J. Appl. Math. & Informatics Vol. **39**(2021), No. 3 - 4, pp. 267 - 275
https://doi.org/10.14317/jami.2021.267

APPLICATION OF SUMUDU TRANSFORM METHOD FOR HYERS-ULAM STABILITY OF PARTIAL DIFFERENTIAL EQUATION

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Abstract. In this study, we investigate the generalized Hyers-Ulam Stability of partial differential equation of the form

$$y_t - ky_{xx} = 0.$$

AMS Mathematics Subject Classification : 34K20, 34K30, 34D04.
Key words and phrases : Partial differential equation, Hyers-Ulam Stability, Sumudu transform.

1. Introduction

Partial differential equations can serve as excellent tools for description of mathematical modelling of systems and processes in the fields of engineering, physics, chemistry, economics, aerodynamics, and polymerrheology, etc. Therefore, the qualitative behaviors of solutions of partial differential equations play an important role in many real world phenomena related to the sciences and engineering technique fields. However, we would not like to give the details of the applications related to partial differential equations here.This information indicates the importance of investigating the qualitative properties, Hyers-Ulam stability (HUS) and Hyers-Ulam Rassias stability (HURS) of partial differential equations. The Sumudu transform is defined over the set of the functions

$$A = \{f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{t}{\tau_j}},\ \text{if } t \in (-1)^j \times [0, \infty)\}$$

by the following formula

$$G(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t}dt$$

Received May 11, 2020. Revised February 15, 2021. Accepted February 25, 2021.

for $u \in (-\tau_1, \tau_2)$ (see [2, 3, 6, 16, 17, 20, 21]). In [16], Sumudu transform defined by

$$S[f(t)] = \int_0^\infty \frac{1}{u} f(t) e^{-\frac{t}{u}} dt.$$

For partial differential equation, the Sumudu transform is defined in the following form:

$$G(x,u) = S[f(x,t)] =: \int_0^\infty \frac{1}{u} f(x,t) e^{-\frac{t}{u}} dt.$$

S.M. Ulam discussed a problem of the stability of homomorphism, in 1940 (see [15]). This problem was answered partially, by Hyers [7] in the Banach spaces. After then, the stability issues of functional differential equations have been studied by many researchers. In the literature, first, Obłoza [12] initiated the study of the Hyers-Ulam stability (HUS) with the ordinary differential equation $x' = f(t,x)$.

Obłoza [13] compared (HUS) and Lyapunov stability, for the ordinary differential equation $x' = f(t,x)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. The author obtained new results on the connection between (HUS) and Lyapunov stability.

Recently, the theory of (HUS) has improved in studying different differential equations with motivating and results (see [1, 4, 5, 8-14, 18, 19]).

In this paper, we investigate the (HURS) of partial differential equation of the form

$$y_t - k y_{xx} = 0 \tag{1}$$

$$y(x,0) = 0, \tag{2}$$

where k is positive real constant and $(x,t) \in D$, $D = (x_0, x] \times (0, \infty)$.

Motivated by the mentioned sources, the aim of this paper is to prove the (HURS) of heat equation given by (1)-(2) by the Sumudu transform method. It is worth mentioning that, to the best of our knowledge, the Sumudu transform method is a very effective method to discuss the Hyers-Ulam stability of equation (1). In addition, to the best of our information till now, the Hyers-Ulam stability of equation (1)-(2) was not discussed in the literature by the Sumudu transform method. This paper is the first attempt in the literature on the topic for the mentioned equation. Our results will also be differ from those obtained in the literature. So, we mean that this paper may be useful for researchers working on the (HUS) to various differential and partially differential equations.

Definition 1.1. We say that equation (1)-(2) has the (HUS), if there exists a constant $K > 0$,with the following property:

for every $\varepsilon > 0, y : D \to \mathbb{R}$, if

$$|y_t - k y_{xx}| \leq \varepsilon,$$

then there exists some $z : D \to \mathbb{R}$ satisfying

$$z_t - kz_{xx} = 0$$

such that

$$|y(x,t) - z(x,t)| \leq K\varepsilon.$$

If the information given above is also true, when we substitute ε and $K\varepsilon$ by $\phi(x,t)$ and $\varphi(x,t)$, respectively, where ϕ, φ are continuous functions not depending on y and z, explicitly, then we call that the corresponding (DE) has the (HURS) (or the generalized (HUS)).

