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FIXED POINT THEOREMS INVOLVING C-CLASS FUNCTIONS IN G_b-METRIC SPACES

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Abstract. The purpose of this paper is to prove some fixed point theorems using (ψ, φ)-contractions via the concept of C-class functions in G_b-metric spaces. Moreover, an example is presented to illustrate the validity of our results.

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1. Introduction and Preliminaries

In 1989, Bakhtin [11] introduced the notion of b-metric spaces which is an interesting generalization of usual metric spaces.

Subsequently, many authors studied b-metric spaces and their topological properties and obtained a number of fixed point theorems for single and multi-valued mappings satisfying different contractive conditions, for more details see ([1], [11], [12], [14], [16], [19], [22], [23], [34], [35], [36]).

On the other hand, in 2006 Mustafa and Sims [25] introduced a new notion of generalized metric spaces named as G-metric space. Based on the notion of a metric space, many fixed point results for different contractive conditions have been presented ([2], [25], [26], [27], [28], [29], [33]).

Recently, based on the two above notions, in [3], Aghajani et al. introduced a new class of generalized metric space, called G_b-metric by combining the notions of b-metric space and G-metric space. In fact, various researchers studied many fixed point theorems for self mappings in this structure (G_b-metric), for example we refer readers to ([3], [10], [13], [18], [20], [24], [30], [31], [32]).

In this paper, in the setting of G_b-metric space, we will obtain fixed point results for single mapping satisfying certain contractive condition. The obtained results extend many recent results in the literature.

We recall some concepts as follows:

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Definition 1.1. [3] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that the mapping $G_b : X \times X \times X \to \mathbb{R}^+$ satisfies:

(G_b1) $G_b(x, y, z) = 0$ if $x = y = z$,

(G_b2) $0 < G_b(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_b3) $G_b(x, x, y) \leq G_b(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_b4) $G_b(x, y, z) = G_b(x, z, y) = G_b(y, z, x) = \cdots$ (symmetry in all three variables), and

(G_b5) $G_b(x, y, z) \leq s\,(G_b(x, a, a) + G_b(a, y, z))$ for all $x, y, z, a \in X$. (rectangle inequality).

It is obvious that G_b-metric space is effectively larger than that of G-metric space. Actually, each G-metric space is a G_b-metric space with $s = 1$.

Example 1.2. [3] Let (X, G) be a G-metric space. Consider $G_*(x, y, z) = G(x, y, z)^p$ where $p > 1$ is a real number. Then, G_* is a G_b-metric with $s = 2^{p-1}$.

Example 1.3. [3] The function defined by

$$G_b(x, y, z) = \frac{1}{9}\left(|x - y| + |y - z| + |x - z|\right)^2$$

is a G_b-metric on $X = \mathbb{R}$.

Example 1.4. Let $X = \mathbb{R}$ and let

$$G_b(x, y, z) = \max\left\{|x - y|^2 + |y - z|^2 + |x - z|^2\right\}.$$

Then (X, G_b) is a G_b-metric space with the coefficient $s = 2$.

Proposition 1.5. [3] *Let (X, G_b) be a G_b-metric space, then for each $x, y, z, a \in X$, we have:*

(1) if $G_b(x, y, z) = 0$, then $x = y = z$,

(2) $G_b(x, y, z) \leq s(G_b(x, x, y) + G_b(x, x, z)$,

(3) $G_b(x, y, y) \leq 2sG_b(y, x, x)$, and

(4) $G_b(x, y, z) \leq s(G_b(x, a, z) + G_b(a, y, z))$.

Definition 1.6. [3] A sequence $\{x_n\}$ in a G_b-metric space X is:

(i) a G_b-Cauchy sequence if, for every $\varepsilon > 0$, there is a natural number n_0 such that for all $n, m, l \geq n_0$, $G_b(x_n, x_m, x_l) < \varepsilon$.

(ii) a G_b-Convergent sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$ such that for all $n, m \geq n_0$, $G_b(x_n, x_m, x) < \varepsilon$.

Definition 1.7. [3] A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.

Proposition 1.8. [3] *Let X be a G_b-metric space. Then the following are equivalent:*

(1) The sequence $\{x_n\}$ is G_b-convergent to x.

(2) $G_b(x_n, x_n, x) \to 0$ as $n \to +\infty$.

