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ON CONGRUENCES INVOLVING EULER POLYNOMIALS AND THE QUOTIENTS OF FERMAT

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Abstract. The aim of this paper is to provide the residues of Euler polynomials modulo p^2 in terms of alternating sums of like powers of numbers in arithmetical progression. Also, we establish the analogue of a classical congruence of Lehmer.

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1. Introduction

The associated Euler number $E_m(0)$ $(m = 0, 1, 2, \ldots)$ is defined by means of the following generating function

$$\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} \frac{E_m(0)}{m!} t^m \tag{1}$$

(see [1, 2, 8, 13]). Moreover, we can write

$$\frac{2}{e^t + 1} = \left(1 - \frac{1 - e^t}{2}\right)^{-1} = \sum_{i=0}^{\infty} \left(\frac{1 - e^t}{2}\right)^i, \tag{2}$$

which converges for $|1 - e^t| < 2$ and hence small values of t. The first few non-zero ones are:

$$E_0(0) = 1, E_1(0) = -\frac{1}{2}, E_3(0) = \frac{1}{4}, E_5(0) = -\frac{1}{2}, E_7(0) = \frac{17}{8}, \ldots.$$

In particular, $E_m(0)$'s are all rational numbers. Since $\frac{2}{e^t+1} - 1$ is an odd function (i.e., if $f(t) = f(t)$, so all of the signs are switched), we see that

$$E_m(0) = 0 \quad \text{for } m \text{ an even integer greater than or equal to 2.} \tag{3}$$

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The Euler polynomials $E_m(t)$ $(m = 0, 1, 2, \ldots)$ is defined by

$$E_m(t) = \sum_{i=0}^{m} \binom{m}{i} E_i(0) t^{m-i}, \tag{4}$$

where $\binom{m}{i} = \frac{n!}{i!(m-i)!}$ (see, for details, [13]). The associated Euler number numbers and Euler polynomials are classical and important in number theory and arise in some combinatorial contexts. For example, see E. Lehmer [9], Sun [13], and Wagstaff [15]. Equation (4) is actually the standard way to define and compute the Euler polynomials inductively. The Euler polynomial $E_m(t)$ of degree m can be rewritten as the unique polynomial solution of the equation

$$E_m(t+1) + E_m(t) = 2t^m, \quad m \geq 0 \tag{5}$$

(see [1, 2, 8, 13]). Also, Euler polynomials satisfy the identity

$$E_m(1-t) = (-1)^m E_m(t), \tag{6}$$

which follows from (1) and (4). In particular, we have

$$E_m = 2^m E_m \left(\frac{1}{2}\right), \tag{7}$$

where E_m are the *ordinary* Euler numbers (see [12, 13, 15]), and as a result of (6), we must have $E_m = E_m\left(\frac{1}{2}\right) = 0$ whenever m is odd. Therefore $E_m \neq E_m(0)$, in fact [13, p. 374, (2.1)]

$$E_m(0) = \frac{2}{m+1}(1 - 2^{m+1}) B_{m+1}, \tag{8}$$

where B_m means the Bernoulli numbers and $m \geq 0$.

Following work of Friedmann and Tamarkin [5], E. Lehmer [9] considered Bernoulli numbers and polynomials modulo primes and prime powers, and showed many identities and combinatorial interpretations involving harmonic numbers. See E. Lehmer [9] for connections between Fermat quotients and Fermat's last theorem.

The motivation of the paper is to generalize the identities of E. Lehmer [9] on Euler polynomials. The results are presented in Section 2 and Section 3, and our proof is based on certain arithmetical identities and congruences for some alternating sums.

2. Results

The following result concerning Euler polynomials was recently presented in [8, p. 2169, Lemma 2.1].

