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ON n-HYPONOHRMALITY FOR BACKWARD EXTENSIONS OF BERGMAN WEIGHTED SHIFTS†

YANWU DONG, GUIJUN ZHENG AND CHUNJI LI*

Abstract. In this paper, we discuss the backward extensions of Bergman shifts $W_{\alpha(m)}$, where

$$\alpha\left(m\right):\sqrt{\frac{m}{m+1}},\sqrt{\frac{m+1}{m+2}},\ldots,\left(m\in\mathbb{N}\right).$$

We obtained a complete description of the n-hynonormality for backward one, two and three step extensions.

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1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $L(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator T in $L(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. For $A, B \in L(\mathcal{H})$, let $[A, B] = AB - BA$. We say that an n-tuple $T = (T_1, \cdots, T_n)$ of operators in $L(\mathcal{H})$ is *hyponormal* if the operator matrix $\left([T_j^*, T_i]\right)_{i,j=1}^n$ is positive on the direct sum of n copies of H. For a positive integer k, $T \in L(\mathcal{H})$ is *k-hyponormal* if $\left(I, T, \cdots, T^k\right)$ is hyponormal. It is well known from Bram-Halmos criterion that T is subnormal if and only if T is k-hyponormal for all $k \in \mathbb{N}$([3], [4]). Thus the implications 'subnormal$\Rightarrow \cdots \Rightarrow$ 2-hyponormal $\Rightarrow$hyponormal' hold, but each converse is not true in general. Since Curto in 1990 introduced a bridge between the hyponormality and subnormality in the concept of k-hyponormality ([2]), many operator theorists have studied these classes of operators until now.

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In the study of these classes, the weighted shifts have played an important roles ([1], [2], [3], [4], [6], [7], [8], [14], etc.).

Let $\{e_n\}_{n=0}^{\infty}$ be the canonical orthonormal basis for Hilbert space $l^2(\mathbb{Z}_+)$ and let $\alpha := \{\alpha_n\}_{n=0}^{\infty}$ be a bounded sequence of positive numbers. Let W_α be a unilateral weighted shift defined by $W_\alpha e_n := \alpha_n e_{n+1}\,(n \geq 0)$. The *moments* of W_α are usually defined by $\gamma_0 := 1, \gamma_i := \alpha_0^2 \cdots \alpha_{i-1}^2\,(i \geq 1)$. We consider k variables $x_i\,(i = 1, \cdots, k)$ satisfying $0 < x_k \leq \cdots \leq x_2 \leq x_1$ and denote an augmented weighted sequence by

$$\alpha(x_1, \cdots, x_k) : x_k, \cdots, x_2, x_1, \alpha_0, \alpha_1, \cdots, (k \geq 1).$$

Let W_α be a p-hyponormal weighted shift and let $k, p, q \in \mathbb{N}$ with $q \leq p$. Then we may consider $W_{\alpha(x_1,\cdots,x_k)}$ as a *backward k-step extension* of W_α and describe the set

$$HE_k(\alpha, q) := \left\{(x_1, \cdots, x_k) : W_{\alpha(x_1,\cdots,x_k)} \text{ is } q\text{-hyponormal}\right\}.$$

Many works have been done in this problem ([7], [8], [10], [11], [13], etc.). In this paper, we discuss the backward extensions of Bergman shifts $W_{\alpha(m)}$, where

$$\alpha(m) : \sqrt{\frac{m}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \ldots, (m \in \mathbb{N}).$$

We obtained a complete description of the n-hynonormality for one, two and three backward extensions.

The calculations in this paper were obtained through computer experiments using the software tool Scientific WorkPlace ([15]). Some lemmas to be used in this paper are as follows.

Lemma 1.1([9, Lemma 3.1]). *Let W_α be a p-hyponormal weighted shift, $q \leq p$. Then $W_{\alpha(x_1,\cdots,x_k)}$ is q-hyponormal if and only if the Hankel matrices*

$$M_{q+1}(k,i) := \begin{bmatrix} \frac{1}{x_1^2\cdots x_{k-i}^2} & \cdots & \frac{1}{x_1^2} & \gamma_0 & \cdots & \gamma_{q-k+i} \\ \frac{1}{x_1^2\cdots x_{k-1-i}^2} & \cdots & \gamma_0 & \gamma_1 & \cdots & \gamma_{q-k+1+i} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_1^2} & \cdots & \gamma_{k-i-2} & \gamma_{k-i-1} & \cdots & \gamma_{q-1} \\ \gamma_0 & \cdots & \gamma_{k-i-1} & \gamma_{k-i} & \cdots & \gamma_q \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{q-k+i} & \cdots & \gamma_{q-1} & \gamma_q & \cdots & \gamma_{2q-k+i} \end{bmatrix} \qquad (1.1)$$

are positive for all i with $0 \leq i \leq k-1$. Therefore we have

$$HE_k(\alpha, q) := \{(x_1, \cdots, x_k) \,|\, M_{q+1}(k,i) \geq 0,\ 0 \leq i \leq k-1\}.$$

