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VERIFIED COMPUTATIONS OF SOLUTIONS FOR SOME UNILATERAL BOUNDARY VALUE PROBLEMS FOR SECOND ORDER EQUATIONS†

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ABSTRACT. In this paper, we propose a new iterative algorithm to automatically prove the existence of solutions for a unilateral boundary value problems for second order equations.

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1. Introduction

In [2], we invesigated some numerical methods for the automatic proof of existence of solutions for a unilateral boundary value problems for second order equations. It is difficulty to apply this method to a problem in which the associated operator is not retractive in a neighborhood of the solution. This is because this method uses the simple iteration method. In order to deal with such an operator, we have to devise a new method like a Newton-like method. In this article, to verify such a solution, we propose a new idea for the numerical verification of generalized obstacle problems. Finally, our aim is to give a solution with an error bound such that existence of the solution within these bounds is automatically verified. We hope to make progress in this direction in the future.

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2. Verification procedures by computer

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary Γ. We define

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx, \quad \nabla u \cdot \nabla v = \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}$$

and the set

$$K = \{v \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0, u \geq 0 \text{ on } \Gamma_+\}.$$

Setting $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}$, and denoting the inner product and norm on V respectively as the follow:

$$(u,v)_V = (\nabla u, \nabla v), \quad \|u\|_V = \|\nabla u\|_{L^2(\Omega)} = |u|_{H^2(\Omega)}.$$

Now, let us consider the following generalized unilateral problem:

$$\text{Find } u \in K \text{ such that } a(u, v-u) \geq (f(u), v-u), \forall v \in K, \tag{1}$$

where, f is a bounded and continuous map from V into $L^2(\Omega)$. In [2], the problem (1) is equivalent to that of finding $u \in V$ such that $u = P_K F(u)$, where $P_K F$ is compact operator on V. From Schauder's fixed point theorem, if we find a nonempty, bounded, convex and closed subset $U \subset V$ satisfying

$$P_K F(U) = \{P_K F(u) : u \in U\} \subset U,$$

there exists an element $u \in P_K F(U)$ such that $u = P_K F(u)$. Our goal is to find a set U.

First, describe the basic verification technique in the present paper. Let V_h be a finite dimensional subspace of V dependent on $0 < h < 1$. We shall denote by $I_\Omega = \{1, 2, 3, \cdots, m_0\}$ the set of all indices i associated with the internal nodes x_i of the domain Ω and we shall denote by $I_\Gamma = \{m_0+1, m_0+2, \cdots, m\}$ the set of all nodes indices i associated with the boundary nodes x_i of the domain Ω and let be $I = I_\Omega \cup I_\Gamma$. Here, for the sake of simplicity, let us assume that $I_{\Gamma_0} = \{m_0+1, m_0+2, \cdots, m_0+j\}$, $I_{\Gamma_+} = \{m_0+j+1, \cdots, m\}$, and $I_\Gamma = I_{\Gamma_0} \cup I_{\Gamma_+}$. We then define K_h, an approximation of K, by

$$K_h = \{v_h \in S_h : v_h(p_i) = 0, \forall i \in I_{\Gamma_0}, v_h(p_i) \geq 0, \forall i \in I_{\Gamma_+}\}.$$

It is easily seen that K_h is closed convex nonempty subset of V_h. Now, we consider the following auxiliary problem associated with (1), concerning any $g \in L^2(\Omega)$:

$$a(u, v-u) \geq (g, v-u), \forall v \in K, u \in K. \tag{2}$$

We then define the approximate problem corresponding to (2) as

$$a(u_h, v_h - u_h) \geq (g, v_h - u_h), \forall v_h \in K_h, u_h \in K_h. \tag{3}$$

Let u be the solution of (2) and $u_h \in K_h$ be the approximate solution of (3).

