Bull. Korean Math. Soc. **58** (2021), No. 3, pp. 699–710
https://doi.org/10.4134/BKMS.b200473
pISSN: 1015-8634 / eISSN: 2234-3016

ON AMPLIATION QUASIAFFINE TRANSFORMS OF OPERATORS

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Abstract. In this paper, we study various connections of local spectral properties, invariant subspaces, and spectra when an operator S in $\mathcal{L}(\mathcal{H})$ is an ampliation quasiaffine transform of an operator T in $\mathcal{L}(\mathcal{H})$.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. As usual, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, and $\sigma_{re}(T)$ for the spectrum, the point spectrum, the approximate point spectrum, the essential spectrum, the left, and the right essential spectrum of T, respectively.

A subspace $\mathcal{M}$ of $\mathcal{H}$ is called an *invariant subspace* for an operator $T \in \mathcal{L}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$. The collection of all subspaces of $\mathcal{H}$ invariant under T is denoted by *Lat* T. We say that $\mathcal{M} \subset \mathcal{H}$ is a *hyperinvariant subspace* for $T \in \mathcal{L}(\mathcal{H})$ if $\mathcal{M}$ is an invariant subspace for every $S \in \mathcal{L}(\mathcal{H})$ commuting with T.

An operator T in $\mathcal{L}(\mathcal{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $\ker(U) = \ker(|T|) = \ker(T)$ and $\ker(U^*) = \ker(T^*)$. Associated with T is a related operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ called the *Aluthge transform* of T, denoted throughout this paper by $\tilde{T}$. In many cases, the Aluthge transforms of T have the better properties than T (see [14, 15] for more details). Another operator transform $|T|U$ of T, denoted $\widehat{T}$, is called the *Duggal transform* of T.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a *quasinormal* operator if T and T^*T commute. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a *p-hyponormal* operator if $(T^*T)^p \geq (TT^*)^p$, where $0 < p < \infty$. If $p = 1$, T is called *hyponormal* (see [8],

Received May 27, 2020; Accepted March 10, 2021.
2010 *Mathematics Subject Classification.* Primary 47A11; Secondary 47A10, 47B20.
Key words and phrases. Ampliation quasisimilarity, local spectral property, invariant subspace.
This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea Government (MSIT) (2019R1F1A1058633) and was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2019R1A6A1A11051177).

[10], and [21]). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a *binormal* operator if T^*T and TT^* commute.

An operator $X \in \mathcal{L}(\mathcal{H})$ is said to be *quasi-invertible* if X has zero kernel and dense range. An operator S in $\mathcal{L}(\mathcal{H})$ is said to be a *quasiaffine transform* of an operator T in $\mathcal{L}(\mathcal{H})$ if there is a quasi-invertible operator X in $\mathcal{L}(\mathcal{H})$ such that $XS = TX$, and this relation of S and T is denoted by $S \prec T$. Operators S and T in $\mathcal{L}(\mathcal{H})$ are said to be *quasisimilar* if there exist quasi-invertible operators X and Y which satisfy the equations $XT = SX$ and $TY = YS$. It is clear that quasisimilarity is an equivalence relation on $\mathcal{L}(\mathcal{H})$.

Definition. An operators S in $\mathcal{L}(\mathcal{H})$ is said to be an ampliation quasiaffine transform of T if there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that the m-th ampliation $S^{(m)} = \underbrace{S \oplus \cdots \oplus S}_{(m)}$ of S is a quasiaffine transform of the n-th ampliation $T^{(n)}$, i.e., $XS^{(m)} = T^{(n)}X$ for some quasi-invertible operator X. The operators S and T in $\mathcal{L}(\mathcal{H})$ are said to be ampliation quasisimilar if there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that the m-th ampliation $S^{(m)} = \underbrace{S \oplus \cdots \oplus S}_{(m)}$ of S is quasisimilar to the n-th ampliation $T^{(n)}$. Moreover, the operators S and T in $\mathcal{L}(\mathcal{H})$ are said to be ampliation similar if there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that the m-th ampliation $S^{(m)} = \underbrace{S \oplus \cdots \oplus S}_{(m)}$ of S is similar to the n-th ampliation $T^{(n)}$.

