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EXPONENTIALLY FITTED NUMERICAL SCHEME FOR SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS INVOLVING SMALL DELAYS†

MERGA AMARA ANGASU, GEMECHIS FILE DURESSA, MESFIN MEKURIA WOLDAREGAY*

Abstract. This paper deals with numerical treatment of singularly perturbed differential equations involving small delays. The highest order derivative in the equation is multiplied by a perturbation parameter ε taking arbitrary values in the interval $(0, 1]$. For small ε, the problem involves a boundary layer of width $O(\varepsilon)$, where the solution changes by a finite value, while its derivative grows unboundedly as ε tends to zero. The considered problem contains delay on the convection and reaction terms. The terms with the delays are approximated using Taylor series approximations resulting to asymptotically equivalent singularly perturbed BVPs. Inducing exponential fitting factor for the term containing the singular perturbation parameter and using central finite difference for the derivative terms, numerical scheme is developed. The stability and uniform convergence of difference schemes are studied. Using a priori estimates we show the convergence of the scheme in maximum norm. The scheme converges with second order of convergence for the case $\varepsilon = O(N^{-1})$ and for the case $\varepsilon \ll N^{-1}$, the scheme converge uniformly with first order of convergence, where N is number of mesh intervals in the domain discretization. We compare the accuracy of the developed scheme with the results in the literature. It is found that the proposed scheme gives accurate result than the one in the literatures.

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*Corresponding author.
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1. Introduction

Delay differential equations(DDEs) are differential equations where the evolution of the system does not only depend on the present state of the system but also on the past history. We call DDEs retarded type if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral DDEs. Singularly perturbed differential difference equations are differential equations in which its highest derivative is multiplied by a small perturbation parameter and having delay parameters on the terms different from the highest derivative.

The estimation of solution of delay differential equations using various numerical approaches is a paramount problem in applied mathematics. The main reason behind the extending importance in study of delay differential equations is their existence in extensive range of application fields such as robotics, control theory, biosciences, particularly in micro scale heat transfer [18], fluid dynamics [5], diffusion in polymers [10], reaction-diffusion equations [3], a lot of model in diseases or physiological processes [11, 19] etc. These equations arises during the mathematical modeling of phenomenons in physical sciences which describes a system involving feedback control with time delays. This time lag appear due to requirement of some definite time to recognize the command and to react to it. The delay models are well known in describing response of immune system, physiological circuit, relation between infection and the production of virus etc.

Notations. Throughout this paper N is denoted for the number of mesh intervals in the discretized domain. The symbol C denotes positive constant independent of c_ε and N. The norm $\|.\|$ denotes the maximum norm.

2. Statement of the Problem: Aim of the Paper

Consider a class of singularly perturbed differential equations having delay in the convection and reaction terms of the form

$$\begin{cases} -\varepsilon u''(x) + a(x)u'(x-\delta) + \beta(x)u(x) + \omega(x)u(x-\delta) = f(x), \ x \in \Omega = (0,1), \\ u(x) = \phi(x), \ x \in \Omega_L = [-\delta, 0], \ u(1) = \psi(1), \end{cases} \tag{1}$$

where $\varepsilon, (0 < \varepsilon \leq 1)$ is singular perturbation parameter and δ is delay satisfying $\delta < \varepsilon$. The functions $a(x), \beta(x), \omega(x)$ and $f(x)$ are assumed to be smooth, bounded and not a function of ε. The values of $\phi(x)$ and $\psi(1)$ are assumed finite values. We assume also the coefficients of non-derivative terms $\beta(x)$ and $\omega(x)$ satisfy

$$\beta(x) + \omega(x) \geq d^* > 0, \ \forall x \in \bar{\Omega}$$

for some constant d^*. This condition ensures that the solution of (1) exhibits boundary layer in the neighbourhood of $x = 0$ or $x = 1$ depending on the sign of the convective term $a(x)$. When $a(x) < 0$ regular boundary layer appears in the neighbourhood of $x = 0$ and for $a(x) > 0$ corresponds to existence of a boundary layer in the neighbourhood of $x = 1$. If $a(x)$ change sign, shock layer will appear on the middle of the domain [21]. The layer is maintained for $\delta \neq 0$ but sufficiently small.

The presence of perturbation parameter ε leads to bad approximation or oscillation in the computed solution while using standard numerical methods [4]. To avoid this oscillations, an unacceptably large number of mesh points are required when ε is very small. This is not practical and leads to rounding error. So, to overcome the drawbacks associated with standard numerical methods, different authors tries to develop numerical schemes that converges free from oscillations.

Lange and Muria in [9] considered the singularly perturbed differential difference equation having delay on convective and reaction terms of the problem. The authors provides insight into the appropriate use of singular perturbation techniques for more general problems. Kadelbajoo and Ramesh [6] used Taylor series approximation for the delay terms and converted the problem into asymptotically equivalent BVPs. The author's developed numerical schemes using upwind, midpoint upwind and hybrid of midpoint upwind on regular region and central finite difference on boundary layer region using piecewise uniform Shishkin mesh. Kumar and Kadalbajoo in [8], used Taylor series approximation for the delay terms and converted the problem into BVPs. The authors computed the numerical solution using B-spline collocation method on piecewise uniform Shishkin mesh.

Adilaxmi et al. in [1], approximate the problem to equivalent BVPs and solved using non standard FDM using exponential fitting factor. Phaneendra and Lula in [16] apply the Gaussian quadrature two-point formula and treat the problems. Bahgat and Hafiz in [2] apply fifth and sixth order finite difference approximation for the derivative terms and develop a finite difference scheme. Their scheme did not satisfy uniform convergence. Melesse et al. in [12, 13] considered a turning layer problem having delay on reaction term. The authors use a hybrid fitted mesh method and initial value technique for treating the problem. In [22, 23, 24] Woldaregay and Duressa have developed different fitted numerical scheme and their convergence analysis for the considered problem of the form in (1).