(3) $G_b(x_n, x, x) \to 0$ as $n \to +\infty$.

Proposition 1.9. [3] *Let X be a G_b-metric space. Then the following are equivalent:*

(1) The sequence $\{x_n\}$ is G_b-Cauchy.

(2) For every for every $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $G(x_n, x_m, x_m) < \varepsilon$.

Lemma 1.10. *Let X be a G_b-metric space with $s \geq 1$. If a sequence $\{x_n\}$ is G_b-convergent, then it has a unique limit point.*

Lemma 1.11. [3] *Let (X, G_b) be a G_b-metric space with $s \geq 1$, and suppose that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are G_b-convergent to x, y and z respectively. Then we have*

$$\frac{1}{s^3}G(x,y,z) \leq \inf G(x_n, y_n, z_n) \leq \sup G(x_n, y_n, z_n) \leq s^3 G(x,y,z).$$

Arslan Hojat Ansari in [4] introduced the concept of a C-class functions which covers a large class of contractive conditions.

Definition 1.12. [4] A continuous function $F : [0, +\infty)^2 \to \mathbb{R}$ is called C-class function if for any $s, t \in [0, +\infty)$; the following conditions hold

$c1$ $F(s,t) \leq s$,

$c2$ $F(s,t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $F(0,0) = 0$ could be imposed in some cases if required. The letter C will denote the class of all C- functions.

Example 1.13. The following examples shows that the class C is nonempty:

1. $F(s,t) = s - t$,
2. $F(s,t) = ms$, for some $m \in (0,1)$,
3. $F(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,1)$,
4. $F(s,t) = \frac{\log(t+a^s)}{(1+t)}$, for some $a > 1$,
5. $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$,
6. $F(s,t) = s\beta(s)$, with $\beta : [0, +\infty) \to (0,1)$ is continuous,
7. $F(s,t) = s - \frac{t}{k+t}$,
8. $F(s,t) = s - (\frac{2+t}{1+t})t$,
9. $F(s,t) = \sqrt[n]{\ln(1+s^n)}$,
10. $F(s,t) = \frac{s}{(1+s)^r}$.

Let Φ_u denote the class of the functions $\varphi : [0, +\infty) \to [0, +\infty)$ which satisfy the following conditions:

a) φ is continuous,

b) $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.

In 1984, Khan et al. [17] introduced altering distance function as follows:

Definition 1.14. [17] A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

i) ψ is non-decreasing and continuous,

ii) $\psi(t) = 0$ if and only if $t = 0$.

Let us suppose that Ψ denote the class of the altering distance functions.

Definition 1.15. A tripled (ψ, φ, F) where $\psi \in \Psi$; $\varphi \in \Phi_u$ and $F \in C$ is said to be a monotone if for any $x, y \in [0, +\infty)$

$$x \leq y \Rightarrow F(\psi(x), \varphi(x)) \leq F(\psi(y), \varphi(y)).$$

Example 1.16. Let $F(s,t) = s - t$, $\varphi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & if \quad 0 \leq x \leq 1 \\ x^2 & if \quad x > 1 \end{cases},$$

then (ψ, φ, F) is monotone.

Example 1.17. Let $F(s,t) = s - t$, $\varphi(x) = x^2$

$$\psi(x) = \begin{cases} \sqrt{x} & if \quad 0 \leq x \leq 1 \\ x^2 & if \quad x > 1 \end{cases},$$

then (ψ, φ, F) is monotone.

Example 1.18. Let $F(s,t) = \frac{s}{1+t}$, $\varphi(x) = \sqrt[3]{x}$

$$\psi(x) = \begin{cases} \sqrt[3]{x} & if \quad 0 \leq x \leq 1 \\ x^3 & if \quad x > 1 \end{cases},$$

then (ψ, φ, F) is monotone.

Example 1.19. Let $F(s,t) = \log\left(\frac{t+e^s}{1+t}\right)$, $\varphi(x) = e^x$ and $\psi(x) = x$
then (ψ, φ, F) is monotone.

Example 1.20. Let $F(s,t) = s - t$, $\varphi(x) = x^3$

$$\psi(x) = \begin{cases} \sqrt[3]{x} & if \quad 0 \leq x \leq 1 \\ x^3 & if \quad x > 1 \end{cases},$$

then (ψ, φ, F) is monotone.