Lemma 1.2([5, Lemma 2.1]). *For $\omega \geq 0$, the determinant $A_n(\omega)$ of the matrix with (i,j) entry $\frac{1}{\omega+i+j-1}(1 \leq i,j \leq n)$ is* [1]

[1] $\Gamma(\cdot)$ is the gamma function.

$$W_n(\omega) = (1!2!\cdots(n-1)!)^2 \frac{\Gamma(\omega+1)\Gamma(\omega+2)\cdots\Gamma(\omega+n)}{\Gamma(\omega+n+1)\Gamma(\omega+n+2)\cdots\Gamma(\omega+2n)}. \quad (1.2)$$

For our convenience, we record the following five determinants which will be useful in the sequel:

$$\Delta_{m,n}^{(1)} = \begin{vmatrix} \frac{1}{m} & \frac{1}{m+1} & \cdots & \frac{1}{m+n-1} \\ \frac{1}{m+1} & \frac{1}{m+2} & \cdots & \frac{1}{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+n-1} & \frac{1}{m+n} & \cdots & \frac{1}{m+2n-2} \end{vmatrix},$$

$$\Delta_{m,n}^{(2)} = \begin{vmatrix} \frac{1}{m} & \frac{1}{m+2} & \cdots & \frac{1}{m+n} \\ \frac{1}{m+1} & \frac{1}{m+3} & \cdots & \frac{1}{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+n-1} & \frac{1}{m+n+1} & \cdots & \frac{1}{m+2n-1} \end{vmatrix},$$

$$\Delta_{m,n}^{(3)} = \begin{vmatrix} \frac{1}{m} & \frac{1}{m+2} & \cdots & \frac{1}{m+n} \\ \frac{1}{m+2} & \frac{1}{m+4} & \cdots & \frac{1}{m+n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+n} & \frac{1}{m+n+2} & \cdots & \frac{1}{m+2n} \end{vmatrix},$$

$$\Delta_{m,n}^{(4)} = \begin{vmatrix} \frac{1}{m} & \frac{1}{m+3} & \cdots & \frac{1}{m+n+1} \\ \frac{1}{m+1} & \frac{1}{m+4} & \cdots & \frac{1}{m+n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+n-1} & \frac{1}{m+n+2} & \cdots & \frac{1}{m+2n} \end{vmatrix},$$

$$\Delta_{m,n}^{(5)} = \begin{vmatrix} \frac{1}{m} & \frac{1}{m+1} & \frac{1}{m+3} & \cdots & \frac{1}{m+n} \\ \frac{1}{m+1} & \frac{1}{m+2} & \frac{1}{m+4} & \cdots & \frac{1}{m+n+1} \\ \frac{1}{m+2} & \frac{1}{m+3} & \frac{1}{m+5} & \cdots & \frac{1}{m+n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+n-1} & \frac{1}{m+n} & \frac{1}{m+n+2} & \cdots & \frac{1}{m+2n-1} \end{vmatrix}.$$

By using Lemma 1.2, we can obtain the following formulas, we omit the tedious calculations here.

Lemma 1.3. *When $n \geq 3$, for the above notation, we have the followings:* [2]

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[2] $\binom{\cdot}{\cdot}$ is the binomial function.

$$\Delta_{m,n}^{(1)} = \tfrac{(1!2!\cdots(n-1)!)^2}{(n!)^n}\left(\prod_{l=n}^{2n-1}\binom{m+l-1}{n}\right)^{-1},$$
$$\Delta_{m,n}^{(2)} = \tfrac{n(1!2!\cdots(n-1)!)^2}{(n!)^n}\binom{m+n}{n}\left(\prod_{l=n-1}^{2n-1}\binom{m+l}{n}\right)^{-1},$$
$$\Delta_{m,n}^{(3)} = \tfrac{(m+n+1)(n!)^2(1!2!\cdots(n-2)!)^2}{m(m+2)((n+1)!)^{n-1}}\left(\prod_{l=n+1}^{2n-1}\binom{m+l+1}{n+1}\right)^{-1},$$
$$\Delta_{m,n}^{(4)} = \tfrac{(n-1)!(n+1)!(m-1)!(1!2!\cdots(n-2)!)^2}{2(m+n-1)!(n!)^{n-1}}\left(\prod_{l=n+1}^{2n-1}\binom{m+l+1}{n}\right)^{-1},$$
$$\Delta_{m,n}^{(5)} = \tfrac{(1!2!\cdots(n-1)!)^2}{2(n!)^{n-1}(n-2)!}\binom{m+n+1}{n}\left(\prod_{l=n-3}^{2n-3}\binom{m+l+2}{n}\right)^{-1}.$$

For the binomial function, we can easily obtain the following formulas through simple calculation.