Lemma 2.1. *Let $m \geq 0$ be integers. Then*

$$E_m(t) = \sum_{i=0}^{m} \left(\frac{1}{2}\right)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j (j+t)^m. \tag{9}$$

In particular, if $m \geq 0$ be integers, then

$$2^m E_m(t) \in \mathbb{Z}[t] \quad \text{and} \quad 2^m E_m(0) \in \mathbb{Z}. \tag{10}$$

This means that we have the following relationship between the ordinary Euler numbers E_{2n} and the associated Euler numbers $E_i(0)$ where $i = 0, 1, \ldots, 2n$:

$$E_{2n} = \sum_{i=0}^{2n} \binom{2n}{i} E_i(0) \in \mathbb{Z}.$$

Remark 2.1. We easily see that $E_m(0)$ will not contain primes p in the denominator when $p > 2$.

Since Euler polynomials satisfy many properties that are similar to those that Bernoulli polynomials satisfy, we would expect a result similar to Kummer's congruence for associated Euler numbers $E_m(0)$. We have the following result.

Lemma 2.2. *For integers $m \geq 1$ and primes $p \geq 3$, we have*

$$E_{m+p-1}(0) \equiv E_m(0) \pmod{p}. \tag{11}$$

Proof. When m and n are positive integers, it follows at once from (4) that

$$E_m(n) \equiv E_m(0) \pmod{n}. \tag{12}$$

Let n be an odd integer with $n \geq 1$. Moreover it is easy to see that

$$E_m(0) + E_m(n) = \sum_{l=0}^{n-1} \left((-1)^l E_m(l) - (-1)^{l+1} E_m(l+1) \right). \tag{13}$$

Using (5) and (13) we have

$$E_m(0) + E_m(n) = 2 \sum_{l=0}^{n-1} (-1)^l l^m. \tag{14}$$

For any odd prime p, by (12), (14) yields the congruence

$$E_{m+p-1}(0) = \sum_{l=0}^{n-1} (-1)^l l^{m+p-1} \pmod{n}. \tag{15}$$

Consequently from Fermat's little Theorem, i.e., $l^{m+p-1} \equiv l^m \pmod{p}$ for $0 \leq l < p$, for $n = p$, (15) imply Kummer's congruence for $E_m(0)$:

$$E_{m+p-1}(0) \equiv E_m(0) \pmod{p}.$$

This completes the proof. □

Remark 2.2. Euler, on page 499 in [4], introduced Euler polynomials to evaluate the alternating sum (14). Carlitz and Levine [2] have also investigated Kummer's congruence for ordinary Euler numbers (7). See Wagstaff, Jr. [15, Theorem 4] for a simple proof of Kummer's congruence for ordinary Euler numbers.

Theorem 2.3. *For integers $n \geq 1$ and primes $p \geq 3$, we have*

$$\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}(p-rn)^{2k} \equiv \frac{n^{2k}}{2}\left\{p\frac{2k}{n}E_{2k-1}(0)+(-1)^{\left[\frac{p}{n}\right]+1}E_{2k}\left(\frac{s}{n}\right)\right\} \pmod{p^3} \tag{16}$$

and

$$\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}(p-rn)^{2k+1} \equiv \frac{n^{2k+1}}{2}\left\{E_{2k+1}(0)+(-1)^{\left[\frac{p}{n}\right]+1}E_{2k+1}\left(\frac{s}{n}\right)\right\} \pmod{p^2}, \tag{17}$$

where s is the least positive residue of $p \pmod{n}$ and $k \geq 1$.

Proof. If in (5), alternating, adding and subtracting this identity with $t = (p - rn)/n$, $r = 1, 2, \ldots, \left[\frac{p}{n}\right]$, where $[x]$ is the greatest integer not exceeding x, n and p are positive integers with $n < p$, for each case, gives the formula

$$E_m\left(\frac{p}{n}\right)+(-1)^{\left[\frac{p}{n}\right]+1}E_m\left(\frac{p}{n}-\left[\frac{p}{n}\right]\right)=2\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}\left(\frac{p-rn}{n}\right)^m.$$