Fixed point theorems involving C-class function have been studied in ([4], [5], [6], [7], [8], [9], [15], [21]).

2. Main results

Now, we are ready to state our main theorem

Theorem 2.1. *Let (X, G_b) be a G_b-metric space with coefficient $s \geq 1$ and suppose mappings $f : X \to X$ satisfying the inequality*

$$\psi\left(2sG_b(fx, fy, fz)\right) \leq F\left(\psi\left(M(x,y,z)\right), \varphi\left(M(x,y,z)\right)\right), \tag{1}$$

where $F : [0, +\infty)^2 \to \mathbb{R}$ is C-class function, $\psi : [0, +\infty) \to [0, +\infty)$ is an altering distance function, $\varphi : [0, +\infty) \to [0, +\infty)$ is an ultra altering distance function and

$$\begin{aligned} M(x,y,z) \quad = \quad & \max\{G_b(x,y,z), G_b(x,fx,fx), G_b(y,fy,fy), G_b(z,fz,fz), \\ & G_b(x,fy,fy), G_b(y,fz,fz), G_b(z,fx,fx)\}, \end{aligned}$$

for all $x, y, z \in X$. Then f has a unique fixed point in X.

Proof. Suppose that x_0 is an arbitrary point in X. Define a sequence $\{x_n\}$ by $x_{n+1} = fx_n$.

Let $x = x_{n-1}$ and $y = z = x_n$. By Equation (1), for any $n \in \mathbb{N}$, we obtain

$$\psi\left(2sG_b(fx_{n-1}, fx_n, fx_n)\right) \leq F\left(\psi\left(M(x_{n-1}, x_n, x_n)\right), \varphi\left(M(x_{n-1}, x_n, x_n)\right)\right),$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n) &= \max\{G_b(x_{n-1}, x_n, x_n), G_b(x_{n-1}, fx_{n-1}, fx_{n-1}), \\ &\quad G_b(x_n, fx_n, fx_n), G_b(x_n, fx_n, fx_n), \\ &\quad G_b(x_{n-1}, fx_n, fx_n), G_b(x_n, fx_n, fx_n), G_b(z, fx_{n-1}, fx_{n-1})\} \\ &= \max\{G_b(x_{n-1}, x_n, x_n), G_b(x_{n-1}, x_n, x_n), \\ &\quad G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_n, x_{n+1}, x_{n+1}), \\ &\quad G_b(x_{n-1}, x_{n+1}, x_{n+1}), G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_n, x_n, x_n)\} \\ &= \max\{G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \\ &\quad G_b(x_{n-1}, x_{n+1}, x_{n+1})\} \\ &\leq \max\{G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \\ &\quad sG_b(x_{n-1}, x_n, x_n) + sG_b(x_n, x_{n+1}, x_{n+1})\} \\ &= s\left(G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1})\right). \end{aligned}$$

Hence

$$\begin{aligned} \psi\left(2sG_b(fx_{n-1}, fx_n, fx_n)\right) &\leq F\left(\psi\left(M(x_{n-1}, x_n, x_n)\right), \varphi\left(M(x_{n-1}, x_n, x_n)\right)\right) \\ &\leq \psi\left(M(x_{n-1}, x_n, x_n)\right), \end{aligned}$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n) &= \max\{G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \\ &\quad G_b(x_{n-1}, x_{n+1}, x_{n+1})\} \end{aligned}$$

implies that

$$\begin{aligned} \psi\left(G_b(x_n, x_{n+1}, x_{n+1})\right) &\leq F\left(\psi\left(M(x_{n-1}, x_n, x_n)\right), \varphi\left(M(x_{n-1}, x_n, x_n)\right)\right) \\ &\leq \psi\left(s\left(G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1})\right)\right). \end{aligned}$$

By the nondecreasing of ψ , it follows that

$$2sG_b(x_n, x_{n+1}, x_{n+1}) \leq s\left(G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1})\right).$$