Let M and N be Sumudu transforms of the functions f and g, respectively. Then

$$S^{-1}[M(u)N(u))] = \int_0^t f(\tau)g'(t-\tau)d\tau. \tag{3}$$

2. Main results

Theorem 2.1. *Let $\varepsilon > 0, \theta : D \to \mathbb{R}$ be a continuous function and $S[\theta(x,t)] = \varphi(x,u)$. Assume that*

$$\int_{x_0}^{x} \exp\left(\frac{1}{\sqrt{u}}(v-x)\right)\varphi(v,u)dv \leq \varphi(x,u). \tag{4}$$

If the function y satisfies the following inequality:

$$|y_t - ky_{xx}| \leq \theta(x,t) \tag{5}$$

for all $(x,t) \in D$, then there exists a solution $z : D \to \mathbb{R}$ of equation (1)-(2) such that

$$|y(x,t) - z(x,t)| \leq \frac{1}{k}\theta(x,t).$$

Proof. Let

$$y(x,t) = X(x)T(t).$$

From this, we can write

$$\begin{aligned} y_{xx}(x,t) &= X''T \\ y_t(x,t) &= XT'. \end{aligned}$$

Then, from inequality (5), we get

$$-\theta(x,t) \leq XT' - kX''T \leq \theta(x,t).$$

Applying Sumudu transform to above inequality, we get

$$\begin{aligned} S(-\theta(x,t)) &\leq S(XT') - S(kX''T) \leq S(\theta(x,t)) \\ -\varphi(x,u) &\leq \int_0^\infty \frac{X}{u}\frac{\partial T}{\partial t}e^{-\frac{t}{u}}dt - k\int_0^\infty \frac{1}{u}X''T(t)e^{-\frac{t}{u}}dt \leq \varphi(x,u) \end{aligned}$$

$$-\varphi(x,u) \leq \frac{X}{u}G(u)-\frac{X}{u}T(0)-kX''G(u)\leq \varphi(x,u),$$

where $S(T(t))=G(u)$.So, we obtain

$$-\frac{1}{k}\varphi(x,u)\leq G(u)X''-G(u)\frac{X}{ku}+\frac{X}{ku}T(0)\leq \frac{1}{k}\varphi(x,u). \tag{6}$$

Since $y(x,0)=0$, we get $T(0)=0$. Thus, we obtain in the following inequality:

$$-\frac{1}{k}\varphi(x,u)\leq G(u)X''-G(u)\frac{X}{ku}\leq \frac{1}{k}\varphi(x,u).$$

Let

$$h(x,u)=G(u)X'-\frac{1}{\sqrt{u}}G(u)X \tag{7}$$

for any $(x,u)\in D$. Then

$$h_x(x,u)=G(u)X''-\frac{1}{\sqrt{u}}G(u)X'.$$

It follows that

$$-\frac{1}{k}\varphi(x,u)\leq h_x(x,u)+\frac{1}{\sqrt{u}}h(x,u)\leq \frac{1}{k}\varphi(x,u).$$

Multiplying the above estimate by $\exp(\frac{1}{\sqrt{u}}(x-x_0))$,we get

$$\begin{aligned} -\frac{1}{k}\exp(\frac{1}{\sqrt{u}}(x-x_0))\varphi(x,u) &\leq \frac{d}{dx}[\exp(\frac{1}{\sqrt{u}}(x-x_0))h(x,u)] \\ &\leq \frac{1}{k}\exp(\frac{1}{\sqrt{u}}(x-x_0))\varphi(x,u). \end{aligned}$$