Theorem 2.1. *Let u and u_h be solutions of (2) and (3), respectively. If $g \in L^2(\Omega)$, then we have*

$$\|u_h - u\|_{L^2(\Omega)} \leq C(g,h)\frac{h^2}{\pi^2}, \tag{4}$$

where,

$$C(g,h) = \frac{h^2}{\pi^2}\|g\|_{L^2(\Omega)}.$$

For any $u \in V$, we now define the rounding $\mathcal{R}(P_K F(u)) \in K_h$ as the solution to the following problem:

$$a(\mathcal{R}(P_K F(u)), v_h - \mathcal{R}(P_K F(u))) \geq (f(u), v_h - \mathcal{R}(P_K F(u))), \forall v_h \in K_h.$$

Next, for a set $U \subset V$, we define the rounding $\mathcal{R}(P_K F(U)) \subset K_h$ as

$$\mathcal{R}(P_K F(U)) = \{u_h \in K_h : u_h = \mathcal{R}(P_K F(u)), u \in U\}.$$

In addition, we define for $U \subset V$ the rounding error $\mathcal{RE}(P_K F(U)) \subset V$ as

$$\mathcal{RE}(P_K F(U)) = \{v \in V : \|v\|_{H_0^1(\Omega)} \leq \sup_{u \in U} C(f(u), h)\}.$$

We choose such a set U of the form $U = U_h + U_\perp$, where $U_h \subset K_h$ and $U_\perp \subset V$. Then the verification condition can be written by

$$\mathcal{R}(P_K F(U)) \subset U_h, \quad \mathcal{RE}(P_K F(U)) \subset U_\perp. \tag{5}$$

Then (5) implies that

$$\mathcal{R}(P_K FU) + \mathcal{RE}(P_K FU) \subset U \tag{6}$$

Next let us introduce the procedure for finding such a set U of (6) using computers. We describe how to obtain such a set of V on a computer. Similarly in [3], we can proceed in the following manner.

Let $\{\phi_j\}_{j=1,\cdots,m}$ be a basis of V_h. A function $v_h \in V_h$ now has the representation

$$v_h(x) = \sum_{j=1}^{m} z_j \phi_j(x), z_j = v_h(x_j), \text{ for } x \in \overline{\Omega}.$$

Let us recall (see [2]) that the problem (2) is equivalent to the following system:

$$\begin{cases} D_{I_\Omega I} z_I - P_{I_\Omega} = 0, \\ (D_{I_\Gamma I} z_I - P_{I_\Gamma}) z_{I_\Gamma} = 0, \\ z_{I_\Gamma} \geq 0, z_{I_{\Gamma_0}} = 0, \\ w_{I_\Gamma} \equiv D_{I_\Gamma I} z_I - P_{I_\Gamma} \geq 0. \end{cases} \tag{7}$$

Here, $D_{II} \equiv (a_{ij})_{i,j \in I}$, with $a_{ij} = (\nabla\phi_i, \nabla\phi_j)$ and z_I is the coefficient vector for $\{\phi_i\}$ corresponding to the function u_h in (4.1). Further, $P_I \equiv \Big((g,\phi_i)\Big)_{i \in I}$ is an m dimensional vector. Condition (7) implies (because $z_{I_\Gamma}, w_{I_\Gamma} \geq 0$) that for I_Γ, either $z_{I_\Gamma} = 0$ or $w_{I_\Gamma} = 0$. Let $(\widetilde{w}_{I_\Gamma}, \widetilde{z}_{I_\Gamma})$ be an approximate solution of (7). Then, delete in (7) every variable $w_{I_\Gamma}, z_{I_\Gamma}$ for which the corresponding component of $\widetilde{w}_{I_\Gamma}, \widetilde{z}_{I_\Gamma}$ is approximately zero. Then, m new equations

$$\widetilde{D}_{II}\widetilde{e}_I - \widetilde{P}_I = 0 \tag{8}$$

remain. Note that the system (8) is linear. In order to find a set U satisfying the above verification condition, we use the simple iterative method as follows:

