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MORE ON FUZZY MAXIMAL, MINIMAL OPEN AND CLOSED SETS

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Abstract. This article is devoted to introduce the notion of fuzzy cleanly covered fuzzy topological spaces; in addition two strong fuzzy separation axioms are studied. By means of fuzzy maximal open sets some properties of fuzzy cleanly covered fuzzy topological spaces are obtained and also by means of fuzzy maximal closed sets few identical results of a fuzzy topological spaces are investigated. Through fuzzy minimal open and fuzzy maximal closed sets, two strong fuzzy separation axioms are discussed.

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1. Introduction

Since the introduction of fuzzy sets by Zadeh[8], several branches of mathematics have been developed by many researchers for many decades. The concept of fuzzy topology introduced by Chang[2] in 1968. Even though various kinds of open sets in topology have been studied for many years, the notions of minimal open[5] and maximal open sets[4] pulled attention to study further. Mukherjee[1] introduced and studied some separation axioms through maximal, minimal open and closed sets. The idea of fuzzy minimal and fuzzy maximal open introduced by Ittanagi and Wali in [3]. In this paper, we study some properties and strong fuzzy separation axioms of fuzzy minimal and fuzzy maximal open sets.

In this paper (X, τ) or X stands for fuzzy topological space. The symbols λ, μ, γ, η,... are used to denote fuzzy sets.

The fuzzy sets having values 0 and 1 respectively at each point of X are denoted by 0_X and 1_X. A fuzzy point in X with support $x \in X$ and value $\alpha(0 < \alpha \leq 1)$ will be denoted by x_α. A fuzzy point x_α belongs to a fuzzy set λ, written as $x_\alpha \in \lambda$ iff $\alpha \leq \lambda(x)$[6].

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2. Fuzzy Maximal and Fuzzy Minimal Open and Closed Sets

Before going to main concepts, we recall some known results:

Definition 2.1. (Ittanagi and Wali[3]) A proper nonzero fuzzy open set λ of X is said to be a fuzzy maximal open set if λ and 1_X are the only fuzzy open sets containing λ.

Definition 2.2. (Ittanagi and Wali[3]) A proper nonzero fuzzy open set λ of X is said to be a fuzzy minimal open set if λ and 0_X are the only fuzzy open sets contained in λ.

Theorem 2.3. *(Ittanagi and Wali[3]) If λ is a fuzzy minimal open set and μ is a fuzzy open set, then either $\lambda \wedge \mu = 0_X$ or $\lambda < \mu$. If μ is a fuzzy minimal open set distinct from λ, then $\lambda \wedge \mu = 0_X$.*

Theorem 2.4. *(Ittanagi and Wali[3]) If λ is a fuzzy maximal open set and μ is a fuzzy open set then, either $\lambda \vee \mu = 1_X$ or $\mu < \lambda$. If μ is a fuzzy maximal open set distinct from λ, then $\lambda \vee \mu = 1_X$.*

Definition 2.5. A fuzzy cover $\mathbf{K}$ of X is said to be a fuzzy minimal cover if for any $\lambda \in \mathbf{K}$, $\mathbf{K} - \{\lambda\}$ is not a fuzzy cover of X. Then $\mathbf{K}$ is said to be a fuzzy minimal open (resp.closed) cover if each member of $\mathbf{K}$ is fuzzy open(resp. closed).

If $\mathbf{K}$ is a fuzzy minimal open cover of X, then there do not exist distinct $\lambda, \mu \in \mathbf{K}$ such that $\mu < \lambda$. Also, suppose a fuzzy open cover $\mathbf{K}$ of X consists of two distinct fuzzy open sets λ,μ such that $\mu < \lambda$, then $\mathbf{K}$ is not a fuzzy minimal open cover of X. Each fuzzy minimal open cover of a fuzzy compact space is finite and each fuzzy open cover of a fuzzy compact space has a finite fuzzy minimal open cover.