It can be shown that for all $n \in \mathbb{N}$,

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq G_b(x_{n-1}, x_n, x_n).$$

This means $\{G_b(x_n, x_{n+1}, x_{n+1})\}$ is a decreasing sequence. Thus it converges and there exists $r \geq 0$ such that

$$\lim_{n\to+\infty} G_b(x_n, x_{n+1}, x_{n+1}) = r.$$

So, there exists $r \geq 0$, such that

$$\begin{aligned}\psi(\lim_{n\to+\infty} G_b(x_n, x_{n+1}, x_{n+1})) &\leq F(\psi(\lim_{n\to+\infty} G_b(x_{n-1}, x_n, x_n)),\\ &\quad \varphi(\lim_{n\to+\infty} G_b(x_{n-1}, x_n, x_n)))\\ &\leq \psi(\lim_{n\to+\infty} G_b(x_{n-1}, x_n, x_n). \qquad (2)\end{aligned}$$

Taking $n \to +\infty$ in (2), we have

$$\psi(r) \leq F(\psi(r), \varphi(r)) \leq \psi(r),$$

so,$\psi(r) = 0$ Consequently $r = 0$, i.e.,

$$\lim_{n\to+\infty} G_b(x_n, x_{n+1}, x_{n+1}) = 0. \qquad (3)$$

Now, we shall show that $\{x_n\}$ is a G_b-Cauchy sequence. It is sufficient to show that $\{x_n\}$ is G_b-Cauchy in X. If it is not, there is $\varepsilon > 0$ and integers n_k, m_k with $n_k > m_k > k$ such that

$$G_b(x_{m_k}, x_{n_k}, x_{n_k}) \geq \varepsilon \quad \text{and} \quad G_b(x_{m_k}, x_{n_k-1}, x_{n_k-1}) < \varepsilon. \qquad (4)$$

By using(3) and (4),we heve

$$\frac{\varepsilon}{s} \leq \lim_{k\to+\infty} \inf G_b(x_{m_k}, x_{n_k}, x_{n_k}), \qquad (5)$$

and by (1), we get

$$\begin{aligned}\psi(2sG_b(x_{n_k}, x_{m_k+1}, x_{m_k+1})) &= \psi(2sG_b(fx_{n_k-1}, fx_{m_k}, fx_{m_k}))\\ &\leq F(\psi(M(x_{n_k-1}, x_{m_k}, x_{m_k})), \qquad (6)\\ &\quad \varphi(M(x_{n_k-1}, x_{m_k}, x_{m_k}))), \qquad (7)\end{aligned}$$

where

$$\begin{aligned}M(x_{n_k-1}, x_{m_k}, x_{m_k}) &= \max\{G_b(x_{n_k-1}, x_{m_k}, x_{m_k}), G_b(x_{n_k-1}, fx_{n_k-1}, fx_{n_k-1}),\\ &\quad G_b(x_{m_k}, fx_{m_k}, fx_{m_k}), G_b(x_{m_k}, fx_{m_k}, fx_{m_k}),\\ &\quad G_b(x_{n_k-1}, fx_{m_k}, fx_{m_k}), G_b(x_{m_k}, fx_{m_k}, fx_{m_k}),\\ &\quad G_b(x_{m_k}, fx_{n_k-1}, fx_{n_k-1})\}\\ &= \max\{G_b(x_{n_k-1}, x_{m_k}, x_{m_k}), G_b(x_{n_k-1}, x_{n_k}, x_{n_k}),\\ &\quad G_b(x_{m_k}, x_{m_k+1}, x_{m_k+1}), G_b(x_{m_k}, x_{m_k+1}, x_{m_k+1}),\\ &\quad G_b(x_{n_k-1}, x_{m_k+1}, x_{m_k+1}), G_b(x_{m_k}, x_{m_k+1}, x_{m_k+1}),\\ &\quad G_b(x_{m_k}, x_{n_k}, x_{n_k})\}.\end{aligned}$$

Taking $k \to +\infty$ in the above inequalities and applying (3), (4) and (5), we get

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to+\infty} \inf M(x_{n_k-1}, x_{m_k}, x_{m_k}) \leq \lim_{k\to+\infty} \sup M(x_{n_k-1}, x_{m_k}, x_{m_k}) \leq \varepsilon. \quad (8)$$