This implies that

$$\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}(p-rn)^m=\frac{n^m}{2}\left\{E_m\left(\frac{p}{n}\right)+(-1)^{\left[\frac{p}{n}\right]+1}E_m\left(\frac{s}{n}\right)\right\}, \tag{18}$$

where we have written s for the least positive residue of p modulo n. Setting $m = 2k,\ k \geq 1$ and $t = p/n$, where p is an odd prime $> n$, in (4), by (3) and (10), we get the congruence

$$E_{2k}\left(\frac{p}{n}\right)=\sum_{r=0}^{2k}\binom{2k}{r}E_r(0)\left(\frac{p}{n}\right)^{2k-r} \equiv 2k\left(\frac{p}{n}\right)E_{2k-1}(0) \pmod{p^3}. \tag{19}$$

Similarly we find for $m = 2k+1$ with $k \geq 1$

$$E_{2k+1}\left(\frac{p}{n}\right) \equiv E_{2k+1}(0)+k(2k+1)E_{2k-1}(0)\left(\frac{p}{n}\right)^2 \pmod{p^3}, \tag{20}$$

since $E_{2k}(0) = 0$ with $k \geq 1$. Substituting these results into (18), we obtain the theorem. □

Corollary 2.4. *For integers $n \geq 1$ and primes $p \geq 3$, we have*

$$\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}r^{2k+1} \equiv -\frac{1}{2}\left(E_{2k+1}(0)+(-1)^{\left[\frac{p}{n}\right]+1}E_{2k+1}\left(\frac{s}{n}\right)\right) + (-1)^{\left[\frac{p}{n}\right]+1}\frac{p}{2n}(2k+1)E_{2k}\left(\frac{s}{n}\right) \pmod{p^2} \tag{21}$$

and

$$\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}r^{2k} \equiv (-1)^{\left[\frac{p}{n}\right]}\left(-\frac{1}{2}E_{2k}\left(\frac{s}{n}\right)+\frac{p}{n}kE_{2k-1}\left(\frac{s}{n}\right)\right) \pmod{p^2}, \quad (22)$$

where s is the least positive residue of $p \pmod{n}$ and $k \in \mathbb{N}$.

Proof. Since

$$\begin{aligned}\sum_{r=1}^{\left[\frac{p}{n}\right]}&(-1)^{r-1}(p-rn)^m\\ &\equiv (-1)^m n^m\left\{\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}r^m-\frac{pm}{n}\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}r^{m-1}\right\} \pmod{p^2},\end{aligned} \quad (23)$$

congruences (16) and (17) may be combined to give sums of like powers of numbers less than $[p/n]$. From (23), we can write

$$\begin{aligned}n^m&\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}r^m\\ &\equiv (-1)^m\left\{\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}(p-rn)^m-pm\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}(p-rn)^{m-1}\right\} \pmod{p^2},\end{aligned} \quad (24)$$

where $m>1$. Now we put $m=2k-1$ with $k>1$. Then, from (16), (17) and (24), we have

$$\begin{aligned}\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}r^{2k+1} \equiv -\frac{1}{2}&\left(E_{2k+1}(0)+(-1)^{\left[\frac{p}{n}\right]+1}E_{2k+1}\left(\frac{s}{n}\right)\right)\\ &+(-1)^{\left[\frac{p}{n}\right]+1}\frac{p}{2n}(2k+1)E_{2k}\left(\frac{s}{n}\right) \pmod{p^2}.\end{aligned} \quad (25)$$

Similarly, for $m=2k$ with $k>1$, we obtain (22). This completes the proof. □

Remark 2.3. Congruences (16) and (17) may be thought of as generalizations of Glaisher's results [7] on Euler polynomials, while congruences (21) and (22) give generalizations of Vandiver's results on Euler polynomials whenever possible. Both sets of formulas depend on the evaluation of $E_m\left(\frac{s}{n}\right)$.

The values of $E_m\left(\frac{s}{n}\right)$ can be tabulated as follows:

$$\begin{aligned}
&E_{2k+1}(1) = -E_{2k+1}(0), \quad k \geq 0,\\
&E_{2k+1}\left(-\frac{1}{2}\right) = -\left(\frac{1}{2}\right)^{2k}, \quad E_{2k+1}\left(\frac{1}{2}\right) = 0, \quad k \geq 0,\\
&E_{2k+1}\left(\frac{1}{3}\right) = -E_{2k+1}\left(\frac{2}{3}\right) = \frac{1}{2}\left(1 - \frac{1}{3^{2k+1}}\right)E_{2k+1}(0), \quad k \geq 0,\\
&E_{2k+1}\left(\frac{1}{4}\right) = -E_{2k+1}\left(\frac{3}{4}\right), \quad k \geq 0.
\end{aligned} \tag{26}$$

These evaluations of $E_m\left(\frac{s}{n}\right)$ are well known.