Definition 2.6. A fuzzy cover $\mathbf{K}$ of X is said to be fuzzy disconnected if for each $\lambda \in \mathbf{K}$, there exists a $\mu \in \mathbf{K}$ such that $\lambda \wedge \mu = 0_X$.

Theorem 2.7. *A fuzzy minimal open cover consists of a fuzzy minimal open set is fuzzy disconnected.*

Proof. Consider $\mathbf{K}$ be a fuzzy minimal open cover of X. Let $\lambda \in \mathbf{K}$ be a fuzzy minimal open set. As λ is a proper nonzero fuzzy open set and $\mathbf{K}$ is a fuzzy cover of X, there exists at least one more fuzzy open set $\mu \in \mathbf{K}$. By Theorem 2.3, we have $\lambda \wedge \mu = 0_X$ or $\lambda \lneqq \mu$. $\mathbf{K}$ being a fuzzy minimal open cover, $\lambda \lneqq \mu$ is not possible. □

Corollary 2.8. *A fuzzy minimal open cover having only fuzzy minimal open sets is fuzzy disconnected.*

We need the following definiton when we have atleast two nonzero fuzzy sets in a fuzzy open cover of a fuzzy topological space.

Definition 2.9. A fuzzy topological space X is said to be fuzzy cleanly covered if each fuzzy open cover of X has a fuzzy minimal open cover consisting of exactly two fuzzy open sets.

Hence, a fuzzy cleanly covered fuzzy topological space is a fuzzy compact space. But the converse need not be true.

Example 2.10. Let $X = \{a, b, c, d\}$. Then fuzzy sets $\gamma_1 = \frac{0}{a} + \frac{0}{b} + \frac{0}{c} + \frac{1}{d}$; $\gamma_2 = \frac{1}{a} + \frac{0}{b} + \frac{0}{c} + \frac{0}{d}$; $\gamma_3 = \frac{0}{a} + \frac{0}{b} + \frac{1}{c} + \frac{0}{d}$; $\gamma_4 = \frac{0}{a} + \frac{1}{b} + \frac{0}{c} + \frac{1}{d}$; $\gamma_5 = \frac{0}{a} + \frac{1}{b} + \frac{1}{c} + \frac{0}{d}$; $\gamma_6 = \frac{1}{a} + \frac{0}{b} + \frac{1}{c} + \frac{0}{d}$; $\gamma_7 = \frac{1}{a} + \frac{0}{b} + \frac{1}{c} + \frac{1}{d}$; $\gamma_8 = \frac{0}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ and $\gamma_9 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{0}{d}$ are defined as follows: Consider the fuzzy topology $\tau = \{0_X, \gamma_1, \gamma_4, \gamma_7, 1_X\}$. Hence (X, τ) is fuzzy cleanly covered and the fuzzy space having no fuzzy maximal and fuzzy minimal open sets.

Example 2.11. From Example 2.10, consider the fuzzy topology $\tau_1 = \{0_X, \gamma_1, \gamma_5, \gamma_8, \gamma_9, 1_X\}$ and the fuzzy space (X, τ_1) is fuzzy compact but not fuzzy cleanly covered because it is having many fuzzy minimal open sets and two fuzzy maximal open sets γ_8 and γ_9 which covers 1_X.

Theorem 2.12. *If each fuzzy open cover of X contains a fuzzy maximal open set, then X is fuzzy cleanly covered.*

Proof. Assume that $\mathbf{K}$ be a fuzzy open cover of X and $\omega \in \mathbf{K}$ be a fuzzy maximal open set. Choose if $\beta \in \mathbf{K}$ is another fuzzy maximal open set distinct from ω, then we have $\omega \vee \beta = 1_X$ by Theorem 2.4 . Hence $\{\omega, \beta\}$ forms fuzzy subcover of $\mathbf{K}$ for X. Let ω be the only fuzzy maximal open set in $\mathbf{K}$. Now ω being a proper nonzero fuzzy open set, there exist at least one more fuzzy open set $\beta \in \mathbf{K}$ distinct from ω to cover X. If $\beta = 1_X$, then $\mathbf{K}$ is a trivial fuzzy open cover of X. Hence we assume that $\beta \neq 1_X$. By Theorem 2.4, we have $\omega \vee \beta = 1_X$ or $\beta \lneqq \omega$. If for all $\Omega \in \mathbf{K}$, we have $\Omega \lneqq \omega$, then $\mathbf{K}$ cannot be a fuzzy open cover of X. Hence it follows that there exists a fuzzy open set $\beta \in \mathbf{K}$ distinct from ω such that $\omega \vee \beta = 1_X$.