Note that

$$G_b(x_{m_k+1}, x_{n_k}, x_{n_k}) \leq 2sG_b(x_{n_k}, x_{m_k+1}, x_{m_k+1}), \tag{9}$$

which implies that by (6), (8) and (9), we obtain

$$\begin{aligned}
\psi(\varepsilon) &\leq F(\lim_{k\to+\infty}\sup\psi(M(x_{n_k-1}, x_{m_k}, x_{m_k})), \\
&\quad \lim_{k\to+\infty}\inf\varphi(M(x_{n_k-1}, x_{m_k}, x_{m_k}))) \\
&\leq F(\psi(\varepsilon), \lim_{k\to+\infty}\inf\varphi(M(x_{n_k-1}, x_{m_k}, x_{m_k}))) \\
&\leq \psi(\varepsilon).
\end{aligned}$$

Hence, $\psi(\varepsilon) = 0$. Consequently, $\varepsilon = 0$.

Which leads to a contradiction because $\varepsilon > 0$.

It follows that $\{x_n\}$ is a G_b-Cauchy sequence and by the G_b-completeness of X, there exists $u \in X$ such that $\{x_n\}$ converges to u as $n \to +\infty$. We claim that $fu = u$. For this, consider

$$\psi(2sG_b(fx_n, fu, fu)) \leq F(\psi(M(x_n, u, u)), \varphi(M(x_n, u, u))),$$

where

$$\begin{aligned}
M(x_n, u, u) &= \max\{G_b(x_n, u, u), G_b(x_n, fx_n, fx_n), \\
&\quad G_b(u, fu, fu), G_b(u, fu, fu), G_b(x_n, fu, fu), \\
&\quad G_b(u, fu, fu), G_b(u, fx_n, fx_n)\} \\
&= \max\{G_b(x_n, u, u), G_b(x_n, x_{n+1}, x_{n+1}), \\
&\quad G_b(u, fu, fu), G_b(u, fu, fu), G_b(x_n, fu, fu), \\
&\quad G_b(u, fu, fu), G_b(u, x_{n+1}, x_{n+1})\}.
\end{aligned}$$

Taking $n \to +\infty$, we obtain that

$$\begin{aligned}
M(x_n, u, u) &= \max\{0, 0, G_b(u, fu, fu), \\
&\quad G_b(u, fu, fu), G_b(u, fu, fu), \\
&\quad G_b(u, fu, fu), 0\} \\
&= G_b(u, fu, fu).
\end{aligned}$$

So,

$$\begin{aligned}
\psi(2sG_b(x_{n+1}, fu, fu)) &\leq F(\psi(G_b(u, fu, fu)), \varphi(G_b(u, fu, fu))) \\
&\leq \psi(G_b(u, fu, fu)).
\end{aligned}$$

By the condition

$$G_b(fu, x_{n+1}, x_{n+1}) \leq 2sG_b(x_{n+1}, fu, fu).$$

We conclud that

$$\begin{aligned}
\psi(G_b(u, fu, fu)) &\leq F(\psi(G_b(u, fu, fu)), \varphi(G_b(u, fu, fu))) \\
&\leq \psi(G_b(u, fu, fu)).
\end{aligned}$$

Hence $fu = u$.

Finally to show the uniqueness of fixed point, suppose that v is another fixed point of f. By Equation(1)

$$\psi\left(2sG_b(fu, fv, fv)\right) \leq F\left(\psi\left(M(u, v, v)\right), \varphi\left(M(u, v, v)\right)\right), \tag{10}$$

where

$$\begin{aligned} M(u, v, v) &= \max\{G_b(u, v, v), G_b(u, fu, fu), \\ &\quad G_b(v, fv, fv), G_b(v, fv, fv), \\ &\quad G_b(u, fv, fv), G_b(v, fv, fv), G_b(v, fu, fu)\} \\ &= \max\{G_b(u, v, v), G_b(u, u, u), \\ &\quad G_b(v, v, v), G_b(v, v, v), \\ &\quad G_b(u, v, v), G_b(v, fv, fv), G_b(v, u, u)\} \\ &= \max\{G_b(u, v, v), 0, 0, 0, \\ &\quad G_b(u, v, v), 0, G_b(v, u, u)\} \\ &\leq 2sG_b(u, v, v). \end{aligned}$$

It follows that

$$\psi\left(2sG_b(fu, fv, fv)\right) \leq F\left(\psi\left(M(u, v, v)\right), \varphi\left(M(u, v, v)\right)\right), \tag{11}$$

$$\begin{aligned} \psi\left(2sG_b(u, v, v)\right) &\leq F\left(\psi\left(G_b(u, v, v)\right), \varphi\left(G_b(u, v, v)\right)\right) \\ &\leq \psi\left(G_b(u, v, v)\right), \end{aligned}$$

which implies that $G_b(u, v, v) = 0$,

As $G_b(u, v, v) = 0$ gives $u = v$, a contradiction.