Lemma 1.4 *If we let* $\Omega = \binom{m+n-3}{n-2}^{-1}$, *then we have the followings:*

(1) $\binom{m+n-1}{n}^{-1} = \frac{n(n-1)\Omega}{(m+n-1)(m+n-2)}$,

(2) $\binom{m+n-1}{n-1}^{-1} = \frac{m(n-1)\Omega}{(m+n-1)(m+n-2)}$,

(3) $\binom{m+n-1}{n-2}^{-1} = \frac{m(m+1)\Omega}{(m+n-1)(m+n-2)}$,

(4) $\binom{m+n-2}{n-1}^{-1} = \frac{(n-1)\Omega}{m+n-2}$,

(5) $\binom{m+n-2}{n-2}^{-1} = \frac{m\Omega}{m+n-2}$.

2. Main results

2.1. One-step backward extensions. For m be a positive and we consider a weight sequence as follows:

$$\alpha(x;m) : \sqrt{x}, \sqrt{\frac{m}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \ldots, (m \geq 2). \tag{2.1}$$

We can rewrite the result as following.

Theorem 2.1([5, Theorem 3.2])**.** *Let* $W_{\alpha(x;m)}$ *be a weighted shift with weight* $\alpha(x;m)$ *in* (2.1). *Then* $W_{\alpha(x;m)}$ *is n-hyponormal if and only if*

$$0 < x \leq H_1(m,n) := \frac{m-1}{m}\left(1 - \binom{m+n-1}{n}^{-2}\right)^{-1}.$$

Remark. $H_1(m,n) \leq \frac{m}{m+1}$, for any $n \in \mathbb{N}$. In fact, let $X := \binom{m+n-1}{n}$, then

$$\begin{aligned} \frac{m}{m+1} - H_1(m,n) &= \frac{m}{m+1} - \frac{m-1}{m}\left(\frac{X^2}{X^2-1}\right) \\ &= \frac{(X-m)(X+m)}{m(X-1)(X+1)(m+1)}, \end{aligned}$$

and since $m \geq 2$, we can show that:

(1) $X = \frac{(m+n-1)!}{n!(m-1)!} \geq m$, by mathematical induction on n,

(2) $X \geq n+1$.

Thus we know $\frac{m}{m+1} \geq H_1(m,n)$, and $\lim_{n\to\infty} X = +\infty$. Therefore, we obtain $\lim_{n\to\infty} H_1(m,n) = \frac{m-1}{m}$.

Proposition 2.2. *Let $W_{\alpha(x;m)}$ be a weighted shift with weight $\alpha(x;m)$ in (2.1). Then $W_{\alpha(x;m)}$ is subnormal if and only if $0 < x \leq \frac{m-1}{m}$.*

We give the following computational values.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$\cdots$	$n=\infty$
$m=2$	$\frac{9}{16}$	$\frac{8}{15}$	$\frac{25}{48}$	$\frac{18}{35}$	$\frac{49}{96}$	$\frac{32}{63} \approx 0.508$	$\to$	$\frac{1}{2}$
$m=3$	$\frac{24}{35}$	$\frac{200}{297}$	$\frac{75}{112}$	$\frac{147}{220}$	$\frac{1568}{2349}$	$\frac{864}{1295} \approx 0.667$	$\to$	$\frac{2}{3}$
$m=4$	$\frac{25}{33}$	$\frac{100}{133}$	$\frac{1225}{1632}$	$\frac{784}{1045}$	$\frac{5292}{7055}$	$\frac{10800}{14399} \approx 0.750$	$\to$	$\frac{3}{4}$
$m=5$	$\frac{45}{56}$	$\frac{245}{306}$	$\frac{3920}{4899}$	$\frac{63504}{79375}$	$\frac{35280}{44099}$	$\frac{87120}{108899} \approx 0.800$	$\to$	$\frac{4}{5}$
$m=6$	$\frac{147}{176}$	$\frac{1568}{1881}$	$\frac{2646}{3175}$	$\frac{52920}{63503}$	$\frac{177870}{213443}$	$\frac{522720}{627263} \approx 0.833$	$\to$	$\frac{5}{6}$
$m=7$	$\frac{224}{261}$	$\frac{6048}{7055}$	$\frac{37800}{44099}$	$\frac{182952}{213443}$	$\frac{731808}{853775}$	$\frac{17667936}{20612585} \approx 0.857$	$\to$	$\frac{6}{7}$
$m=8$	$\frac{162}{185}$	$\frac{1800}{2057}$	$\frac{27225}{31114}$	$\frac{78408}{89609}$	$\frac{368082}{420665}$	$\frac{10306296}{11778623} \approx 0.875$	$\to$	$\frac{7}{8}$
$m=9$	$\frac{225}{253}$	$\frac{3025}{3403}$	$\frac{27225}{30628}$	$\frac{184041}{207046}$	$\frac{1002001}{1127251}$	$\frac{4601025}{5176153} \approx 0.889$	$\to$	$\frac{8}{9}$
$m=10$	$\frac{605}{672}$	$\frac{14520}{16133}$	$\frac{306735}{340816}$	$\frac{6012006}{6680005}$	$\frac{5010005}{5566672}$	$\frac{13087360}{14541511} \approx 0.900$	$\to$	$\frac{9}{10}$