Recall that a collection $\mathbf{K}$ of fuzzy subsets of X is called fuzzy locally finite if each $x_\alpha \in X$ has a fuzzy neighborhood meeting only fuzzy finitely many members of $\mathbf{K}$. Also if each $\{\Omega_k \mid k \in \Lambda\}$ is a fuzzy locally finite collection of fuzzy sets in X, then $Cl(\bigvee_{k \in \Lambda} \Omega_k) = \bigvee_{k \in \Lambda} Cl(\Omega_k)$. □

Theorem 2.13. *If $\{\Omega_k \mid k \in \Lambda\}$ is a collection of distinct fuzzy minimal open sets in a fuzzy locally finite space X, then $Cl(\bigvee_{k \in \Lambda} \Omega_k) = \bigvee_{k \in \Lambda} Cl(\Omega_k)$.*

Proof. As Ω_k is a fuzzy minimal open set for each $k \in \Lambda$, we have $\Omega_{k_1} \wedge \Omega_{k_2} = 0_X$ for $k_1, k_2 \in \Lambda$ with $k_1 \neq k_2$. For each $x_\alpha \in X$, there exists a finite fuzzy open set Ω such that $x_\alpha \in \Omega$. Since Ω is a finite fuzzy open set and $\Omega_{k_1} \wedge \Omega_{k_2} = 0_X$ for $k_1, k_2 \in \Lambda$ with $k_1 \neq k_2$, Ω intersects only finitely many members of $\{\Omega_k \mid k \in \Lambda\}$. Therefore, $\{\Omega_k \mid k \in \Lambda\}$ is fuzzy locally finite. Hence the result follows. □

Definition 2.14. (Ittanagi and Wali[3]) A proper nonzero fuzzy closed γ of X is said to be a fuzzy minimal closed set if any fuzzy closed set which is contained in γ is 0_X or γ.

Theorem 2.15. *(Ittanagi and Wali*[3]*) If γ is a fuzzy minimal closed set and ϑ is any fuzzy closed set, then either $\gamma \wedge \vartheta = 0_X$ or $\gamma < \vartheta$.*

Definition 2.16. (Ittanagi and Wali[3]) A proper nonzero fuzzy closed γ of X is said to be a fuzzy maximal closed set if any fuzzy closed set which contains γ is 1_X or γ.

Theorem 2.17. *Ittanagi and Wali*[3]*) If γ is a fuzzy maximal closed set and ϑ is a fuzzy closed set, then either $\gamma \vee \vartheta = 1_X$ or $\vartheta < \gamma$.*

As Theorem 2.18, Corollary 2.19, Theorem 2.20 and Theorem 2.21 are similar to Theorem 2.7, Corollary 2.8, Theorem 2.12 and Theorem 2.13 , the proofs are omitted.

Theorem 2.18. *A fuzzy minimal closed cover consists of a fuzzy minimal closed set is fuzzy disconnected.*

Corollary 2.19. *A fuzzy minimal closed cover consists of only fuzzy minimal closed sets is fuzzy disconnected.*

Theorem 2.20. *If each fuzzy closed cover $\mathfrak{F}$ of X contains a fuzzy maximal closed set γ, then there exists an $\vartheta \in \mathfrak{F}$ such that $\gamma \vee \vartheta = 1_X$.*