For any x_0, integrating the above inequality from x_0 to x, and dividing by $\exp(\frac{1}{\sqrt{u}}(x-x_0))$, respectively, we obtain

$$\begin{aligned} -\frac{1}{k}\int_{x_0}^{x}\exp(\frac{1}{\sqrt{u}}(v-x_0))\varphi(v,u)dv &\leq \exp(\frac{1}{\sqrt{u}}(x-x_0))h(x,u)-h(x_0,u) \\ &\leq \frac{1}{k}\int_{x_0}^{x}\exp(\frac{1}{\sqrt{u}}(v-x_0))\varphi(v,u)dv, \end{aligned}$$

and

$$\begin{aligned} -\frac{1}{k}\int_{x_0}^{x}\exp(\frac{1}{\sqrt{u}}(v-x))\varphi(v,u)dv &\leq h(x,u)-h(x_0,u)\exp(-\frac{1}{\sqrt{u}}(x-x_0)) \\ &\leq \frac{1}{k}\int_{x_0}^{x}\exp(\frac{1}{\sqrt{u}}(v-x))\varphi(v,u)dv. \end{aligned}$$

Then from (4), we get

$$-\frac{1}{k}\varphi(x,u) \leq h(x,u) - h(x_0,u)\exp(-\frac{1}{\sqrt{u}}(x-x_0)) \leq \frac{1}{k}\varphi(x,u) \tag{8}$$

Since

$$h(x,u) = G(u)X' - \frac{1}{\sqrt{u}}G(u)X,$$

inequality (8) can be written as follows:

$$-\frac{1}{k}\varphi(x,u) \leq G(u)[X' - \frac{1}{\sqrt{u}}X] - h(x_0,u)\exp(-\frac{1}{\sqrt{u}}(x-x_0)) \leq \frac{1}{k}\varphi(x,u).$$

Similarly, multiplying the above inequality by $\exp(-\frac{1}{\sqrt{u}}(x-x_0))$, and integrating x_0 to x, we get

$$\begin{aligned} &-\frac{1}{k}\int_{x_0}^{x}\varphi(v,u)\exp(-\frac{1}{\sqrt{u}}(v-x_0))dv \\ \leq\ & G(u)[X\exp(-\frac{1}{\sqrt{u}}(x-x_0)) - X(x_0)] \\ &-\int_{x_0}^{x} h(x_0,u)\exp(-\frac{2}{\sqrt{u}}(v-x_0))dv \\ \leq\ & \frac{1}{k}\int_{x_0}^{x}\varphi(v,u)\exp(-\frac{1}{\sqrt{u}}(v-x_0))dv \end{aligned}$$

and then

$$\begin{aligned} &-\frac{1}{k}\int_{x_0}^{x}\varphi(v,u)\exp(-\frac{1}{\sqrt{u}}(v-x))dv \\ \leq\ & G(u)[X - X(x_0)\exp(\frac{1}{\sqrt{u}}(x-x_0))] \\ &-\int_{x_0}^{x} h(x_0,u)\exp(-\frac{1}{\sqrt{u}}(2v-x-x_0))dv \\ \leq\ & \frac{1}{k}\int_{x_0}^{x}\varphi(v,u)\exp(-\frac{1}{\sqrt{u}}(v-x))dv. \end{aligned}$$

From this, we get

$$\begin{aligned} &-\frac{1}{k}\varphi(x,u) \\ \leq\ & G(u)[X - X(x_0)\exp(\frac{1}{\sqrt{u}}(x-x_0))] \end{aligned}$$

$$
\begin{aligned}
& -\int_{x_0}^{x} h(x_0,u)\exp(-\frac{1}{\sqrt{u}}(2v-x-x_0))dv \\
\leq \quad & \frac{1}{k}\varphi(x,u),
\end{aligned}
$$

and consequently

$$
\begin{aligned}
-\frac{1}{k}\varphi(x,u) \quad \leq \quad & G(u)[X - X(x_0)\exp(\frac{1}{\sqrt{u}}(x-x_0))] \\
& +h(x_0,u)\frac{\sqrt{u}}{2}[\exp(-\frac{1}{\sqrt{u}}(x-x_0)) - \exp(-\frac{1}{\sqrt{u}}(x_0-x))] \\
\leq \quad & \frac{1}{k}\varphi(x,u).
\end{aligned}
$$