The main contribution in this paper is developing exponentially fitted numerical scheme which converges uniformly in maximum norm; develop the uniform convergence analysis of the scheme.

2.1. Approximations for terms with the delay. For the case delay parameter is smaller than the perturbation parameter, approximating $u'(x-\delta)$ and $u(x-\delta)$ using Taylor's series approximation is valid [17]. Since, we assumed above $\delta < \varepsilon$, the terms $u'(x-\delta)$ and $u(x-\delta)$ approximated as

$$\begin{cases} u'(x-\delta) \approx u'(x) - \delta u''(x) + O(\delta^2), \\ u(x-\delta) \approx u(x) - \delta u'(x) + \frac{\delta^2}{2} u''(x) + O(\delta^3). \end{cases} \tag{2}$$

Substituting (2) into (1) results to

$$\begin{cases} -c_{\varepsilon}(x)u''(x)+p(x)u'(x)+d(x)u(x)=f(x),\ x\in\Omega=(0,1),\\ u(0)=\phi(0),\ \ u(1)=\psi(1). \end{cases} \tag{3}$$

where $c_{\varepsilon}(x)=\varepsilon+\delta a(x)-\frac{\delta^2}{2}\omega(x)$, $p(x)=a(x)-\delta\omega(x)$ and $d(x)=\beta(x)+\omega(x)$. For small values of ε, (3) is asymptotically equivalent to (1). We assume $0 < c_{\varepsilon}(x)\leq\varepsilon-\delta M_1-\delta^2 M_2=c_{\varepsilon}$, where $a(x)\geq M_1$ and $\omega(x)\geq 2M_2$ for M_1 and M_2 are constants.

We consider first the case $p(x)\geq p^*>0$, which imply occurrence of boundary layer near the right side of the domain. The problem obtained by setting $c_{\varepsilon}=0$ in (3) is called reduced problem and given as

$$\begin{cases} p(x)u_0'(x)+d(x)u_0(x)=f(x),\ x\in\Omega=(0,1),\\ u_0(0)=\phi(0),\ \ u(1)\neq\psi(1). \end{cases} \tag{4}$$

It is a first order IVPs. For small values of c_{ε} the solution of (4) is very close to the solution of (3). The other case $p(x)\leq p^*<0$, imply the occurrence of the boundary layer on the left side of the domain.

Let us denote differential operator L for the differential equation in (3) as $Lz(x)=-c_{\varepsilon}z''(x)+p(x)z'(x)+d(x)z(x)$.

2.2. Properties of the Continuous Solution.

Lemma 2.1. *(Maximum Principle) Let z be a sufficiently smooth function defined on Ω which satisfies $z(0)\geq 0$ and $z(1)\geq 0$. Then, $Lz(x)\geq 0,\ \forall x\in\Omega$ implies that $z(x)\geq 0,\ \forall x\in\bar{\Omega}$.*

Proof. Let x^* be such that $z(x^*)=\min_{(x)\in\bar{\Omega}} z(x)$ and suppose that $z(x^*)<0$. It is clear that $x^*\notin\{0,1\}$. Since $z(x^*)=\min_{(x)\in\bar{\Omega}} z(x)$ which implies $z'(x^*)=0$ and $z''(x^*)\geq 0$ and giving that $Lz(x^*)<0$ which is contradiction to the assumption made above $Lz(x^*)\geq 0,\ \forall x\in\Omega$. Therefore $z(x)\geq 0,\ \forall x\in\bar{\Omega}$. □

Lemma 2.2. *(Stability estimate) Let $u(x)$ be the solution of (3). Then, it satisfies the bound*

$$|u(x)|\leq\frac{\|f\|}{d^*}+\max\{|\phi(0)|,|\psi(1)|\}. \tag{5}$$

where d^* is lower bound of $d(x)$ on $[0,1]$.

Proof. By defining barrier functions $\vartheta^{\pm}(x,t)=\frac{\|f\|}{d^*}+\max\{\phi(0),\psi(1)\}\pm u(x)$ and applying the maximum principle, we obtain the required bound. At the boundary points, we obtain $\vartheta^{\pm}(0)\geq 0,\quad \vartheta^{\pm}(1)\geq 0$ and for the differential operator

$$\begin{aligned}
L\vartheta^{\pm}(x) =& -c_\varepsilon \vartheta''_{\pm}(x) + p(x)\vartheta'_{\pm}(x) + d(x)\vartheta_{\pm}(x)\\
=& \mp c_\varepsilon u''(x) \pm p(x)u'(x) + d(x)\big(\frac{\|f\|}{q^*} + \max\{\phi(0), \psi(1)\} \pm u(x)\big)\\
=& d(x)\big(\frac{\|f\|}{d^*} + \max\{\phi(0), \psi(1)\}\big) \pm f(x)\\
\geq& 0, \text{since } d(x) \geq d^* > 0.
\end{aligned}$$

Implying that $L\vartheta_{\pm}(x) \geq 0,\ \forall x \in \Omega$. Hence, by maximum principle in Lemma 2.1, we obtain $\vartheta_{\pm}(x) \geq 0,\ \forall x \in \bar{\Omega}$, which gives the required bound. □

Lemma 2.3. *The derivatives of the solution of* (3) *is bounded as*

$$\big|\, u^{(k)}(x) \,\big| \leq \begin{cases} C\big(1 + c_\varepsilon^{-k} \exp(-p^* x/c_\varepsilon)\big), x \in \bar{\Omega},\ \textit{for left layer},\\ C\big(1 + c_\varepsilon^{-k} \exp(-p^*(1-x)/c_\varepsilon)\big), x \in \bar{\Omega},\ \textit{for right layer}. \end{cases} \tag{6}$$

where p^* *is lower bound of* $p(x)$ *and* $k = 0, 1, 2, 3, 4$.