Theorem 2.21. *If $\{\Omega_k \mid k \in \Lambda\}$ is a collection of distinct fuzzy minimal closed sets in a fuzzy locally finite space X, then $Cl(\bigvee_{k\in\Lambda} \Omega_k) = \bigvee_{k\in\Lambda} Cl(\Omega_k)$.*

Theorem 2.22. *Let η, γ be fuzzy closed sets in X such that $\eta \wedge \gamma \neq 0_X$. Then $\eta \wedge \gamma$ is a fuzzy minimal closed set in $(\eta, \mathfrak{F}_\eta)$ if γ is a fuzzy minimal closed set in $(X, \mathfrak{F})$.*

Proof. Similar to Proof of Theorem 4.11 [7]. □

3. Two strong fuzzy separation axioms

It is observed that if there exists two fuzzy minimal open sets λ, μ in a fuzzy topological space X, then $\lambda \wedge \mu = 0_X$[3]. By means of this result, we introduce the following two strong fuzzy separation axioms.

Definition 3.1. A fuzzy topological space X is said to be fuzzy strongly regular iff for every singleton in X and every fuzzy closed set ϑ in X such that $x_\alpha \leq co\, \vartheta$, there exists distinct fuzzy minimal open sets λ, μ such that $x_\alpha \leq \lambda,\ \vartheta \leq \mu$ and $\lambda \leq co\ \mu$.

Clearly, a fuzzy strongly regular topological space is fuzzy regular.

Definition 3.2. A fuzzy topological space X is said to be fuzzy strongly normal iff for any two closed sets γ, ϑ such that $\gamma \leq co\ \vartheta$, there exists two disjoint fuzzy minimal open sets λ, μ such that $\gamma < \lambda, \vartheta \leq \mu$ and $\lambda \leq co\ \mu$.

For a subset ω of a fuzzy topological space X, we define

$$MinInt(\omega) = \begin{cases} 0_X \text{ if } \omega \text{ contains no fuzzy minimal open set} \\ \bigvee \{\alpha \mid \alpha \text{ is a fuzzy minimal open set contained in } \omega\} \end{cases}$$

$$MaxCl(\omega)=\begin{cases}1_X \text{ if } \omega \text{ contained in no fuzzy maximal closed set}\\ \bigwedge\{\gamma \mid \gamma \text{ is a fuzzy maximal closed set containing } \omega\}\end{cases}$$

The union of finitely many distinct fuzzy minimal open sets is not fuzzy minimal open set and the intersection of finitely many distinct fuzzy maximal closed sets is not fuzzy maximal closed set. From this point of view, $MinInt(\omega)$ (resp. $MaxCl(\omega)$) may not be fuzzy minimal open (resp. fuzzy maximal closed).

Example 3.3. From Example 2.10, consider the fuzzy topology $\tau_2=\{0_X,\gamma_2,\gamma_3,\gamma_6,\gamma_8,1_X\}$. Now $MinInt(\gamma_7)=\gamma_6$ which is not fuzzy minimal open set and $MaxCl(\gamma_4)=\gamma_4$ which is not fuzzy maximal closed set.

Theorem 3.4. *For a fuzzy subset ω of X, $1_X-MinInt(\omega)=MaxCl(1_X-\omega)$.*

Proof. Let us assume that $MinInt(\omega)=0_X$. This means that ω contains no fuzzy minimal open set. If possible, suppose that $1_X-\omega$ contains a fuzzy maximal closed set γ. Then $1_X-\gamma$ is a fuzzy minimal open set contained in ω, a contradiction. So in this case we have $1_X-MinInt(\omega)=MaxCl(1_X-\omega)$. Now suppose that ω contains a fuzzy minimal open set α. Then $1_X-\omega<1_X-\alpha$. In addition, $1_X-\alpha$ is a fuzzy maximal closed set. Hence we get $MaxCl(1_X-\omega)\neq 1_X$ if and only if $MinInt(\omega)\neq 0_X$. It is easy to see that if $\{\alpha\}$ is a collection of all fuzzy minimal open sets contained in ω, then $\{1_X-\alpha\}$ is the collection of all fuzzy maximal closed sets containing $1_X-\omega$ and vice-versa. Hence we have

$$\begin{aligned}&1_X-MinInt(\omega)=1_X-\bigvee\{\alpha \mid \alpha \text{ is fuzzy minimal open contained in } \omega\}\\ &=\bigwedge\{1_X-\alpha \mid 1_X-\alpha \text{ is fuzzy maximal closed containing } 1_X-\omega\}\\ &=MaxCl(1_X-\omega).\end{aligned}$$ □