Applying inverse Sumudu transform to above inequality, we obtain

$$
\begin{aligned}
& S^{-1}[-\frac{1}{k}\varphi(x,u)] \\
\leq \quad & S^{-1}[G(u)X] - S^{-1}[G(u)X(x_0)\exp(\frac{1}{\sqrt{u}}(x-x_0))] \\
& +S^{-1}[h(x_0,u)\frac{\sqrt{u}}{2}(\exp(-\frac{1}{\sqrt{u}}(x-x_0))] \\
& -S^{-1}[\exp(-\frac{1}{\sqrt{u}}(x-x_0)))] \\
\leq \quad & S^{-1}[\frac{1}{k}\varphi(x,u)].
\end{aligned}
$$

We know that

$$
S^{-1}\left[\exp(-\frac{a}{\sqrt{u}})\right] = erfc\left(\frac{a}{2\sqrt{t}}\right),
$$

from the Table of the Special Sumudu transforms in [9]. By (3), we get

$$
S^{-1}[G(u)X(x_0)\exp(\frac{1}{\sqrt{u}}(x-x_0))] = X(x_0)\int_0^t erfc(\frac{(x_0-x)}{2\sqrt{\tau}})T'(t-\tau)d\tau.
$$

Similarly, for the $S^{-1}[h(x_0,u)\frac{\sqrt{u}}{2}(\exp(-\frac{1}{\sqrt{u}}(x-x_0)) - \exp(-\frac{1}{\sqrt{u}}(x-x_0)))]$, we know that

$$
S^{-1}[\frac{\sqrt{u}}{2}(\exp(-\frac{1}{\sqrt{u}}(x-x_0)))] = \frac{1}{\sqrt{\pi t}(x-x_0)^2}\int_0^\infty \sqrt{u}\exp(\frac{-u^2}{4(x-x_0)^2 t}J(2\sqrt{u})du
$$

and

$$
S^{-1}[\frac{\sqrt{u}}{2}(\exp(-\frac{1}{\sqrt{u}}(x_0-x)))] = \frac{1}{\sqrt{\pi t}(x-x_0)^2}\int_0^\infty \sqrt{u}\exp(\frac{-u^2}{4(x-x_0)^2 t}J(2\sqrt{u})du.
$$

from the Table of the Special Sumudu transforms in [9]. Let

$$\frac{1}{\sqrt{\pi t}(x-x_0)^2}\int_0^\infty \sqrt{u}\exp(\frac{-u^2}{4(x-x_0)^2 t}J(2\sqrt{u})du = A(x,t)$$

and

$$S^{-1}[h(x_0,u)] = H(x,t).$$

Then from (3), we get

$$S^{-1}[h(x_0,u)\frac{\sqrt{u}}{2}(\exp(-\frac{1}{\sqrt{u}}(x-x_0)))]$$

$$= \int_0^t A(x,\tau)H'(x,t-\tau)d\tau.$$

So, we obtain

$$S^{-1}[h(x_0,u)\frac{\sqrt{u}}{2}(\exp(-\frac{1}{\sqrt{u}}(x-x_0)) - \exp(-\frac{1}{\sqrt{u}}(x-x_0)))] = 0$$

$$-\frac{1}{k}\theta(x,t) \le X(x)T(t) - X(x_0)\int_0^t erfc(\frac{(x_0-x)}{2\sqrt{\tau}})T'(t-\tau)d\tau \le \frac{1}{k}\theta(x,t).$$

Then

$$-\frac{1}{k}\theta(x,t) \le y(x,t) - X(x_0)\int_0^t erfc(\frac{(x_0-x)}{2\sqrt{\tau}})T'(t-\tau)d\tau \le \frac{1}{k}\theta(x,t).$$

From above inequality, we get

$$|y(x,t)-z(x,t)| \le \frac{1}{k}\theta(x,t),$$

where

$$z(x,t) = X(x_0)\int_0^t erfc(\frac{(x_0-x)}{2\sqrt{\tau}})T'(t-\tau)d\tau.$$

□

Conclusion

Second order partial differential equation (weat equation) was considered. The (HURS) of this equation was investigated. We benefited from Sumudu transform method in proving our main results thus extending and improving some recent results found in the literature.