Proof. See on [7] or [14]. □

3. Numerical Scheme

The domain $[0, 1]$ is discretized into N equal number of subintervals each of length h. Let $x_i = ih, i = 0, 1, 2, ..., N$ for $h = 1/N$ with $x_0 = 0, x_N = 1$. To find the numerical solution of the problem in (3), we apply a exponentially fitted operator finite difference method. To determine the exponential fitting factor we use the theory used in asymptotic method for solving singularly perturbed BVPs.

We consider and treat separately the right and left boundary layer problems.

3.1. Case I: Right boundary layer problem. In this case, the boundary layer is near $x = 1$. From the theory of singular perturbations given in [15] we get the asymptotic solution up to zeroth order approximation as

$$u(x) = u_0(x) + \frac{p(1)}{p(x)}(\psi(1) - u_0(1)) \exp\big(-\int_x^1 \big(\frac{p(x)}{c_\varepsilon} - \frac{d(x)}{p(x)}\big)dx\big) + O(c_\varepsilon). \tag{7}$$

Using Taylor series about $x = 1$ for $p(x)$ and considering c_ε is small enough and simplifying we obtain

$$u(x) = u_0(x) + (\psi(1) - u_0(1)) \exp\big(-\frac{p(1)(1-x)}{c_\varepsilon}\big), \tag{8}$$

where u_0 is the solution of the reduced problems (obtained by setting $c_\varepsilon = 0$). Considering h is small enough, the discretized form of (8) becomes

$$u(x_i) = u(ih) = u_0(ih) + (\psi(1) - u_0(1)) \exp(-p(1)(1/c_\varepsilon - i\rho)), \tag{9}$$

where $\rho = h/c_\varepsilon,\ h = 1/N$. Using Taylor's series approximation for $u_0((i+1)h)$ and $u_0((i-1)h)$ up to first order, we obtain

$$\begin{aligned} u(x_{i-1}) = u_{i-1} =& u_0(ih) + (\psi(1) - u_0(1)) \exp(-p(1)(1/c_\varepsilon - (i-1)\rho)), \\ u(x_{i+1}) = u_{i+1} =& u_0(ih) + (\psi(1) - u_0(1)) \exp(-p(1)(1/c_\varepsilon - (i+1)\rho)). \end{aligned} \tag{10}$$

To handle the influence of the singular perturbation parameter exponentially fitting factor $\sigma_1(\rho)$ is induced on the term containing the singular perturbation parameter. Using central finite difference method we write the numerical scheme as

$$L^h U_i \equiv -c_\varepsilon \sigma_1(\rho) \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + p(x_i) \frac{U_{i+1} - U_{i-1}}{2h} + d(x_i) U_i = f_i, \tag{11}$$

for $i = 1, 2, ..., N-1$, where U_i is denoted for the approximation for $u(x)$ using the above discretization at mesh point x_i and f_i is denoted for $f(x_i)$. Considering h is small, multiplying (11) by h and truncating the term $h(f_i - d(x_i)U_i)$ gives

$$-\frac{\sigma_1(\rho)}{\rho}(U_{i-1} - 2U_i + U_{i+1}) + \frac{p(x_i)}{2}(U_{i+1} - U_{i-1}) = 0. \tag{12}$$

Since

$$\begin{aligned} U_{i-1} - 2U_i + U_{i+1} =& (\psi(1) - u_0(1)) \exp(-p(1)(1/c_\varepsilon - i\rho))[e^{p(1)\rho} - 2 + e^{-p(1)\rho}], \\ U_{i+1} - U_{i-1} =& (\psi(1) - u_0(1)) \exp(-p(1)(1/c_\varepsilon - i\rho))[e^{-p(1)\rho} - e^{p(1)\rho}]. \end{aligned} \tag{13}$$

Substituting the results in (13) into (12) the fitting factor is obtained as

$$\sigma_1(\rho) = \frac{\rho p(x_i)}{2} \coth\big(\frac{\rho p(1)}{2}\big). \tag{14}$$

Hence, the required finite difference scheme becomes

$$L^h U_i = E_i U_{i-1} + F_i U_i + G_i U_{i+1} = H_i,\ i = 1, 2, ..., N-1, \tag{15}$$

where

$E_i = -\dfrac{c_\varepsilon \sigma_1(\rho)}{h^2} - \dfrac{p(x_i)}{2h},\ \ F_i = \dfrac{2c_\varepsilon \sigma_1(\rho)}{h^2} + d(x_i),\ \ G_i = -\dfrac{c_\varepsilon \sigma_1(\rho)}{h^2} + \dfrac{p(x_i)}{2h}$ and $H_i = f_i$.

3.2. Case II: Left boundary layer problem. In this case, the boundary layer is near $x = 0$. From the theory of singular perturbations given in [15] the asymptotic solution up to zeroth order approximation is given as

$$u(x) = u_0(x) + \frac{p(0)}{p(x)}(\phi(0) - u_0(0)) \exp\Big(-\int_0^x \big(\frac{p(x)}{c_\varepsilon} - \frac{d(x)}{p(x)}\big)dx\Big) + O(c_\varepsilon). \tag{16}$$

Using Taylor series at $x = 0$ for $p(x)$ and simplifying we obtain

$$u(x) = u_0(x) + (\phi(0) - u_0(0)) \exp(-\frac{p(0)x}{c_\varepsilon}), \tag{17}$$

where u_0 is the solution of the reduced problems. Using the same procedure as the right boundary layer case, the fitting factor is obtained as

$$\sigma_2(\rho) = \frac{\rho p(x_i)}{2} \coth\big(\frac{\rho p(0)}{2}\big), \tag{18}$$

and the required finite difference scheme becomes

$$L^h U_i = E_i U_{i-1} + F_i U_i + G_i U_{i+1} = H_i,\ i = 1, 2, ..., N-1, \tag{19}$$

where

$$E_i = -\frac{c_\varepsilon \sigma_2(\rho)}{h^2} - \frac{p(x_i)}{2h},\ \ F_i = \frac{2c_\varepsilon \sigma_2(\rho)}{h^2} + d(x_i),\ \ G_i = -\frac{c_\varepsilon \sigma_2(\rho)}{h^2} + \frac{p(x_i)}{2h}$$

and $H_i = f_i$.