Example 2.5. It follows readily from (17) and (26) that

$$\sum_{r=1}^{(p-1)/2} (-1)^{r-1}(p-2r)^{2k+1} \equiv 2^{2k}E_{2k+1}(0) \pmod{p^2}, \tag{27}$$

$$\begin{aligned}
&\sum_{r=1}^{\left[\frac{p}{3}\right]}(-1)^{r-1}(p-3r)^{2k+1}\\
&\equiv \frac{3^{2k+1}}{2}\begin{cases} E_{2k+1}(0) - E_{2k+1}\left(\frac{1}{3}\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod 3\\ E_{2k+1}(0) + E_{2k+1}\left(\frac{2}{3}\right) \pmod{p^2} & \text{if } p \equiv 2 \pmod 3\end{cases}\\
&\equiv \frac{3^{2k+1}}{2}\left(E_{2k+1}(0) - E_{2k+1}\left(\frac{1}{3}\right)\right) \pmod{p^2}, \quad p > 3,
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\sum_{r=1}^{\left[\frac{p}{4}\right]}(-1)^{r-1}(p-4r)^{2k+1}\\
&\equiv 2\cdot 4^{2k}\begin{cases} E_{2k+1}(0) - (-1)^{\left[\frac{p}{4}\right]}E_{2k+1}\left(\frac{1}{4}\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod 4\\ E_{2k+1}(0) - (-1)^{\left[\frac{p}{4}\right]}E_{2k+1}\left(\frac{3}{4}\right) \pmod{p^2} & \text{if } p \equiv 3 \pmod 4\end{cases}\\
&\equiv 2\cdot 4^{2k}\left(E_{2k+1}(0) \mp (-1)^{\left[\frac{p}{4}\right]}E_{2k+1}\left(\frac{1}{4}\right)\right)\\
&\qquad\qquad \pmod{p^2} \quad \text{if } p \equiv \pm 1 \pmod 4, \quad p > 5.
\end{aligned} \tag{29}$$

Example 2.6. Next we will give the results of substituting $E_m\left(\frac{s}{n}\right)$ in (21) and (22) for $n = 1$ and 2. If $n = 1$, (21) and (22) are of course the same as $(-1)\times(17)$ and $(-1)\times(16)$, respectively. We obtain easily

$$\begin{aligned}
&\sum_{r=1}^{p}(-1)^{r-1}r^{2k+1} \equiv -E_{2k+1}(0) \pmod{p^2},\\
&\sum_{r=1}^{p}(-1)^{r-1}r^{2k} \equiv -pkE_{2k-1}(0) \pmod{p^2}.
\end{aligned} \tag{30}$$

If $n = 2$, (21) becomes

$$\sum_{r=1}^{(p-1)/2} (-1)^{r-1} r^{2k+1} \equiv -\frac{1}{2}\left(E_{2k+1}(0) + (-1)^{\frac{p-1}{2}} \frac{p}{2}(2k+1)E_{2k}\left(\frac{1}{2}\right)\right) \pmod{p^2}, \tag{31}$$

since $E_{2k+1}(1/2) = 0$ with $k \geq 0$. Similarly, (22) becomes

$$\sum_{r=1}^{(p-1)/2} (-1)^{r-1} r^{2k} \equiv (-1)^{\frac{p+1}{2}} \frac{1}{2} E_{2k}\left(\frac{1}{2}\right) \pmod{p^2}, \tag{32}$$

while (16) gives

$$\sum_{r=1}^{(p-1)/2} (-1)^{r-1} (p-2r)^{2k} \equiv 2^{2k-1}\left(pkE_{2k-1}(0) + (-1)^{\frac{p+1}{2}} E_{2k}\left(\frac{1}{2}\right)\right) \pmod{p^3}, \tag{33}$$

where $k > 1$. Hence, (33) reduces to the congruence

$$\sum_{r=1}^{(p-1)/2} (-1)^{r-1} (p-2r)^{2k} \equiv \frac{(-1)^{\frac{p+1}{2}}}{2} E_{2k} \pmod{p}, \tag{34}$$

where E_k is the kth ordinary Euler numbers (see (7)) and $k > 1$.