References

1. Q.H. Alqifiary and S.M. Jung, *Laplace transform and generalized Hyers-Ulam stability of linear differential equations*, Electron. J. Differential Equations **80** (2014), 1-11.
2. F.B.M. Belgacem, A.A. Karaballi, S.L. Kalla, *Analytical investigations of the Sumudu transform and applications to integral production equations*, Mathemetical Problems in Engineering **3** (2003), 103-118.
3. F.B.M. Belgacem, *Introducing and analysing deeper Sumudu properties*, Nonlinear Studies **13** (2006), 23-41.
4. E. Bicer and C. Tunc, *On the Hyers-Ulam stability of certain partial differential equations of second order*, Nonlinear Dyn. Syst. Theory **17** (2017), 150-157.
5. E. Bicer, *Hyers-Ulam stability for the wave equation*, Karaelmas Science and Engineering Journal **8** (2018), 264-267.
6. H. Eltayeb and A. Kılıçman, *A note on the Sumudu transforms and differential equations*, Applied Mathematical Sciences **4** (2010), 1089-1098.
7. Donald H. Hyers, *On the Stability of the Linear Functional Equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222-224.
8. S.M. Jung, *A Fixed Point Approach to the Stability of Differential Equations* $y' = F(x, y)$, Bull. Malays. Math. Sci. Soc. **33** (2010), 47-56.
9. F.B.M. Belgacem and A.A. Karaballi, *Sumudu Transform fundamental properties investigations and applications*, J. Appl. Math. Stoch. Anal. (2006), 1-23.
10. H. Liu and X. Zhao, *Hyers-Ulam-Rassias stability of second order partial differential equations*, Ann. Differential Equations **29** (2013), 430-437.
11. N. Lungu and D. Popa, *Hyers-Ulam stability of a first order partial differential equation*, J. Math. Anal. Appl. **385** (2012), 86-91.
12. M. Obłoza, *Hyers Stability of the Linear Differential Equation*, RocznikNauk.-Dydakt. Prace Mat. **13** (1993), 259-270.
13. M. Obłoza, *Connections between Hyers and Lyapunov Stability of the Ordinary Differential Equations*, Rocznik Nauk-Dydakt. Prace Mat. **14** (1997), 141-146.
14. C. Tunc and E. Bicer, *Hyers-Ulam-Rassias stability for a first order functional differential equation*, J. Math. Fundam. Sci. **47** (2015), 143-153.
15. S.M. Ulam, *HProblems in modern Mathematics*, Science Editions John Wiley& Sons, Inc. New York, 1964.
16. G.K. Watugala, *Sumudu transform: a new integral transform to solve differential equations and control engineering problems*, International Journal of Mathematical Education in Science and Technology **1** (1993), 35-43.
17. S. Weerakoon, *Application of Sumudu transform to partial differential equations*, Int. J. Math. Educ. Sci. Technol. **25** (1994), 277-283.
18. E. Bicer and C. Tunc, *On the Hyers -Ulam stability of Laguerre and Bessel equations by Laplace transform method*, Nonlinear Dyn. Syst. Theory **17** (2017), 340-346.
19. E. Bicer and C. Tunc, *New Theorems for Hyers-Ulam Stability of Lienard Equation with Variable Time Lags*, International Journal of Mathematics and Computer Science **3** (2018), 231-242.
20. A. Golmankhaneh and C. Tunç, *Sumudu Transform in Fractal Calculus*, Applied Mathematics and Computation **350** (2019), 386-401.
21. H. Khan, C. Tunc, R.A. Khan, A.G. Shirzoi, S.A. Gul, A. Khan, *Approximate analytical solutions of space-fractional telegraph equations by Sumudu Adomian decomposition method*, Appl. Appl. Math. **13** (2018), 781-802.

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