3.3. Convergence Analysis. Let us denote the difference operators for approximating the first and second derivatives as

$$D^- u(x_i) = \frac{u_i - u_{i-1}}{h},\ D^+ u(x_i) = \frac{u_{i+1} - u_i}{h},\ D^0 u(x_i) = \frac{u_{i+1} - u_{i-1}}{2h}$$
$$\text{and } D^+D^- u(x_i) = \frac{u_{i-1} - u_i + u_{i+1}}{h^2}. \tag{20}$$

First, we need to prove the discrete comparison principle for the scheme in (15).

Lemma 3.1. *Discrete comparison principle(Existence and Uniqueness of Discrete solution). The discrete scheme $L^h U_i = H_i, i = 1, 2, ..., N-1$ and for given U_0 and U_N has a solution. If $L^h U_i \le L^h B_i$ for $i = 1, 2, ..., N-1$, and if $U_0 \le B_0$ and $U_N \le B_N$ then $U_i \le B_i$, $\forall i = 0, 1, 2, ..., N$.*

Proof. The matrix associated with operator L^h is of size $(N+1) \times (N+1)$ and satisfies the property of M matrix. See the detail proof in [7]. □

Lemma 3.2. *Let $z_i = 1 + x_i$ for $0 \le i \le N$. Then, there exist a positive constant C such that $L^h z_i \ge C$, for $1 \le i \le N-1$.*

Proof. Using the discrete operator on the mesh function z_i. For $1 \le i \le N-1$ we have

$$\begin{aligned} L^h z_i =& - c_\varepsilon \sigma(\rho) D_x^+ D_x^- z_i + p(x_i) D_x^0 z_i + d(x_i) z_i, \\ =& - c_\varepsilon \sigma(\rho) D_x^+ D_x^- (1 + x_i) + p(x_i) D_x^0 (1 + x_i) + d(x_i)(1 + x_i), \\ =& 0 + p(x_i) + d(x_i)(1 + x_i), \\ \ge& C, \text{ since } d(x_i) \text{ is bounded function.} \end{aligned}$$

□

Recalling that the maximum principle holds for L^h if it is positive.

Lemma 3.3 (*Discrete Stability Estimate*)**.** *The solution U_i of the discrete scheme in* (15) *satisfy the following bound.*

$$|U_i| \le \frac{\max|L^h U_i|}{d^*} + \max\{|\phi(0)|, |\psi(1)|\}. \tag{21}$$

Proof. Let $p = \frac{\max|L^h U_i|}{d^*} + \max\{|\phi(0)|, |\psi(1)|\}$ and define the barrier function $\vartheta_i^{\pm}$ by $\vartheta_i^{\pm} = p \pm U_i$. On the boundary points, we obtain $\vartheta_0^{\pm} = p \pm U_0 \geq 0$ and $\vartheta_N^{\pm} = p \pm U_N \geq 0$. On the discretized domain $x_i,\ \ 0 < i < N$, we obtain

$$\begin{aligned}
L^h \vartheta_i^{\pm} =& - c_\varepsilon \sigma(\rho)\big(\frac{p \pm U_{i+1} - 2(p \pm U_i) + p \pm U_{i-1}}{h^2}\big) \\
&+ p(x_i)\big(\frac{p \pm U_{i+1} - p \pm U_{i-1}}{2h}\big) + d(x_i)(p \pm U_i) \\
=& d(x_i)p \pm L^h U_i \\
=& d(x_i)\big(\frac{\max|L^h U_i|}{d^*} + \max\{|\phi(0)|, |\psi(1)|\}\big) \pm f_i \geq 0, \ \ \text{since} \ \ d(x_i) \geq d^*.
\end{aligned}$$

Using the result of Lemma 3.1 and 3.2, we obtain $\vartheta_i^{\pm} \geq 0$, $\forall x_i \in \bar{\Omega}^N$. Hence the required bound is obtained. □

The following theorem gives the truncation error bound of the developed scheme.

Theorem 3.4. *Let $u(x_i)$ and U_i be respectively the solution of the continuous problem* (3) *and the discrete scheme* (15). *Then, for sufficiently large N, the truncation error bounded as*

$$|L^h(u(x_i) - U_i)| \leq \frac{CN^{-2}}{N^{-1} + c_\varepsilon}\big(1 + c_\varepsilon^{-3} \exp(-p^*(1 - x_i)/c_\varepsilon)\big). \tag{22}$$

Proof. Let us consider the truncation error

$$\begin{aligned}
L^h(u(x_i) - U_i) =& - c_\varepsilon \sigma(\rho)(u''(x_i) - D^+ D^- u(x_i)) \\
&+ p(x_i)(u'(x_i) - D^0 u(x_i)) \\
=& - c_\varepsilon [p(x_i)\frac{\rho}{2} \coth\big(p(1)\frac{\rho}{2}\big) - 1] D^+ D^- u(x_i) \\
&+ c_\varepsilon(u''(x_i) - D^+ D^- u(x_i)) \\
&+ p(x_i)(u'(x_i) - D^0 u(x_i)),
\end{aligned}$$

since $\sigma(\rho) = p(x_i)\dfrac{\rho}{2} \coth\big(p(1)\dfrac{\rho}{2}\big)$.