From the table, we can obtain the k-hyponormalities easily, for example, $W_{\alpha(x;5)}$ is 4-hyponormal if and only if $0 < x \leq \frac{3920}{4899}$, where $\alpha(x;5) : \sqrt{x}, \sqrt{\frac{5}{6}}, \sqrt{\frac{6}{7}}, \ldots$.

2.2. Two-step backward extensions. Now we discuss the two-step backward extensions of weighted shift. For m be a positive and we consider a weight sequence as follows:

$$\alpha(x,y;m) : \sqrt{y}, \sqrt{x}, \sqrt{\frac{m}{m+1}}, \sqrt{\frac{m+1}{m+2}}, \ldots, (m \geq 3). \tag{2.2}$$

Theorem 2.3. *Let $W_{\alpha(x,y;m)}$ be a weighted shift with weight $\alpha(x,y;m)$ in (2.2). Then $W_{\alpha(x,y;m)}$ is n-hyponormal if and only if*

(i) $0 < x \le \frac{m-1}{m}\left(1-\binom{m+n-1}{n}^{-2}\right)^{-1}$,

(ii) $0 < y \le \min\left\{\frac{x}{A_2x^2+A_1x+A_0}, x\right\}$, *where*

$$A_2 = \tfrac{m}{m-2} - \tfrac{2mn(m+n-1)}{(m-1)^2} + \tfrac{m^2}{(m-1)^2}\binom{m+n-1}{m}^2 - \tfrac{m}{m-2}\binom{m+n-2}{n}^{-2},$$
$$A_1 = \tfrac{2}{1-m}\left(m\binom{m+n-1}{m}^2 - n(m+n-1)\right), A_0 = \binom{m+n-1}{m}^2. \tag{2.3}$$

Proof. Let $\alpha(x,y;m)$ be given in (2.2). Then the moments of W_α are as follows:

$$\gamma_0 = 1, \text{ and } \gamma_k = \frac{m}{m+k}\,(k \ge 1).$$

From Lemma 1.1, we know that $W_{\alpha(x,y;m)}$ is n-hyponormal if and only if two Hankel matrices $M_{n+1}(2,0)$ and $M_{n+1}(2,1)$ are positive. First we consider the positivity of matrix $M_{n+1}(2,1)$, where

$$M_{n+1}(2,1) = \begin{bmatrix} \frac{1}{x} & 1 & \frac{m}{m+1} & \cdots & \frac{m}{m+n-1} \\ 1 & \frac{m}{m+1} & \frac{m}{m+2} & \cdots & \frac{m}{m+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{m}{m+n-1} & \frac{m}{m+n} & \frac{m}{m+n+1} & \cdots & \frac{m}{m+2n-1} \end{bmatrix}.$$

Since

$$\det M_{n+1}(2,1) = m^{n+1}\left(\left(\frac{1}{mx} - \frac{1}{m-1}\right)\Delta^{(1)}_{m+1,n} + \Delta^{(1)}_{m-1,n+1}\right),$$

and by Lemma 1.3, we have $\det M_{n+1}(2,1) \ge 0$ if and only if

$$0 < x \le \frac{(m-1)\,\Delta^{(1)}_{m+1,n}}{m\left(\Delta^{(1)}_{m+1,n} - (m-1)\,\Delta^{(1)}_{m-1,n+1}\right)} = \frac{m-1}{m}\left(1-\binom{m+n-1}{n}^{-2}\right)^{-1}.$$