Set

$$U_h = \sum_{j=1}^{m} \left[\ \underline{A}_j,\ \ \overline{A}_j\ \right] \phi_j \tag{9}$$

and

$$U_\perp = \{\phi \in V : \|\phi\|_V \le \alpha\}, \tag{10}$$

respectively. Here $\sum_{j=1}^{m} \left[\ \underline{A}_j,\ \ \overline{A}_j\ \right] \phi_j$ is interpreted as the set of functions in which each element is a linear combination of $\{\phi_j\}_{j=1}^{m}$ whose coefficient of ϕ_j belongs to the corresponding interval $\left[\ \underline{A}_j,\ \ \overline{A}_j\ \right]$ for each $1 \le j \le m$. As stated above, we usually find a candidate set of the form

$$U = U_h + U_\perp \tag{11}$$

Let $\mathbf{R}^+$ denote the set of all nonnegative real numbers. For $\alpha \in \mathbf{R}^+$ we set

$$[\alpha] \equiv \{\phi \in V : \|\phi\|_V \le \alpha\}.$$

Now, let $A_j(\ 1 \le j \le m)$ be intervals on $\mathbf{R}$ and let $\sum_{j=1}^{m} A_j\phi_j$ be a linear combination of $\{\phi_j\}$. We denote the set of such interval functions by IS_h, i.e.,

$$IS_h \equiv \{\tilde{\phi} \in V_h; \tilde{\phi} = \sum_{j=1}^{m} A_j\phi_j\}.$$

Then, setting $U = \sum_{j=1}^{m} A_j\phi_j + [\alpha]$ and $g = f(U)$ in (2), we consider the linear system

$$\widetilde{D}_{II}\widetilde{e}_I = (f(U), \phi_j), 1 \le j \le m. \tag{12}$$

(12) is in fact a linear system of equations whose right-hand side consists of intervals. It can be easily seen that $\mathcal{R}(P_K F(U))$ is directly computed or enclosed from U_h and $U_\perp$ by solving a linear system of equations with right-hand side consists of intervals using some interval arithmetic approaches. On the other hand, $\mathcal{RE}(P_K F(U))$ is unknown but can be evaluated by the following constructive a priori error estimates:

$$\|u - u_h\|_V \le \sup_{u \in U} C(f(u), \psi, h).$$

Following [1], we have

Theorem 2.2. *Let E^* be interval solutions of the linear system (12) containing the actual solutions. Then, the following is true: If $\inf(E^*) \ge 0$, the problem (7) has an optimal solution $z_I \in \mathbf{R}^m$. The non-zero components of z_I are included in E^* and the others are zero.*

In the actual computation, we fix an approximate solution $u_h^{(0)} \in K_h$ of (1) such that $u_h^{(0)} = \sum_{j=1}^{M} u_j\phi_j$. We consider the set of functions $U_h \in IS_h$ of the form

$$U_h = \sum_{j=1}^{m} (u_j + A_j)\phi_j, \tag{13}$$

where A_j are intervals centered at 0 and $U_\perp$ is the same (10).

Then we take an interval vector(B_j) satisfying

$$\mathcal{R}(P_K F(U)) \subset \sum_{j=1}^{m} B_j \phi_j, \tag{14}$$

where (B_j) is determined by a solution of the linear system of equations with interval right hand side. That is, for the $m \times m$ matrix and the m dimensional interval vector $\mathbf{b} = (f(U), \phi_j)$, the interval vector (B_j) can be computed as a solution of the following equation

$$G \cdot (B_j) = \mathbf{b}. \tag{15}$$

Here, $(f(U), \phi_j)$ stands for the interval enclosure of the set $\{(f(u), \phi_j) \in \mathbf{R}; u \in U\}$. Next, we set

$$\beta = \sup_{u \in U} C(f(u), h). \tag{16}$$

Now the computable verification condition is described as

Theorem 2.3. *For the sets defined by (10), (11) and (13), if the following conditions hold*

$$B_j \subset u_j + A_j, \quad \beta \leq \alpha, \quad j = 1 \cdots m, \tag{17}$$

then there exists a solution u of $u = P_K F(u)$ in U.

Based on Theorem 2.3, we have the following verification algorithm by using simple iteration method with δ-inflation. In actual calculation, we use the following algorithm.