Now for $z > 0$, C_1 and C_2 are constants we have

$$C_1 \frac{z^2}{z+1} \leq z \coth(z) - 1 \leq C_2 \frac{z^2}{z+1} \ \ \text{and} \ \ c_\varepsilon \frac{(N^{-1}/c_\varepsilon)^2}{N^{-1}/c_\varepsilon + 1} = \frac{N^{-2}}{N^{-1} + c_\varepsilon}. \tag{23}$$

Using Taylor series expansion for $u(x_{i-1})$ and $u(x_{i+1})$ at x_i as

$$u(x_{i\pm1}) = u(x_i) \pm h u'(x_i) + \frac{h^2}{2!} u''(x_i) \pm \frac{h^3}{3!} u^{(3)}(x_i) + \frac{h^4}{4!} u^{(4)}(x_i) + ...$$

we obtain the bound as

$$
\begin{aligned}
\left|u'(x_i) - D^0 u(x_i)\right| &\leq CN^{-2}\left\|u^{(3)}(x_i)\right\|, \\
|D^+D^-u(x_i)| &\leq C\|u''(x_i)\| \text{ and} \\
|u''(x_i) - D^+D^-u(x_i)| &\leq CN^{-2}\|u^{(4)}(x_i)\|,
\end{aligned} \tag{24}
$$

where $\left\|u^{(k)}(x_i)\right\| = \max_{x_i \in (x_0, x_N)} |u^{(k)}(x_i)|, k = 2, 3, 4$. Using these bounds in (24) and (23) we obtain

$$
\begin{aligned}
|L^h(u(x_i) - U_i)| \leq & C\frac{N^{-2}}{N^{-1} + c_\varepsilon}\|u''(x_i)\| + c_\varepsilon CN^{-2}\|u^{(4)}(x_i)\| + CN^{-2}\|u^{(3)}(x_i)\| \\
\leq & \frac{N^{-2}}{N^{-1} + c_\varepsilon}\|u''(x_i)\| + CN^{-2}[c_\varepsilon\|u^{(4)}(x_i)\| + \|u^{(3)}(x_i)\|]
\end{aligned}
$$

Using the bounds for the derivatives of the solution in Lemma 2.3 gives

$$
\begin{aligned}
|L^h(u(x_i) - U_i)| \leq & \frac{CN^{-2}}{N^{-1} + c_\varepsilon}\big(1 + c_\varepsilon^{-2}\exp(-p^*(1 - x_i)/c_\varepsilon)\big) \\
& + CN^{-2}[c_\varepsilon\big(1 + c_\varepsilon^{-4}\exp(-p^*(1 - x_i)/c_\varepsilon)\big) \\
& + \big(1 + c_\varepsilon^{-3}\exp(-p^*(1 - x_i)/c_\varepsilon)\big)] \\
\leq & \frac{CN^{-2}}{N^{-1} + c_\varepsilon}\big(1 + c_\varepsilon^{-2}\exp(-p^*(1 - x_i)/c_\varepsilon)\big) \\
& + CN^{-2}\big[\big(c_\varepsilon + c_\varepsilon^{-3}\exp(-p^*(1 - x_i)/c_\varepsilon)\big) \\
& + \big(1 + c_\varepsilon^{-3}\exp(-p^*(1 - x_i)/c_\varepsilon)\big)\big]
\end{aligned}
$$

which simplifies to

$$
|L^h(u(x_i) - U_i)| \leq \frac{CN^{-2}}{N^{-1} + c_\varepsilon}\big(1 + c_\varepsilon^{-3}\exp(-p^*(1 - x_i)/c_\varepsilon)\big), \tag{25}
$$

since $c_\varepsilon^{-3} \geq c_\varepsilon^{-2}$. □

Lemma 3.5. *For $c_\varepsilon \to 0$ and for given fixed N, we obtain*

$$
\lim_{c_\varepsilon \to 0} \max_j \frac{\exp(-p^* x_j/c_\varepsilon)}{c_\varepsilon^m} = 0, \ \lim_{c_\varepsilon \to 0} \max_j \frac{\exp(-p^*(1 - x_j)/c_\varepsilon)}{c_\varepsilon^m} = 0, \tag{26}
$$

where $x_j = jh, h = 1/N, \forall j = 1, 2, ..., N - 1$ for $m = 1, 2, 3, ...$

Proof. See on [20]. □

Theorem 3.6. *Let u and U be respectively the solution of continuous problem in* (3) *and the discrete scheme in* (15)*. Then, for sufficiently large N, the error satisfies the bounded*

$$
\|u - U\| \leq \frac{CN^{-2}}{N^{-1} + c_\varepsilon}. \tag{27}
$$

Proof. Substituting the results in Lemma 3.5 into (25) results to

$$|L^h(u(x_i) - U_i)| \leq \frac{CN^{-2}}{N^{-1} + c_\varepsilon}. \tag{28}$$

By the discrete maximum principle we obtain

$$\|u - U\| \leq \frac{CN^{-2}}{N^{-1} + c_\varepsilon}. \tag{29}$$

□

Remark 3.1. Form equation (27) one can observe that for the case $c_\varepsilon \geq N^{-1}$ the scheme secures second order convergence and we expect to lose an order of convergence for $c_\varepsilon \to 0$, and in fact it turns out that the scheme is first-order uniformly convergent.

4. Numerical Examples and Discussion

To demonstrate the efficiency of the proposed scheme, we solved three examples having boundary layers.

Example 4.1. Consider the problem

$$-\varepsilon u''(x) - (1+x)u'(x-\delta) - \sin(2x)u(x-\delta) + \exp(-x)u(x) = -\sin(2x) - 3\exp(-x)$$

with interval boundary conditions $u(x) = -1,\ -\delta \leq x < 0$ and $u(1) = 1$.