3. Applications

Let p be an odd prime and a an integer not divisible by p. The quotient

$$q_p(a) = \frac{a^{p-1} - 1}{p} \tag{35}$$

is called the Fermat quotient of p with base a, which is an integer according to the Fermat Little Theorem. This quotient has been extensively studied because of its links to numerous question in number theory. A classical congruence, due to F.G. Eisenstein [3] in 1850, asserts that for a prime $p \geq 3$,

$$q_p(2) \equiv \frac{1}{2} \sum_{r=1}^{p-1} (-1)^{r-1} \frac{1}{r} \pmod{p}, \tag{36}$$

which was extended in 1861 by J.J. Sylvester [14] and in 1901 by Glaisher [6, pp. 21–22] as

$$q_p(2) \equiv -\frac{1}{2} \sum_{r=1}^{(p-1)/2} \frac{1}{r} \pmod{p}. \tag{37}$$

The above congruence was generalized in 1905 by M. Lerch in the first paper of substance on Fermat quotients [10] (see [11, pp. 949–950]).

Theorem 3.1. *Let* $t(p-1) \not\equiv 2 \pmod{p-1}$ *and* $(p, t) = 1$. *Then*

$$\sum_{a=1}^{p-1} (-1)^a q_p(a) \equiv -\frac{1}{2} E_{t(p-1)-1}(0) \pmod{p}. \tag{38}$$

Proof. Observe that

$$2\sum_{r=1}^{(p-1)/2}(2r-1)^{2k}=\sum_{r=1}^{p-1}r^{2k}+\sum_{r=1}^{p-1}(-1)^{r-1}r^{2k}. \tag{39}$$

As early as 1938, Lehmer [9, (15)] established the following interesting congruence

$$\sum_{r=1}^{p-1}r^{2k}\equiv pB_{2k}\pmod{p^2}\quad\text{if } 2k\not\equiv 2\pmod{p-1}, \tag{40}$$

where B_k is the kth Bernoulli numbers (see [13]). From (30), (35) and (40), it follows that

$$\sum_{r=1}^{(p-1)/2}(2r-1)^{2k}\equiv\frac{1}{2}\left(pB_{2k}-pkE_{2k-1}(0)\right)\pmod{p^2} \tag{41}$$

if $2k\not\equiv 2 \pmod{p-1}$. Further, in virtue of (35), we obtain that

$$\sum_{a=1}^{p-1}(-1)^a q_p(a)=\sum_{a=1}^{p-1}q_p(a)-2\sum_{a=1}^{(p-1)/2}(-1)^a q_p(2a-1). \tag{42}$$

Recall ([9, p. 354]) that

$$a^{t(p-1)}=1+ptq_p(a)\pmod{p^2}, \tag{43}$$

where $t\in\mathbb{N}$. Thus by (42) and (43), we have

$$\begin{aligned}
&pt\sum_{a=1}^{p-1}(-1)^a q_p(a)\\
&\equiv\sum_{a=1}^{p-1}\left(a^{t(p-1)}-1\right)-2\sum_{a=1}^{(p-1)/2}\left((2a-1)^{t(p-1)}-1\right)\pmod{p^2}\\
&\equiv\sum_{a=1}^{p-1}a^{t(p-1)}-2\sum_{a=1}^{(p-1)/2}(2a-1)^{t(p-1)}\pmod{p^2}\\
&\equiv\frac{p(p-1)t}{2}E_{t(p-1)-1}(0)\pmod{p^2}\quad\text{if } t(p-1)\not\equiv 2\pmod{p-1}\\
&\equiv-\frac{pt}{2}E_{t(p-1)-1}(0)\pmod{p^2}\quad\text{if } t(p-1)\not\equiv 2\pmod{p-1},
\end{aligned} \tag{44}$$