3. Algorithm

Step 1. Setting $A_j^{(0)} = [0, 0], j = 1 \cdots m$ and $\alpha^{(0)} = 0$, initial candidate set is defined by $U^{(0)} = \{u_h^{(0)}\}$. Usually, $u_h^{(0)}$ is determined as

$$a(u_h^{(0)}, v_h - u_h^{(0)}) \geq (f(u_h^{(0)}), v_h - u_h^{(0)}), \forall v_h \in K_h, u_h^{(0)} \in K_h.$$

Step 2. For the candidate set $U^{(k)}$ determined by $A_j^{(k)}$ and $\alpha^{(k)}$, we compute $(B_j^{(k)})$ and $\beta^{(k)}$ from (15) and (16), respectively. Here, $(B_j^{(k)})$ is determined as the solution set of (15), as described above. Of course, $(B_j^{(k)})$ satisfies the conditions of Theorem 2 in application to the case in which $U = U^{(k)}$.

Step 3. If (17) holds, then there exists a desired solution in the set

$$U^{(k)} = u_h^{(0)} + \sum_{j=1}^{m} A_j^{(k)} \phi_j + [\alpha^{(k)}];$$

stop; if not, go to the next step.

Step 4. Using some fixed constant $\delta > 0$, we set

$$A_j^{(k+1)} = [-\delta, \delta] + B_j^{(k)}, j = 1 \cdots m,$$

$$\alpha^{(k+1)} = (1+\delta)\beta^{(k)}$$

and goto Step 2. Here δ is an inflation parameter.

4. A Numerical Example

We now provide a numerical example of verification in the one-dimensional case following the procedure described in the previous section.

Let $\Omega = (0,1)$. We consider the case $f(u) = u + (\dfrac{4-9\pi^2}{4}) sin \frac{3\pi}{2} x$ and use a uniform partition of Ω, that is, $x_i = \frac{i}{m}$, $0 \le i \le m$. Set $e_i = (x_{i-1}, x_i)$; then we have $h = \frac{1}{m}$. We take

$$V_h \equiv \{v_h \in C^0(0,1) : v_h|_{e_i} \in P_1(e_i), 1 \le i \le m\},$$

where $P_1(e_i)$ is the space of polynomials of degree ≤ 1 on e_i. We now choose the basis $\{\phi_i\}_{i=1}^m$ of V_h as the usual hat functions.

The execution conditions are as follows.

Numbers of elements $= 100$,
Extension parameters : $\epsilon = 10^{-3}$,
Initial values : $u_h^{(0)} =$ Galerkin approximation in Step 1. $\alpha_0 = 0$.

Results are as follows:

Iteration numbers for verification : $k = 3$,
$L^2 -$ error bound : 0.0012317,
Maximum width of coefficient intervals in $\{A_j^{(N)}\}$: 0.0088614.

In Figure 1, the values of α, $\max|A_j^{(n)}|$ for various of m are shown.

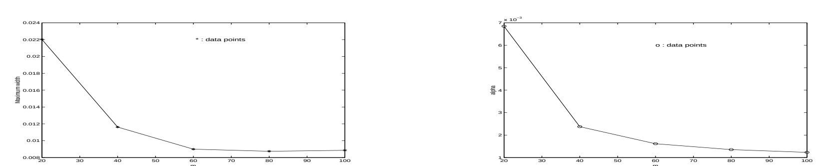

FIGURE 1. Value of $\alpha, \max|A_j^{(n)}|$

5. Concluding Remarks

In this paper, numerical verification of solutions for some unilateral boundary value problems for second order equations using a finite element method have been discussed only for the simple iteration method. Hence, if the compact operator $P_K F$ is not retractive operator in the neighborhood of the solution, it is difficult to use the scheme proposed in this paper. In order to verify the existence of solutions for a general problem, we need a Newton-like type method. By applying such an idea from this paper, we are able to establish solutions for some unilateral boundary value problems for second order equations without any restrictions on the associated operator. In order to use a Newton-like type method, a major difficulty in solving the fixed point formulation $u = P_K F(u)$ numerically is the treatment of the non-differentiable operator $P_K F$. However, using (12), we cannot only define a Newton-like operator, but also devise a Newton-like method.