Next we consider the positivity of matrix $M_{n+1}(2,0)$, where

$$M_{n+1}(2,0) = \begin{bmatrix} \frac{1}{xy} & \frac{1}{x} & 1 & \cdots & \frac{m}{m+n-2} \\ \frac{1}{x} & 1 & \frac{m}{m+1} & \cdots & \frac{m}{m+n-1} \\ 1 & \frac{m}{m+1} & \frac{m}{m+2} & \cdots & \frac{m}{m+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{m}{m+n-2} & \frac{m}{m+n-1} & \frac{m}{m+n} & \cdots & \frac{m}{m+2n-2} \end{bmatrix}.$$

Since

$$\frac{\det M_{n+1}(2,0)}{m^{n+1}} = \left(\frac{1}{mxy} - \frac{1}{m-2}\right)\Delta^{(1)}_{m,n} - \left(\frac{1}{mx} - \frac{1}{m-1}\right)^2\Delta^{(1)}_{m+2,n-1}$$

$$-2\left(\frac{1}{mx}-\frac{1}{m-1}\right)\Delta_{m-1,n}^{(2)}+\Delta_{m-2,n+1}^{(1)},$$

and by Lemma 1.3, we have $\det M_{n+1}(2,0)\geq 0$ if and only if $0<y\leq \frac{x}{A_2x^2+A_1x+A_0}$, where A_0, A_1 and A_2 are as in (2.3). The proof is complete. □

By Theorem 2.3, we can obtain the following results.

Corollary 2.4([5, Theorem 3.6]). *Let* $\alpha(x,y;3):\sqrt{y},\sqrt{x},\sqrt{\frac{3}{4}},\sqrt{\frac{4}{5}},\ldots$ *be a weighted shift. Then* $W_{\alpha(x,y;3)}$ *is n-hyponormal if and only if*

(1) $0<x\leq \frac{2}{3}\frac{(n+1)^2(n+2)^2}{n(n+3)(n^2+3n+4)}$,

(2) $0<y\leq \min\left\{\frac{x}{A_2x^2+A_1x+A_0},x\right\}$, *where*

$$\begin{aligned}
A_0 &= \frac{1}{36}n^2(n+1)^2(n+2)^2,\\
A_1 &= -\frac{1}{12}n(n-1)(n+2)(n+3)\left(n^2+2n+4\right),\\
A_2 &= \frac{n(n+2)(n-1)(n+3)\left(n^4+4n^3+9n^2+10n-8\right)}{16(n+1)^2}.
\end{aligned}$$

Corollary 2.5. *Let* $\alpha(x,y;4):\sqrt{y},\sqrt{x},\sqrt{\frac{4}{5}},\sqrt{\frac{5}{6}},\ldots$ *be a weighted shift. Then* $W_{\alpha(x,y;4)}$ *is n-hyponormal if and only if*

(1) $0<x\leq \frac{3}{4}\frac{(n+1)^2(n+2)^2(n+3)^2}{n(n+4)(n^2+2n+3)(n^2+6n+11)}$,

(2) $0<y\leq \min\left\{\frac{x}{A_2x^2+A_1x+A_0},x\right\}$, *where*

$$\begin{aligned}
A_0 &= \frac{1}{576}n^2(n+1)^2(n+2)^2(n+3)^2,\\
A_1 &= -\frac{1}{216}n(n-1)(n+3)(n+4)\left(n^4+6n^3+17n^2+24n+36\right),\\
A_2 &= \frac{n(n-1)(n+4)(n+3)}{324(n+2)^2(n+1)^2}\\
&\quad\times\left(n^8+12n^7+66n^6+216n^5+477n^4+756n^3+680n^2+96n-360\right).
\end{aligned}$$

2.3. Three-step backward extensions. Next we discuss the three-step backward extensions of weighted shift. For m be a positive and we consider a weight sequence as follows:

$$\alpha(x,y,z;m):\sqrt{z},\sqrt{y},\sqrt{x},\sqrt{\frac{m}{m+1}},\sqrt{\frac{m+1}{m+2}},\ldots,(m\geq 4). \tag{2.4}$$

Theorem 2.6. *Let* $0<z\leq y\leq x$ *and* $W_{\alpha(x,y,z;m)}$ *be a weighted shift with weight* $\alpha(x,y,z;m)$ *in* (2.4). *Then* $W_{\alpha(x,y,z;m)}$ *is n-hyponormal if and only if*

(i) $0 < x \leq \frac{m-1}{m}\left(1-\binom{m+n-1}{n}^{-2}\right)^{-1}$,

(ii) $0 < y \leq \min\left\{\frac{x}{A_2x^2+A_1x+A_0}, x\right\}$, *where* A_0, A_1 *and* A_2 *are as in* (2.3),