that is,

$$\sum_{a=1}^{p-1}(-1)^a q_p(a)\equiv-\frac{1}{2}E_{t(p-1)-1}(0)\pmod{p} \tag{45}$$

if $t(p-1)\not\equiv 2 \pmod{p-1}$ and $(p,t)=1$, and the proof is completed. □

Theorem 3.2. *For integers $k \geq 1$ and primes $p \geq 3$, we have*

$$\sum_{r=1}^{p-1}(-1)^{r-1}r^{2k+1}q_p(r) \equiv \frac{1}{p}\left(E_{2k+1}(0) - E_{2k+p}(0)\right) \pmod{p} \tag{46}$$

and

$$\sum_{r=1}^{p-1}(-1)^{r-1}r^{2k}q_p(r) \equiv -\frac{p-1}{2}E_{2k-1}(0) \pmod{p}. \tag{47}$$

Proof. Now we in a position to transform the above sums into sums involving Fermat's quotients by means of the relation

$$(-1)^{a-1}(a^{m+p-1} - a^m) = (-1)^{a-1}pa^m q_p(a). \tag{48}$$

Hence, (48) may be written as

$$p\sum_{r=1}^{p-1}(-1)^{r-1}r^m q_p(r) = \sum_{r=1}^{p-1}(-1)^{r-1}r^{m+p-1} - \sum_{r=1}^{p-1}(-1)^{r-1}r^m. \tag{49}$$

Putting $m = 2k+1$ in (49), from (30), we find

$$\begin{aligned} p\sum_{r=1}^{p-1}(-1)^{r-1}r^{2k+1}q_p(r) &= \sum_{r=1}^{p-1}(-1)^{r-1}r^{2k+p} - \sum_{r=1}^{p-1}(-1)^{r-1}r^{2k+1} \\ &\equiv E_{2k+1}(0) - E_{2k+p}(0) \pmod{p^2}, \end{aligned} \tag{50}$$

that is,

$$\sum_{r=1}^{p-1}(-1)^{r-1}r^{2k+1}q_p(r) \equiv \frac{1}{p}\left(E_{2k+1}(0) - E_{2k+p}(0)\right) \pmod{p}, \tag{51}$$

where $k \in \mathbb{N}$. Putting $m = 2k$ in (49), from (30), we find

$$\begin{aligned} p\sum_{r=1}^{p-1}(-1)^{r-1}r^{2k}q_p(r) &= \sum_{r=1}^{p-1}(-1)^{r-1}r^{2k+p-1} - \sum_{r=1}^{p-1}(-1)^{r-1}r^{2k} \\ &\equiv p\left(kE_{2k-1}(0) - \frac{2k+p-1}{2}E_{2k-1+p-1}(0)\right) \pmod{p^2} \\ &\equiv p\left(kE_{2k-1}(0) - \frac{2k+p-1}{2}E_{2k-1}(0)\right) \pmod{p^2} \\ &\quad \text{(by (11))} \\ &\equiv p\left(k - \frac{2k+p-1}{2}\right)E_{2k-1}(0) \pmod{p^2}, \end{aligned} \tag{52}$$

that is,

$$\sum_{r=1}^{p-1}(-1)^{r-1}r^{2k}q_p(r) \equiv -\frac{p-1}{2}E_{2k-1}(0) \pmod{p}, \tag{53}$$

where $k \in \mathbb{N}$. This completes the proof. □

Theorem 3.3. *Let $n > 1, \alpha > 1$ and $p > 3$. Then*

$$\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}\frac{1}{p-nr} \equiv \frac{n^{\phi(p^\alpha)-1}}{2}\begin{cases} E_{\phi(p^\alpha)-1}\left(\frac{s}{n}\right)-(-1)^{\left[\frac{p}{n}\right]}E_{\phi(p^\alpha)-1}(0) \\ \qquad (\bmod\ p^2)\ \textit{if}\ \left[\frac{p}{n}\right]\ \textit{is odd} \\ -E_{\phi(p^\alpha)-1}\left(\frac{s}{n}\right)+(-1)^{\left[\frac{p}{n}\right]}E_{\phi(p^\alpha)-1}(0) \\ \qquad (\bmod\ p^2)\ \textit{if}\ \left[\frac{p}{n}\right]\ \textit{is even.} \end{cases} \tag{54}$$