References

1. N.W. Bazley, D.W. Fox, *Comparision operators for lower bounds to eigenvalues*, J. reine angew. Math. **223** (1966), 142-149.
2. J. Cea, *Optimisation, théorie et algorithmes*, Paris: Dunod, 1971.
3. X. Chen, *A verification method for solutions of nonsmooth equations*, Computing **58** (1997), 281-294.
4. P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
5. R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*. Springer, New York, 1984.
6. I. Hlaváček, J. Haslinger, J. Nečas, J. Lovíšek, *Solution of Variational Inequalities in Mechanics*, Springer Ser. Appl. Math. Sci. **66** (1988), 221-265.
7. T. Kato, *On the upper and lower bounds of eigenvalues*, J. Phys. Soc. Japan **4** (1949), 334-339.
8. N.J. Lehmann, *Optimale Eigenwerteinschlieungen*, Numer. Math. **5** (1963), 246-272.
9. U. Mosco, *Approximation of the solutions of some variational inequalities*, Roma: Press of Roma University, 1971.
10. M.T. Nakao, *A numerical verification method for the existence of weak solutions for nonlinear boundary value problems*, Journal of Math. Analysis and Appl. **164** (1992), 489-507.
11. Y. Watanabe, N. Yamamoto, M.T. Nakao, *Verified computations of solutions for nondifferential elliptic equations related to MHD equilibria*. Nonlinear Analysis, Theory, Mathods & Applications **28** (1997), 577-587.
12. F. Natterer, *Optimale L^2-Konvergenz finiter Elemente bei Variationsungleichungen*, Bonn. Math. Schr. **89** (1976), 1-12.
13. M. Plum, *Explict H_2-Estimates and Pointwise Bounds for Solutions of Second-Order Elliptic Bounday Value Problems*. Journal of Mathematical Analysis and Appl. **165** (1992), 36-61.
14. M. Plum, *Enclosures for weak solutions of nonlinear elliptic boundary value problems*, WSSIAA, World Scientific Publishing Company **3** (1994), 505-521.
15. M. Plum, *Existence and Enclosure Results for Continua of Solutions of Parameter - Dependent Nonlinear Boundary Value Problems*, Journal of Computational and Applied Mathematics **60** (1995), 187-200.

16. S.M. Rump, *Solving algebraic problems with high accuracy*, A new approach to scientific computation (Edited by U.W. KULISH and W.L. MIRANKER), Academic Press, New York, 1983.
17. C.S. Ryoo, *Numerical verification of solutions for a simplified Signorini problem*, Comput. Math. Appl. **40** (2000), 1003-1013.
18. C.S. Ryoo, *An approach to the numerical verification of solutions for obstacle problems*, Comput. Math. Appl. **53** (2007), 842-850.
19. C.S. Ryoo, *Numerical verification of solutions for Signorini problems using Newton-like mathod*, International Journal for Numerical Methods in Engineering **73** (2008), 1181-1196.
20. C.S. Ryoo and R.P. Agarwal, *Numerical inclusion methods of solutions for variational inequalities*, International Journal for Numerical Methods in Engineering **54** (2002), 1535-1556.
21. C.S. Ryoo and R.P. Agarwal, *Numerical verification of solutions for generalized obstacle problems*, Neural, Parallel & Scientific Compuataions **11** (2003), 297-314.
22. C.S. Ryoo and M.T. Nakao, *Numerical verification of solutions for variational inequalities*, Numerische Mathematik **81** (1998), 305-320.
23. F. Scarpini, M.A. Vivaldi, *Error estimates for the approximation of some unilateral problems*, R.A.I.R.O. Numerical Analysis **11** (1977), 197-208.
24. G. Strang, G. Fix, *An analysis of the finite element method*, Englewood Cliffs, New Jersey, Prentice-Hall, 1973.

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