Example 4.2. Consider the problem

$$-\varepsilon u''(x) + (1+x)u'(x-\delta) - \exp(-2x)u(x-\delta) + \exp(-x)u(x) = 0$$

with interval boundary conditions $u(x) = 1,\ -\delta \leq x < 0$ and $u(1) = -1$.

Example 4.3. Consider the problem

$$\varepsilon u''(x) + \exp(x)u'(x-\delta) + xu(x) = 0$$

with interval boundary conditions $u(x) = 1,\ -\delta \leq x < 0$ and $u(1) = 1$.

Since the exact solution of the considered problems are not known, the maximum absolute errors are estimated using the double mesh principle [20] defined by

$$E^N_{\varepsilon,\delta} = \max_{0 \leq i \leq N} |U^N_i - U^{2N}_i|$$

where U^N_i stands for the numerical solution of the problem on N number of mesh points and U^{2N}_i stands for the numerical solution of the problem on $2N$ number of mesh points by including the mid-points $x_{i+1/2}$ into the mesh numbers. The ε-uniform error is calculated using

$$E^N = \max_{\varepsilon,\delta} |E^N_{\varepsilon,\delta}|.$$

The rate of convergence of the scheme is calculated using

$$r^N_{\varepsilon,\delta} = \log_2 \left(E^N_{\varepsilon,\delta} / E^{2N}_{\varepsilon,\delta} \right)$$

and the ε- uniform rate of convergence is calculated using

$$r^N = \log_2 \left(E^N / E^{2N} \right).$$

Table 1. Maximum absolute error of Example 4.1 using the scheme without the exponential fitting factor.

$\varepsilon \downarrow$	N= 32	64	128	256	512
2^0	1.7449e-04	4.3578e-05	1.0892e-05	2.7228e-06	6.8068e-07
2^{-4}	4.0306e-03	9.5618e-04	2.3814e-04	5.9301e-05	1.4821e-05
2^{-8}	3.0451e-01	1.6889e-01	6.7718e-02	1.9185e-02	4.0210e-03
2^{-12}	1.8563e+00	7.5150e-01	5.3496e-01	4.4055e-01	3.0947e-01
2^{-16}	1.3023e+01	5.5450e+00	1.9091e+00	8.1004e-01	6.2861e-01
2^{-20}	2.3093e+01	1.9549e+01	1.2837e+01	5.5662e+00	1.9241e+00
2^{-24}	2.4312e+01	2.3342e+01	2.2422e+01	1.9511e+01	1.2853e+01
2^{-28}	2.4392e+01	2.3625e+01	2.3521e+01	2.3326e+01	2.2474e+01
2^{-32}	2.4397e+01	2.3643e+01	2.3593e+01	2.3614e+01	2.3587e+01

Table 2. Example 4.1: Maximum absolute error of proposed scheme for $\delta = 0.3\varepsilon$.

$\varepsilon \downarrow$	N= 32	64	128	256	512	1024
2^0	2.0279e-04	5.0688e-05	1.2672e-05	3.1679e-06	7.9197e-07	1.9799e-07
2^{-4}	4.5004e-03	1.1766e-03	2.9768e-04	7.4646e-05	1.8676e-05	4.6698e-06
2^{-8}	1.2358e-02	6.1567e-03	2.7834e-03	9.9970e-04	2.8941e-04	7.5616e-05
2^{-12}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8192e-04	3.8711e-04
2^{-16}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
2^{-20}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
2^{-24}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
2^{-28}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
2^{-32}	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
E^N	1.2360e-02	6.2211e-03	3.1206e-03	1.5628e-03	7.8203e-04	3.9117e-04
r^N	0.9904	0.9953	0.9977	0.9988	0.9994	-

We consider three model examples: Example 4.1 and 4.3 exhibit a boundary layer on the left end of the domain whereas Example 4.2 exhibit a boundary layer on the right side of the domain with layer width of $O(c_\varepsilon)$. The maximum absolute error of Example 4.1 is given in Table 1 and 2. The result in Table 1 is the maximum absolute error of the central difference scheme without the exponential fitting factor and the result in Table 2 is by applying the exponential fitting factor $\sigma(\rho)$ on the term containing the perturbation parameter c_ε. In Table 2, 3 and 4 one observes in each columns, as $\varepsilon \to 0$ the maximum absolute error becomes stable and uniform, which indicates that the proposed scheme

TABLE 3. Example 4.2: Maximum absolute error of proposed scheme for $\delta = 0.3\varepsilon$.

$\varepsilon \downarrow$	N= 32	64	128	256	512	1024
2^{0}	1.2923e-05	3.2328e-06	8.0833e-07	2.0210e-07	5.0524e-08	1.2631e-08
2^{-4}	3.0544e-04	7.6883e-05	1.9319e-05	4.8333e-06	1.2089e-06	3.0224e-07
2^{-8}	1.0065e-03	3.8159e-04	1.8183e-04	7.2180e-05	1.8429e-05	4.6372e-06
2^{-12}	1.0771e-03	5.5397e-04	2.8088e-04	1.4103e-04	6.6043e-05	2.4279e-05
2^{-16}	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
2^{-20}	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
2^{-24}	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
2^{-28}	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
2^{-32}	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
E^N	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
r^N	0.9592	0.9799	0.9902	0.9950	0.9976	-

TABLE 4. Example 4.3: Maximum absolute error of proposed scheme for $\delta = 0.3\varepsilon$.