(iii) $0 < z \leq \min\left\{\frac{9m^3(m-1)^2(m+1)(m-2)^2(m-3)(n+1)^2xy(B_1x-B_0)}{(C_6x^3+C_5x^2+C_4x+C_3)y^2+C_2x^2y+C_1xy+C_0x}, y\right\}$, *where*

$$
\begin{aligned}
B_0 &= m-1, \quad B_1 = m\left(1-\frac{(n-1)^2\,\Omega^2}{(m+n-2)^2}\right),\\
C_0 &= -9m^3\,(m-1)^3\,(m-2)^2\,(m-3)\,(m+1)\,(n+1)^2\,,\\
C_1 &= 18m^2\,(m-1)^3\,(m-2)^2\,(m-3)\,(m+1)\,(n-1)\,(m+n-1)\,(n+1)^2\,,\\
C_2 &= -18m^4\,(m-1)^2\,(m-2)\,(m-3)\,(m+1)\,(n+1)^2\\
&\quad\times\left(\frac{(m-2)\,n^2+(m-2)^2\,n-2\,(m-1)^2}{m}+\frac{n\,(n-1)^2\,\Omega^2}{(m+n-2)}\right),\\
C_3 &= -9\,(m-1)^3\,(m-2)^2\,(m-3)\,(n+1)^2\,(m+n-1)^2\,(m+n-2)^2\,\Omega^{-2},\\
C_4 &= -9m\,(m-1)^2\,(m-2)^2\,(m-3)\,(n+1)^2\,(m+n-1)^2\,(m+n-2)^2\\
&\quad\times\left(\frac{m\,(n-1)\left((2m+1)\,n^2+(2m+1)\,(m-2)\,n-m\,(m-1)\right)}{(m+n-1)\,(m+n-2)^2}-3\Omega^{-2}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
C_5 &= 9m^2\,(m-1)\,(m-2)\,(n+1)^2\,(m+n-1)^2\,(m+n-2)^2\\
&\quad\times\left(c_{51}\Omega^2-3\,(m-2)\,(m-3)\,\Omega^{-2}+c_{52}\right),\\
C_6 &= 9m^2\,(n+1)^2\,(m+n-1)^2\,(m+n-2)^2\\
&\quad\times\left(\frac{\left(c_{61}-c_{62}\Omega^2+c_{63}\Omega^4\right)}{(m+n-1)^2\,(m+n-2)^4}+m\,(m-2)^2\,(m-3)\,\Omega^{-2}\right),
\end{aligned}
$$

with

$$
\begin{aligned}
c_{51} &= \frac{m^2n\,(m+1)\,(n-1)^2\left((2m-5)\,n^2+(m-2)\,(2m-5)\,n-(m-1)\,(m-3)\right)}{(m+n-1)^2\,(m+n-2)^3},\\
c_{52} &= \frac{m\left(q_1(m)\,n^4+q_2(m)\,n^3+q_3(m)\,n^2-q_4(m)\,n+q_5(m)\right)}{(m+n-2)^2\,(m+n-1)^2},
\end{aligned}
$$

$$
\begin{aligned}
q_1(m) &= 2\,(m-3)\,(2m+1)\,(m-2)\,,\\
q_2(m) &= 4\,(2m+1)\,(m-3)\,(m-2)^2\,,\\
q_3(m) &= 2\,(m-3)\left(2m^4-15m^3+27m^2-6m-11\right),\\
q_4(m) &= 2\,(m-1)\,(m-2)\,(m-3)\left(4m^2-5m-3\right),
\end{aligned}
$$

$$q_5(m) = 3\left(m^3 - 5m^2 + 4m + 2\right)(m-1)^2,$$

$$\begin{aligned}
c_{61} &= -m^2(m+n-2)^2\left(p_1(m)n^4 + p_2(m)n^3 + p_3(m)n^2 - p_4(m)n + p_5(m)\right), \\
c_{62} &= m^3(m+1)(n-1)^2\left(p_6(m)n^4 + p_7(m)n^3 + p_8(m)n^2 - p_9(m)n + p_{10}(m)\right), \\
c_{63} &= m^3n^2(m+1)(n-1)^4,
\end{aligned}$$

$$\begin{aligned}
p_1(m) &= (2m+1)(m-3)(m-2)^2, \\
p_2(m) &= 2(2m+1)(m-3)(m-2)^3, \\
p_3(m) &= (m-2)(m-3)\left(2m^4 - 16m^3 + 28m^2 - 5m - 12\right), \\
p_4(m) &= (m-1)(m-3)\left(5m^2 - 5m - 4\right)(m-2)^2, \\
p_5(m) &= \left(3m^4 - 17m^3 + 25m^2 - 3m - 12\right)(m-1)^2, \\
p_6(m) &= (m-2)(2m-5), \\
p_7(m) &= 2(2m-5)(m-2)^2, \\
p_8(m) &= (m-4)\left(2m^3 - 12m^2 + 22m - 13\right), \\
p_9(m) &= (m-1)(m-2)(m-3)(3m-4), \\
p_{10}(m) &= (m-1)^2(m-2)^2.
\end{aligned}$$