Theorem 3.5. *For a fuzzy subset ω of X, $MinInt(\omega)$ is fuzzy minimal open if and only if ω contains one and only one fuzzy minimal open set.*

Proof. The sufficiency follows easily. Let $MinInt(\omega)$ be fuzzy minimal open set and ω contains two fuzzy minimal open sets α,β. Then, $MinInt(\omega)=\alpha\vee\beta<\omega$. Since $\alpha,\beta<\alpha\vee\beta$ and $MinInt(\omega)$ is fuzzy minimal open , we have $\alpha=\alpha\vee\beta$ and $\beta=\alpha\vee\beta$. $\alpha=\alpha\vee\beta$ implies that $\beta<\alpha$ and $\beta=\alpha\vee\beta$ implies that $\alpha<\beta$. Hence we have $\alpha=\beta$. □

Theorem 3.6 is a dual of Theorem 3.5. The proof of the theorem is omitted as the proof is similar to that of Theorem 3.5.

Theorem 3.6. *For a fuzzy subset ω of X,$MaxCl(\omega)$ is fuzzy maximal closed if and only if ω contained in one and only one fuzzy maximal closed set.*

Theorem 3.7. *A fuzzy topological space X is fuzzy strongly regular if and only if for each fuzzy open set β and each $x_\alpha\in\beta$, there exists a fuzzy minimal open set λ and a fuzzy maximal closed set γ such that $x_\alpha\in\lambda<MaxCl(\lambda)<\gamma<\beta$.*

Proof. Let X be a fuzzy strongly regular space and β be a fuzzy open set. For $x_\alpha\in\beta$, we obtain by fuzzy strong regularity of X two distinct fuzzy minimal open sets λ, μ such that $x_\alpha \in \lambda$, $1_X - \beta < \mu$. λ, μ being distinct fuzzy minimal open sets, we have $\lambda \wedge \mu = 0_X$. Now $\lambda \wedge \mu = 0_X \Rightarrow \lambda < 1_X - \mu$. Since μ is fuzzy minimal open, $1_X - \mu$ is fuzzy maximal closed and so $MaxCl(1_X - \mu) = 1_X - \mu$. Thus $MaxCl(\lambda) < MaxCl(1_X - \mu) = 1_X - \mu < \beta$. Putting $\gamma = 1_X - \mu$, we see that γ is a fuzzy maximal closed and $x_\alpha \in \lambda < MaxCl(\lambda) < \gamma < \beta$.

Conversely, let γ be a fuzzy closed set and $x_\alpha \in X$ such that $x_\alpha \notin \gamma$. As $1_X - \gamma$ is a fuzzy open set with $x_\alpha \in 1_X - \gamma$, there exist a fuzzy minimal open set λ and a fuzzy maximal closed set ϑ such that $x_\alpha \in \lambda < MaxCl(\lambda) < \vartheta < 1_X - \gamma$. We put $\mu = 1_X - \vartheta$. Then μ is a fuzzy minimal open set with $\gamma < \mu$ and $\lambda \wedge \mu = 0_X$. □

Theorem 3.8. *A fuzzy topological space X is fuzzy strongly normal if and only if for a fuzzy closed set γ and for a fuzzy open set β with $\gamma < \beta$, there exist a fuzzy minimal open set λ and a fuzzy maximal closed set ϑ such that $\gamma < \lambda < MaxCl(\lambda) < \vartheta < \beta$.*

Proof. Similar to the proof of Theorem 3.7. □

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