$\varepsilon \downarrow$	N= 32	64	128	256	512	1024
2^{0}	2.6752e-05	6.7167e-06	1.6806e-06	4.2024e-07	1.0506e-07	2.6261e-08
2^{-4}	7.2923e-04	2.0234e-04	5.2463e-05	1.3249e-05	3.3212e-06	8.3085e-07
2^{-8}	2.5410e-03	1.2424e-03	5.2756e-04	1.8637e-04	5.6431e-05	1.5386e-05
2^{-12}	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6740e-04	8.0935e-05
2^{-16}	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04	8.3963e-05
2^{-20}	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04	8.3963e-05
2^{-24}	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04	8.3963e-05
2^{-28}	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04	8.3963e-05
2^{-32}	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04	8.3963e-05
E^N	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04	8.3963e-05
r^N	0.9591	0.9797	0.9899	0.9950	0.9975	-

converges uniformly independent of the effect of ε whereas one observes from the result in Table 1, the scheme without exponential fitting factor does not converges. In Tables 5 and 6, we observe the comparison of the maximum absolute error using the proposed exponentially fitted scheme and the results in [6], [8] and [16] from the literature. As one see the results, the proposed scheme is more accurate than the result given in the literature. In addition it converges independent of the perturbation parameter in maximum norm. In Table 7, the rate of convergence of the scheme is given for each examples. As it is discussed in remark 3.1, the scheme secures second order convergence for $\varepsilon \geq N^{-1}$ and for $\varepsilon < N^{-1}$ the scheme converges uniformly with linear order of convergence.

Table 5. Example 4.2: Comparison of maximum absolute error of proposed scheme and result in [6] and [8] for $\delta = 0.3\varepsilon$.

$\varepsilon \downarrow$	N= 32	64	128	256	512	1024
	Proposed	Scheme				
2^{-4}	3.0544e-04	7.6883e-05	1.9319e-05	4.8333e-06	1.2089e-06	3.0224e-07
2^{-8}	1.0065e-03	3.8159e-04	1.8183e-04	7.2180e-05	1.8429e-05	4.6372e-06
2^{-12}	1.0771e-03	5.5397e-04	2.8088e-04	1.4103e-04	6.6043e-05	2.4279e-05
2^{-16}	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
2^{-20}	1.0771e-03	5.5398e-04	2.8088e-04	1.4140e-04	7.0944e-05	3.5532e-05
	B-Spline	in [8]				
2^{-4}	1.1491e-02	2.7950e-03	6.9407e-04	1.7323e-04	4.3289e-05	1.0821e-05
2^{-8}	3.7010e-02	1.2611e-02	4.2314e-03	1.4187e-03	4.6812e-04	1.5501e-04
2^{-12}	3.6669e-02	1.2541e-02	4.2449e-03	1.4298e-03	4.7508e-04	1.5924e-04
2^{-16}	3.6701e-02	1.2463e-02	4.1666e-03	1.4086e-03	4.7513e-04	1.5946e-04
2^{-20}	3.6709e-02	1.2478e-02	4.1763e-03	1.3898e-03	4.5369e-04	1.5379e-04
	Upwind	in [6]				
2^{-4}	3.72e-2	2.18e-2	1.27e-2	7.28e-3	4.12e-3	2.30e-3
2^{-8}	4.22e-2	2.56e-2	1.49e-2	8.31e-3	4.53e-3	2.45e-3
2^{-12}	3.93e-2	2.36e-2	1.38e-2	7.89e-3	4.49e-3	4.49e-3
2^{-16}	3.90e-2	2.33e-2	1.35e-2	7.66e-3	4.30e-3	2.39e-3
2^{-20}	3.90e-2	2.33e-2	1.35e-2	7.65e-3	4.28e-3	2.37e-3

Table 6. Example 4.3: Comparison of maximum absolute error of proposed scheme and result in [16].

$\varepsilon \downarrow$	N=16	32	64	128	256	512
	Proposed	Scheme				
10^{-9}	4.8030e-03	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04
10^{-10}	4.8030e-03	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04
10^{-11}	4.8030e-03	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04
10^{-12}	4.8030e-03	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04
10^{-13}	4.8030e-03	2.5439e-03	1.3085e-03	6.6353e-04	3.3410e-04	1.6763e-04
	Result	in [16]				
10^{-9}	0.66e-02	0.42e-02	0.23e-02	0.12e-02	0.64e-03	0.32e-03
10^{-10}	0.66e-02	0.42e-02	0.23e-02	0.12e-02	0.64e-03	0.32e-03
10^{-11}	0.66e-02	0.42e-02	0.23e-02	0.12e-02	0.64e-03	0.32e-03
10^{-12}	0.66e-02	0.42e-02	0.23e-02	0.12e-02	0.64e-03	0.32e-03
10^{-13}	0.66e-02	0.42e-02	0.23e-02	0.12e-02	0.64e-03	0.32e-03

For left boundary layer problems, one can observe from Figure 2 as the values of the delay increases the size of the boundary layer decreases. For the case of the right layer problems as seen in Figure 1, as the values of the delay increases the size of the boundary layer increases. In Figure 3, we observe the effect of the

TABLE 7. Rate of convergence of Example 4.1, 4.2 and 4.3.

