Theorem 3.10. *Let F be a higher order strongly exponentially sharply pseudo preinvex function on K_η with a constant $\mu > 0$. Then*

$$-\langle e^{F(v)}F'(v), \eta(v,u)\rangle \geq \mu\|\eta(v,u)\|^p, \quad \forall u, v \in K_\eta.$$

Proof. Let F be a higher order strongly exponentially sharply pseudo preinvex function on K_η. Then

$$e^{F(v)} \geq e^{F(v+\lambda\eta(v,u))} + \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p, \quad \forall u, v \in K_\eta, \lambda \in [0,1],$$

from which we have

$$\frac{e^{F(v+\lambda\eta(v,u))} - e^{F(v)}}{\lambda} + \mu\{\lambda^{p-1}(1-\lambda) + (1-\lambda)^p\}\|\eta(v,u\|^p) \leq 0.$$

Taking limit in the above-mentioned inequality, as $\lambda \to 0$, we have

$$-\langle e^{F(v)}F'(v), \eta(v,u)\rangle \geq \mu\|\eta(v,u)\|^p,$$

the required result. □

Definition 3.2. *A function F is said to be a pseudo preinvex function with respect to strictly positive bifunction $W(.,.)$, if*

$$\begin{aligned} & e^{F(v)} < e^{F(u)} \\ & \Rightarrow \\ & e^{F(u+\lambda\eta(v,u))} < e^{F(u)} + \lambda(\lambda-1)W(v,u), \forall u, v \in K_\eta, \lambda \in [0,1]. \end{aligned}$$

Theorem 3.11. *If the function F is higher order strongly exponentially preinvex function such that $F(v) < F(u)$, then the function F is higher order strongly exponentially pseudo preinvex.*

Proof. Since $F(v) < F(u)$ and F is higher order strongly preinvex function, then
$\forall u, v \in K_\eta, \quad \lambda \in [0,1]$, we have

$$\begin{aligned} e^{F(u+\lambda\eta(v,u))} &\leq e^{F(u)} + \lambda(e^{F(v)} - e^{F(u)}) \\ &\quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p \\ &< e^{F(u)} + \lambda(1-\lambda)(e^{F(v)} - e^{F(u)}) \\ &\quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p \\ &= e^{F(u)} + \lambda(\lambda-1)(e^{F(u)} - e^{F(v)}) \\ &\quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p \\ &< e^{F(u)} + \lambda(\lambda-1)W(u,v) \\ &\quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p, \forall u, v \in K_\eta, \end{aligned}$$

□

where $W(u,v) = e^{F(u)} - e^{F(v)} > 0$. This shows that the function F is higher order strongly exponentially pseudo preinvex.

We now discuss the optimality for the differentiable higher order strongly exponentially preinvex functions.

Theorem 3.12. *Let F be a differentiable higher order strongly exponentially preinvex function with modulus $\mu > 0$. If $u \in K_\eta$ is the minimum of the function F, then*

$$e^{F(v)} - e^{F(u)} \geq \mu \|\eta(v,u)\|^p, \quad \forall u, v \in K_\eta. \tag{15}$$

Proof. Let $u \in K_\eta$ be a minimum of the function F. Then

$$e^{F(u)} \leq e^{F(v)}, \forall v \in K_\eta. \tag{16}$$

Since K_η is an invex set, so, $\forall u, v \in K_\eta, \quad \lambda \in [0,1]$,

$$v_\lambda = u + \lambda \eta(v,u) \in K_\eta.$$

Taking $v = v_\lambda$ in (16), we have

$$0 \leq \lim_{\lambda \to 0} \{\frac{e^{F(u+\lambda\eta((v.u))} - e^{F(u)}}{\lambda}\} = \langle F'(u), \eta(v.u)\rangle. \tag{17}$$

Since F is differentiable higher order strongly exponentially preinvex function, so

$$\begin{aligned} e^{F(u+\lambda\eta(v,u))} &\leq e^{F(u)} + \lambda(e^{F(v)} - e^{F(u)}) \\ &\quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v.u)\|^p, \forall u, v \in K_\eta, \end{aligned}$$

from which, using (17), we have

$$\begin{aligned} e^{F(v)} - e^{F(u)} &\geq \lim_{\lambda \to 0} \{\frac{e^{F(u+\lambda\eta(v,u))} - e^{F(u)}}{\lambda}\} \\ &\quad +\mu\{\lambda^{p-1}(1-\lambda) + (1-\lambda)^p\}\|\eta(v,u)\|^p \\ &= \langle e^{F(u)} F'(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^p, \end{aligned}$$

the required result (15). □

Remark: We would like to mention that, if $u \in K_\eta$ satisfies the inequality

$$\langle e^{F(u)} F'(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^p \geq 0, \quad \forall u, v \in K_\eta, \tag{18}$$

then $u \in K_\eta$ is the minimum of the function F. The inequality of the type (18) is called the strongly exponentially variational-like inequality and appears to new one.

It is worth mentioning that inequalities of the type (18) may not arise as the minimization of the preinvex functions. This motivated us to consider a more general variational-like inequality of which (18) is a special case. To be more precise, for given operator T, bifunction $\eta(.,.)$ and a constant $\mu > 0$, consider the problem of finding $u \in K_\eta$, such that

$$\langle e^{Tu}, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^p \geq 0, \forall v \in K_\eta, p \geq 1, \tag{19}$$

which is called higher order strongly variational-like inequality. It is an interesting problem from both analytically and numerically point of view.