Proof. Let $\alpha(x,y,z;m)$ be given in (2.4). Then the moments of W_α are as follows:

$$\gamma_0 = 1, \text{ and } \gamma_k = \frac{m}{m+k}\,(k \geq 1).$$

Also from Lemma 1.1 we know that $W_{\alpha(x,y,z;m)}$ is n-hyponormal if and only if three Hankel matrices $M_{n+1}(3,0)$, $M_{n+1}(3,1)$ and $M_{n+1}(3,2)$ are positive. We have discussed the positivity of matrices $M_{n+1}(3,1)\,(= M_{n+1}(2,0))$ and $M_{n+1}(3,2)\,(= M_{n+1}(2,1))$ in Theorem 2.3. Therefore we just need to consider the positivity of matrix $M_{n+1}(3,0)$.

Since $\det M_{n+1}(3,0) = m^{n+1}\Lambda_{n+1}(x,y,z;m)$, with

$$\begin{aligned}
\Lambda_{n+1}(x,y,z;m) &= \begin{vmatrix}
\frac{1}{mxyz} & \frac{1}{mxy} & \frac{1}{mx} & \cdots & \frac{1}{m+n-3} \\
\frac{1}{mxy} & \frac{1}{mx} & \frac{1}{m} & \cdots & \frac{1}{m+n-2} \\
\frac{1}{mx} & \frac{1}{m} & \frac{1}{m+1} & \cdots & \frac{1}{m+n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{m+n-3} & \frac{1}{m+n-2} & \frac{1}{m+n-1} & \cdots & \frac{1}{m+2n-3}
\end{vmatrix} \\
&= \left(\frac{1}{mxyz} - \frac{1}{m-3}\right)\left(\left(\frac{1}{mx} - \frac{1}{m-1}\right)\Delta_1 + \Delta_2\right) \\
&\quad +2\left(\frac{1}{mxy} - \frac{1}{m-2}\right)\left(\left(\frac{1}{mx} - \frac{1}{m-1}\right)\Delta_3 - \Delta_4\right)
\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{1}{mx}-\frac{1}{m-1}\right)^3\Delta_5-\left(\frac{1}{mxy}-\frac{1}{m-2}\right)^2\Delta_1\\
&-\left(\frac{1}{mx}-\frac{1}{m-1}\right)^2(\Delta_6+2\Delta_7)\\
&+\left(\frac{1}{mx}-\frac{1}{m-1}\right)(2\Delta_8+\Delta_9)+\Delta_{10},
\end{aligned}
$$

where

$$
\begin{array}{llll}
\Delta_1=\Delta^{(1)}_{m+1,n-1}, & \Delta_2=\Delta^{(1)}_{m-1,n}, & \Delta_3=\Delta^{(2)}_{m,n-1}, & \\
\Delta_4=\Delta^{(2)}_{m-2,n}, & \Delta_5=\Delta^{(1)}_{m+3,n-2}, & \Delta_6=\Delta^{(3)}_{m-1,n-1}, & \\
\Delta_7=\Delta^{(4)}_{m-1,n-1}, & \Delta_8=\Delta^{(5)}_{m-2,n}, & \Delta_9=\Delta^{(3)}_{m-3,n}, & \Delta_{10}=\Delta^{(1)}_{m-3,n+1},
\end{array}
$$

and by Lemma 1.2, Lemma 1.3 and Lemma 1.4, we obtain $\det M_{n+1}(3,0)\geq 0$ if and only if

$$
0<z\leq\frac{9m^3(m-1)^2(m+1)(m-2)^2(m-3)(n+1)^2xy(B_1x-B_0)}{(C_6x^3+C_5x^2+C_4x+C_3)y^2+C_2x^2y+C_1xy+C_0x},
$$

where B_0, B_1 and $C_i\,(i=0,1,2,\cdots,6)$ are described in (iii). The proof is complete. □

The authors in [12] obtained the following results, which is the case of $m=4$ in Theorem 2.6.