Proof. Also $\alpha \in \mathbb{N}$ and $n > 1$. If, by a slight change in notation, we set

$$\begin{aligned} \sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}\left(\frac{p}{n}-r\right)^m &= \begin{cases} \sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}\left(\frac{s}{n}+r-1\right)^m & \text{if } \left[\frac{p}{n}\right] \text{ is odd} \\ \sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r}\left(\frac{s}{n}+r-1\right)^m & \text{if } \left[\frac{p}{n}\right] \text{ is even} \end{cases} \\ &= \begin{cases} \sum_{r=0}^{\left[\frac{p}{n}\right]-1}(-1)^{r}\left(\frac{s}{n}+r\right)^m & \text{if } \left[\frac{p}{n}\right] \text{ is odd} \\ \sum_{r=0}^{\left[\frac{p}{n}\right]-1}(-1)^{r-1}\left(\frac{s}{n}+r\right)^m & \text{if } \left[\frac{p}{n}\right] \text{ is even,} \end{cases} \end{aligned} \tag{55}$$

then we have

$$\begin{aligned} &\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}\frac{1}{p-nr} \\ &\equiv \sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}(p-nr)^{\phi(p^\alpha)-1} \quad (\bmod\ p^\alpha) \\ &\equiv n^{\phi(p^\alpha)-1}\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}\left(\frac{p}{n}-r\right)^{\phi(p^\alpha)-1} \quad (\bmod\ p^\alpha) \\ &= n^{\phi(p^\alpha)-1}\begin{cases} \sum_{r=0}^{\left[\frac{p}{n}\right]-1}(-1)^{r}\left(\frac{s}{n}+r\right)^{\phi(p^\alpha)-1} \quad (\bmod\ p^\alpha) & \text{if } \left[\frac{p}{n}\right] \text{ is odd} \\ \sum_{r=0}^{\left[\frac{p}{n}\right]-1}(-1)^{r-1}\left(\frac{s}{n}+r\right)^{\phi(p^\alpha)-1} \quad (\bmod\ p^\alpha) & \text{if } \left[\frac{p}{n}\right] \text{ is even,} \end{cases} \end{aligned} \tag{56}$$

where $\phi(n)$ denotes Euler's totient function. It is well known that (cf. [13])

$$\sum_{r=0}^{n-1}(-1)^r(t+r)^m = \frac{1}{2}(E_m(t)+(-1)^{n-1}E_m(t+n)). \tag{57}$$

From (56) and (57), it follows that

$$\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}\frac{1}{p-nr} \equiv \frac{n^{\phi(p^\alpha)-1}}{2}\begin{cases} E_{\phi(p^\alpha)-1}\left(\frac{s}{n}\right)+(-1)^{\left[\frac{p}{n}\right]-1}E_{\phi(p^\alpha)-1}\left(\frac{s}{n}+\left[\frac{p}{n}\right]\right) \\ \qquad\qquad \pmod{p^\alpha} \text{ if } \left[\frac{p}{n}\right] \text{ is odd} \\ -E_{\phi(p^\alpha)-1}\left(\frac{s}{n}\right)+(-1)^{\left[\frac{p}{n}\right]}E_{\phi(p^\alpha)-1}\left(\frac{s}{n}+\left[\frac{p}{n}\right]\right) \\ \qquad\qquad \pmod{p^\alpha} \text{ if } \left[\frac{p}{n}\right] \text{ is even.} \end{cases} \tag{58}$$