If $M(u, v, v) = G_b(u, u, v)$, then

$$\begin{aligned} \psi\left(G_b(u, v, v)\right) &\leq F\left(\psi\left(G_b(u, u, v)\right), \varphi\left(G_b(u, u, v)\right)\right) \\ &\leq \psi\left(2sG_b(u, v, v)\right). \end{aligned}$$

So, by nondecreasing of ψ, it follows that

$$\psi\left(G_b(u, v, v)\right) = 0.$$

So, $G_b(u, v, v) = 0$

Thus the contradiction implies that the fixed point is unique.Therefore, the proof is completed □

Corollary 2.2. *Let f be self maps on a complete G_b-metric space with coefficient $s \geq 1$, satisfying the inequality*

$$\psi\left(2sG_b(fx, fy, fz)\right) \leq \psi\left(M(x, y, z)\right) - \varphi\left(M(x, y, z)\right)$$

where $\varphi \in \Psi$, $\psi \in \Psi$ and

$$\begin{aligned} M(x, y, z) &= \max\{G_b(x, y, z), G_b(x, fx, fx), \\ &\quad G_b(y, fy, fy), G_b(z, fz, fz), \end{aligned}$$

$$G_b(x, fy, fy), G_b(y, fz, fz), G_b(z, fx, fx)\},$$

for all $x, y, z \in X$. Then f has a unique fixed point in X.

Proof. Set $F(s,t) = s - t$ in Theorem (2.1). □

Corollary 2.3. *Let (X, G_b) be a G_b-metric space with coefficient $s \geq 1$ and suppose mappings $f : X \to X$ satisfying the inequality*

$$G_b(fx, fy, fz) \leq kM(x, y, z),$$

where

$$\begin{aligned} M(x,y,z) &= \max\{G_b(x,y,z), G_b(x, fx, fx), \\ &\quad G_b(y, fy, fy), G_b(z, fz, fz), \\ &\quad G_b(x, fy, fy), G_b(y, fz, fz), G_b(z, fx, fx)\}, \end{aligned}$$

for all $x, y, z \in X$. Then f has a unique fixed point in X.

Proof. Put $\psi(t) = t$, $F(s,t) = ks$ with $k \in (0,1)$, we can find the result of corollary □

Now, we give an example to support our result (2.1).

Example 2.4. Let $X = \left[0, \frac{1}{2}\right]$ and $G_b(x,y,z) = (\max\{x,y,z\})^2$ be a G_b- metric on X.

Then it is clear that (X, G_b) is a complete G_b-metric with $s = 2$. Also, define the mapping $f : X \to X$ by

$$f(x) = \frac{x^2}{3}.$$

We take $\psi(t) = t$ and $F(t,s) = \frac{2}{3}t$ for $t \in [0, +\infty)$, so that

$$F\left(\psi\left(M(x,y,z)\right), \varphi\left(M(x,y,z)\right)\right) = \frac{2}{3}\psi\left(M(x,y,z)\right) = \frac{2}{3}M(x,y,z),$$

where

$$\begin{aligned} M(x,y,z) &= \max\{G_b(x,y,z), G_b(x, fx, fx), \\ &\quad G_b(y, fy, fy), G_b(z, fz, fz), \\ &\quad G_b(x, fy, fy), G_b(y, fz, fz), G_b(z, fx, fx)\}. \end{aligned}$$

For all $x \leq y \leq z$, we have

$$\begin{aligned} \psi\left(2sG_b(fx, fy, fz)\right) &= 2sG_b(fx, fy, fz) \\ &= 2s\left(\max\{\frac{x^2}{3}, \frac{y^2}{3}, \frac{z^2}{3}\}\right)^2 \\ &= 2s\frac{z^4}{9} \leq \frac{2}{3}z^2. \end{aligned}$$

So, the axioms of Theorem (2.1) are satisfied, and 0 is the unique fixed point of f.

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