Corollary 2.7([12, Theorem 3.6])**.** *Let* $0<z\leq y\leq x$ *and*

$$
\alpha(x,y,z):\sqrt{z},\sqrt{y},\sqrt{x},\sqrt{\frac{4}{5}},\sqrt{\frac{5}{6}},\cdots.
$$

Then $W_{\alpha(x,y,z)}$ *is n-hyponormal if and only if*

(i) $0<x\leq\frac{3}{4}\frac{(n+1)^2(n+2)^2(n+3)^2}{n(n+4)(n^2+2n+3)(n^2+6n+11)}$,

(ii) $0<y\leq\min\left\{\frac{x}{A_2x^2+A_1x+A_0},x\right\}$, *where*

$$
\begin{aligned}
A_0 &= \frac{1}{576}n^2(n+1)^2(n+2)^2(n+3)^2,\\
A_1 &= -\frac{1}{216}n(n-1)(n+3)(n+4)r_1(n),\\
A_2 &= \frac{n(n-1)(n+4)(n+3)}{324(n+2)^2(n+1)^2}r_2(n),
\end{aligned}
$$

with

$$
\begin{aligned}
r_1(n) &= n^4+6n^3+17n^2+24n+36,\\
r_2(n) &= n^8+12n^7+66n^6+216n^5+477n^4+756n^3+680n^2+96n-360,
\end{aligned}
$$

(iii) $0<z\leq\min\left\{\frac{103680(n+1)^2xy(B_1x-B_0)}{(C_6x^3+C_5x^2+C_4x+C_3)y^2+C_2x^2y+C_1xy+C_0x},y\right\}$, *where*

$$
\begin{aligned}
B_0 &= 3, \quad B_1 = \frac{4(n-1)(n+3)\left(n^2+2\right)\left(n^2+4n+6\right)}{n^2(n+2)^2(n+1)^2}, \\
C_0 &= -311\,040\,(n+1)^2, \quad C_1 = 155\,520\,(n-1)(n+1)^2(n+3), \\
C_2 &= -\frac{207\,360\,(n-1)(n-2)(n+3)(n+4)\left(n^2+2n+3\right)}{n(n+2)}, \\
C_3 &= -27n^2(n-1)^2(n+1)^4(n+2)^2(n+3)^2, \\
C_4 &= 108\,(n-1)(n-2)(n+1)^2(n+3)(n+4)\,r_3(n), \\
C_5 &= -\frac{144\,(n-1)(n-2)(n+3)(n+4)}{n(n+2)}r_4(n), \\
C_6 &= \frac{64\,(n-2)(n-1)^2(n+3)^2(n+4)}{n(n+1)^2(n+2)}r_5(n),
\end{aligned}
$$

with

$$
\begin{aligned}
r_3(n) &= n^6+6n^5+18n^4+32n^3+69n^2+90n-72, \\
r_4(n) &= n^{10}+10n^9+47n^8+136n^7+299n^6+562n^5 \\
&\quad +265n^4-1428n^3-3060n^2-2448n+2160, \\
r_5(n) &= n^{10}+10n^9+51n^8+168n^7+435n^6+930n^5 \\
&\quad +701n^4-1540n^3-5076n^2-5616n+4320.
\end{aligned}
$$

Remark. (1) In Theorem 2.1, we assume $m \geq 2$, the authors in [5, Theorem 3.2] obtained a result for $m = 1$. That is, let $\alpha(x;1) : \sqrt{x}, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \ldots$, then $W_{\alpha(x;1)}$ is n-hyponormal if and only if $0 < x \leq \frac{1}{2\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}$.
(2) In Theorem 2.3, we assume $m \geq 3$, and in Theorem 2.6, we assume $m \geq 4$. We can obtain similar results $m = 1, 2$ for two-step backward extensions, and $m = 1, 2, 3$ for three-step backward extensions. We leave them to interested readers.

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Yanwu Dong received M.Sc. from Northeastern University. His research interests include optimization and unilateral weighted shifts.

Department of Mathematics, Zhanjiang Preschool Education College(Fundamental Education College of Lingnan Normal University), Zhanjiang 524084, Guangdong, P.R. China.
e-mail: long__feng@sina.com

Guijun Zheng received M.Sc. from Northeastern University. Her research interests include optimization and unilateral weighted shifts.

Department of Mathematics, Zhanjiang Preschool Education College(Fundamental Education College of Lingnan Normal University), Zhanjiang 524084, Guangdong, P.R. China.
e-mail: zhengguijun1981@163.com

Chunji Li received Ph.D. degree from Kyungpook National University, Korea. His research interests focus on the control theory, moment method, and unilateral weighted shifts.

Department of Mathematics, Northeastern University, Shenyang 110819, P.R. China.
e-mail: lichunji@mail.neu.edu.cn