Since $E_{\phi(p^\alpha)-2}(0)=0$ with $\alpha>1$ and $p>3$, we have

$$\begin{aligned} E_{\phi(p^\alpha)-1}\left(\frac{s}{n}+\left[\frac{p}{n}\right]\right) &= E_{\phi(p^\alpha)-1}\left(\frac{p}{n}\right) \\ &\equiv E_{\phi(p^\alpha)-1}(0)+\frac{p}{n}E_{\phi(p^\alpha)-2}(0) \pmod{p^2} \\ &\equiv E_{\phi(p^\alpha)-1}(0) \pmod{p^2}, \end{aligned}$$

which has the paraphrase

$$\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}\frac{1}{p-nr} \equiv \frac{n^{\phi(p^\alpha)-1}}{2}\begin{cases} E_{\phi(p^\alpha)-1}\left(\frac{s}{n}\right)-(-1)^{\left[\frac{p}{n}\right]}E_{\phi(p^\alpha)-1}(0) \\ \qquad\qquad \pmod{p^2} \text{ if } \left[\frac{p}{n}\right] \text{ is odd} \\ -E_{\phi(p^\alpha)-1}\left(\frac{s}{n}\right)+(-1)^{\left[\frac{p}{n}\right]}E_{\phi(p^\alpha)-1}(0) \\ \qquad\qquad \pmod{p^2} \text{ if } \left[\frac{p}{n}\right] \text{ is even,} \end{cases} \tag{59}$$

where $\alpha>1$ and $p>3$. This completes the proof. □

Example 3.4. (54) implies that, for $n=2$ and 3,

$$\sum_{r=1}^{(p-1)/2}(-1)^{r-1}\frac{1}{p-2r} \equiv 2^{\phi(p^\alpha)-2}E_{\phi(p^\alpha)-1}(0) \pmod{p^2}, \tag{60}$$

$$\sum_{r=1}^{\left[\frac{p}{3}\right]}(-1)^{r-1}\frac{1}{p-3r} \equiv \frac{3^{\phi(p^\alpha)-1}}{2}\left(E_{\phi(p^\alpha)-1}(0)-E_{\phi(p^\alpha)-1}\left(\frac{1}{3}\right)\right) \pmod{p^2}, \tag{61}$$

where we use the fact that for $p>3$,

$$p\equiv 1 \pmod 3 \Leftrightarrow \exists \text{ even integer } k>1 \text{ satisfying } p=3k+1 \Leftrightarrow \left[\frac{p}{3}\right] \text{ is even,}$$

$$p\equiv 2 \pmod 3 \Leftrightarrow \exists \text{ odd integer } k\geq 1 \text{ satisfying } p=3k+2 \Leftrightarrow \left[\frac{p}{3}\right] \text{ is odd}$$

and $E_m(1-t)=(-1)^m E_m(t)$.

Example 3.5. By Fermat's quotients (35), we obtain

$$\frac{1}{a}=a^{p-2}-p\frac{q_p(a)}{a},$$

so that, after a is replaced by $p-nr$,

$$p\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}\frac{1}{p-nr}q_p(p-nr)=\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}(p-nr)^{p-2}-\sum_{r=1}^{\left[\frac{p}{n}\right]}(-1)^{r-1}\frac{1}{p-nr}. \tag{62}$$

Now, if we write $n=2$ in (17), then

$$\sum_{r=1}^{(p-1)/2}(-1)^{r-1}(p-2r)^{p-2}\equiv 2^{p-3}E_{p-2}(0) \pmod{p^2}, \tag{63}$$

where we have used $E_{2k+1}(1/2)=0$ with $k\geq 0$. Next, we put $n=2$ in (62) and use (60), (63), we get

$$\begin{aligned}\sum_{r=1}^{(p-1)/2}(-1)^{r-1}\frac{1}{p-2r}q_p(p-2r)\\ \equiv\frac{1}{2p}\left(2^{p-2}E_{p-2}(0)-2^{\phi(p^\alpha)-1}E_{\phi(p^\alpha)-1}(0)\right) \pmod{p},\end{aligned} \tag{64}$$

where $\alpha>1$ and $p>3$. Simiarly, the evaluation (62) with $n=3$ provides a new expression for

$$\sum_{r=1}^{\left[\frac{p}{3}\right]}(-1)^{r-1}\frac{1}{p-3r}q_p(p-3r)$$

by using (28) and (61).

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