It is well known that each strongly convex functions is of the form $f \pm \|.\|^2$, where f is a convex function. Similar result is proved for the higher order strongly exponentially preinvex convex functions. In this direction, we have:

Theorem 3.13. *Let f be a higher order strongly exponentially affine preinvex function. Then F is a higher order strongly exponentially preinvex convex function, if and only if, $H = F - f$ is a exponentially preinvex function.*

Proof. Let f be strongly exponentially affine preinvex function. Then

$$e^{f(u+\lambda(v-u)} = (1-\lambda)e^{f(u)} + \lambda e^{f(v)} - \mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p. \quad (20)$$

From the higher order strongly exponentially preinvexity of F, we have

$$\begin{aligned} e^{F((u+\lambda\eta(v-u))} &\leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} \\ &\quad -\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p. \end{aligned} \quad (21)$$

From (20) and (21), we have

$$e^{F(u+\lambda\eta(v,u)} - e^{f((u+\lambda\eta(v,u)} \leq (1-\lambda)(e^{F(u)} - e^{f(u)}) + \lambda(e^{F(v)} - e^{f(v)}), \quad (22)$$

from which it follows that

$$\begin{aligned} e^{H(u+\lambda\eta(v,u)} &= e^{F(u+\lambda\eta(v,u))} - e^{f((u+\lambda\eta(v,u))})^p \\ &\leq (1-\lambda)e^{F(u)} + \lambda e^{F(v)} - (1-\lambda)e^{f(u)} - \lambda e^{f(v)}, \\ &= (1-\lambda)(e^{F(u)} - e^{f(u)}) + \lambda(e^{F(v)} - e^{f(v)}), \end{aligned}$$

which show that $g = F - f$ is an exponentially preinvex function.
The inverse implication is obvious. □

We would like to remark that one can show that a function F is a strongly exponentially preinvex function, if and only if, F is strongly exponentially affine preinvex function essentially using the technique of Adamek [1] and Noor et al. [**?**].

We would like to point that the higher order strongly exponentially convex is also a Wright higher order strongly convex functions. From the definition 2.2, we have

$$\begin{aligned} e^{F((u+\lambda)(v,u)} &+ e^{F(v+\lambda(u,v)} \leq e^F(u) + e^F(v) \\ &- 2\mu\{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\}\|\eta(v,u)\|^p, \forall u,v \in K_\eta, \lambda \in [0,1], \end{aligned}$$

which is called the Wright higher order strongly exponentially convex function. It is an interesting problem to study the properties and applications of the Wright higher order strongly exponentially convex functions.

We now derive new parallelogram laws for uniformly Banach spaces as a novel application of higher order strongly exponentially preinvex functions.

Taking $e^{F(u)} = \|u\|^p$ in Definition 2.3, we have

$$\begin{aligned} \|u+\lambda\eta(v,u)\|^p &= (1-\lambda)\|u\|^p + \lambda\|v\|^p \\ &- \mu\{\lambda^p(1-\lambda)+\lambda(1-\lambda)^p\}\|\eta(v,u)\|^p, \\ & \quad \forall u,v \in K_\eta, \lambda \in [0,1], p \geq 1. \end{aligned} \tag{23}$$

Taking $\lambda = \frac{1}{2}$ in 23, we have

$$\|\frac{2u+\eta(v,u)}{2}\|^p + \mu\frac{1}{2^p}\|\eta(v,u)\|^p = \frac{1}{2}\|u\|^p + \frac{1}{2}\|v\|^p, \forall u,v \in K_\eta, \tag{24}$$

which implies that

$$\|2u+\eta(v,u)\|^p + \mu\|\eta(v,u)\|^p = 2^{p-1}\{\|u\|^p + \|v\|^p\}, \forall u,v \in K_\eta, p \geq 1, \tag{25}$$

which is known as the parallelogram-like laws for the Banach spaces involving bifunction $\eta(v,u)$.

If $\eta(v,u) = v-u$, then (25) reduces to the parallelogram-like law as:

$$\|v+u\|^p + \mu\|v-u\|^p = 2^{p-1}\{\|u\|^p + \|v\|^p\}, \forall u,v \in K, \tag{26}$$

which are known as the parallelogram-like law for the uniform Banach spaces involving the preinvex functions. Xi [55] obtained these characterizations of p-uniform convexity and q-uniform smoothness of a Banach space via the functionals $\|.\|^p$ and $\|.\|^q$, respectively. Bynum [13] and Chen et al [14, 15] have studied the properties and applications of the parallelogram laws for the Banach spaces. For the applications of the parallelogram laws in Banach spaces in prediction theory and applied sciences, see [13, 14, 15] and the references therein. For $p = 2$, one can obtain the following parallelogram:

$$\|v+u\|^2 + \mu\|v-u\|^2 = 2\{\|u\|^2 + \|v\|^2\}, \forall u,v \in K,$$

which characterizes the inner product spaces(Hilbert space).

Conclusion

In this paper, we have introduced and studied a new class of preinvex functions with respect to any arbitrary bifunction. which is called the higher order strongly exponentially generalized preinvex function. It is shown that several new classes of strongly preinvex and convex functions can be obtained as special cases of the higher order strongly exponentially generalized preinvex functions. Some basic properties of these functions are explored. The ideas and techniques of this paper may motivate further research.

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