ε $\downarrow$	N= 32 r_1	64 r_2	128 r_3	256 r_4	512 r_5
	Example	4.1			
2^0	**2.0003**	**2.0000**	**2.0000**	**2.0000**	**2.0000**
2^{-4}	**1.9354**	**1.9828**	**1.9956**	**1.9989**	**1.9998**
2^{-8}	1.0052	1.1453	1.4773	**1.7884**	**1.9364**
2^{-12}	0.9904	0.9953	0.9977	0.9990	1.0143
2^{-16}	0.9904	0.9953	0.9977	0.9988	0.9994
2^{-20}	0.9904	0.9953	0.9977	0.9988	0.9994
	Example	4.2			
2^0	**1.9991**	**1.9998**	**1.9999**	**2.0000**	**2.0000**
2^{-4}	**1.9902**	**1.9926**	**1.9989**	**1.9993**	**1.9999**
2^{-8}	1.3993	1.0694	1.3329	**1.9696**	**1.9907**
2^{-12}	0.9593	0.9799	0.9940	1.0945	1.4437
2^{-16}	0.9592	0.9799	0.9902	0.9950	0.9976
2^{-20}	0.9592	0.9799	0.9902	0.9950	0.9976
	Example	4.3			
2^0	**1.9938**	**1.9988**	**1.9997**	**2.0000**	**2.0002**
2^{-4}	**1.8496**	**1.9474**	**1.9854**	**1.9961**	**1.9990**
2^{-8}	1.0323	1.2357	1.5012	**1.7236**	**1.8749**
2^{-12}	0.9591	0.9797	0.9899	0.9950	0.9975
2^{-16}	0.9591	0.9797	0.9899	0.9950	0.9975
2^{-20}	0.9591	0.9797	0.9899	0.9950	0.9975

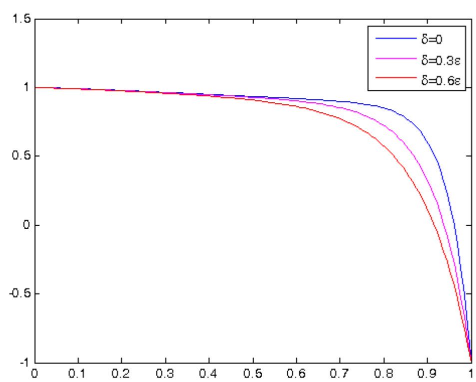

FIGURE 1. Effect of delay on the solution of Example 4.2 for $\varepsilon = 0.1$.

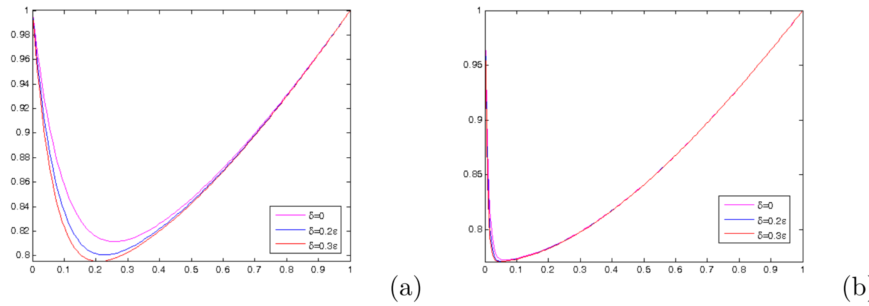

FIGURE 2. Effect of delay on the solution of Example 4.3 on (a) $\varepsilon = 0.1$ on (b) $\varepsilon = 0.01$.

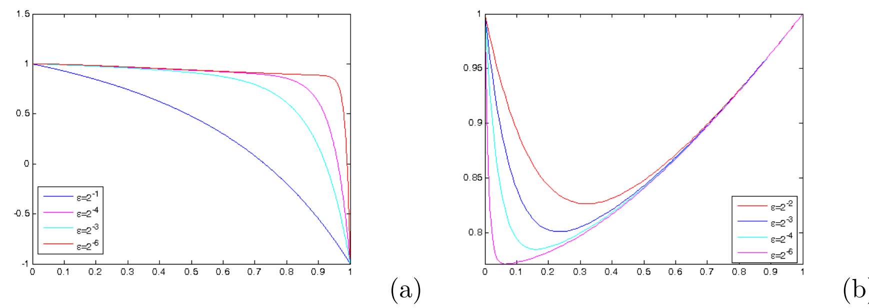

FIGURE 3. Effect of ε on the solution of Example 4.2 and 4.3 on (a) and (b) respectively for $\delta = 0.3\varepsilon$.

perturbation parameter on the behaviour of the solution for different values of the perturbation parameter. In this figure, we observe that as the perturbation parameter goes small the boundary layer becomes more strong.

5. Conclusion

In this paper, we considered a singularly perturbed differential equations having delay on the convection and reaction terms of the equation. The considered problem exhibits a boundary layer on the left or right end of the domain. We applied a Taylor series approximation for the terms containing delay. The resulting singularly perturbed BVPs are solved using exponentially fitted finite difference method. The stability of the scheme is investigated using the maximum principle and bound on the solution. The detail convergence analysis is shown by considering the truncation error of the discretization. The results obtained in this paper gives more accurate than the schemes in the literature and it is a parameter uniformly convergent.

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Merga Amara Angasu received M.Sc. from Jimma University. Since 2017 he has been at Jimma University. His research interests include numerical solution of singularly perturbed problems.

Department of Mathematics, College of Natural Sciences, Jimma University, Jimma, Ethiopia.
e-mail: merga.am86@gmail.com

Gemechis File Duressa received his M.Sc. degree from Addis Ababa University, Ethiopia and Ph.D. degree from National Institute of Technology, Warangal, India. He is currently working as an Associate professor of Mathematics at Jimma University. His research interests include Numerical Methods for Singularly Perturbed Differential Equations(both ODE and PDE). So far, he has published more than 67 research articles in reputable journals.

Department of Mathematics, College of Natural Sciences, Jimma University, Jimma, Ethiopia.
e-mail: gammeef@gmail.com

Mesfin Mekuria Woldaregay received M.Sc. from Addis Ababa University, and currently he is a Ph.D. scholar in Jimma University. He is currently a lecturer at adama Science and Technology University since 2010. His research interests are computational mathematics, numerical solution of singularly perturbed problems. He has published more than 10 research articles in reputable journals.

Department of Applied Mathematics, College of Natural Sciences, Adama Science and Technology University, Adama, Ethiopia.
e-mail: msfnmkr02@gmail.com