It is well known from [9] that ampliation quasisimilarity is an equivalence relation on $\mathcal{L}(\mathcal{H})$ more general than quasisimilarity, with the property that if S and T in $\mathcal{L}(\mathcal{H})$ are ampliation quasisimilar, then S has a nontrivial hyperinvariant subspace if and only if T does. However, the ampliation quasisimilarity does not imply the quasisimilarity. For example, if $N \in \mathcal{L}(\mathcal{H})$ is normal of multiplicity one, then N and $N \oplus N$ are ampliation quasisimilar, but N and $N \oplus N$ are not quasisimilar. Indeed, if N and $N \oplus N$ are quasisimilar, by [11] N and $N \oplus N$ are unitarily equivalent. So we have a contradiction.

In this paper, we study various connections of local spectral properties, invariant subspaces, and spectra when an operator S in $\mathcal{L}(\mathcal{H})$ is an ampliation quasiaffine transform of an operator T in $\mathcal{L}(\mathcal{H})$.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property modulo a closed set* $\mathcal{R} \subset \mathbb{C}$ if for every open subset G of $\mathbb{C}\backslash\mathcal{R}$ and any analytic function $f : G \to \mathcal{H}$ such that $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. In particular, when $\mathcal{R} = \emptyset$, an operator T is said to have the *single-valued extension property*, abbreviated SVEP. For an operator $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, the *resolvent set* $\rho_T(x)$ of T at x is defined to consist of z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ on a neighborhood of z_0, with values in

$\mathcal{H}$, which verifies $(T-z)f(z) \equiv x$. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. Using local spectra, we define the *local spectral subspace* of T by $\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$, where F is a subset of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *Bishop's property* (β) *modulo a closed set* $\mathcal{R} \subset \mathbb{C}$ if for every open subset G of $\mathbb{C}\backslash\mathcal{R}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. In particular, when $\mathcal{R} = \emptyset$, an operator T is said to have the Bishop's property (β). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable modulo a closed set* $\mathcal{R} \subset \mathbb{C}$ if for every open cover $\{U, V\}$ of $\mathbb{C}\backslash\mathcal{R}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\mathcal{H} = \mathcal{X} + \mathcal{Y}, \quad \sigma(T|_{\mathcal{X}}) \subset \overline{U}, \text{ and } \quad \sigma(T|_{\mathcal{Y}}) \subset \overline{V}.$$

In particular, when $\mathcal{R} = \emptyset$, an operator T is said to be decomposable. It is well known from [2], [17], and [20] that

Decomposable $\Rightarrow$ Bishop's property (β) $\Rightarrow$ Dunford's property (C) $\Rightarrow$ SVEP.

It can be shown that the converse implications do not hold in general as can be seen from [6], [16], and [17].

3. Main results

In this section we investigate various local spectral connections between S and T when they are ampliation quasisimilar. First of all, we begin with the following lemma.

Lemma 3.1. *Assume that operators S and T in $\mathcal{L}(\mathcal{H})$ are ampliation quasisimilar. Then, for a closed set $\mathcal{R} \subset \mathbb{C}$, the following statements hold.*

(i) *S has the single valued extension property modulo $\mathcal{R}$ if and only if T does.*

(ii) *S^* has the single valued extension property modulo $\mathcal{R}$ if and only if T^* does.*

Proof. (i) Assume that S has the single valued extension property modulo $\mathcal{R}$. If S and T are ampliation quasisimilar, then there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that $S^{(n)}$ and $T^{(m)}$ are quasisimilar, i.e.,

$$\begin{cases} XS^{(n)} = T^{(m)}X \quad \text{and} \\ S^{(n)}Y = YT^{(m)}, \end{cases}$$

where X and Y are quasi-invertible. Since S has the single valued extension property modulo $\mathcal{R}$, it is obvious that $S^{(n)}$ has the single valued extension property modulo $\oplus_1^n \mathcal{R}$ from [6]. Let $\oplus_1^m G$ be any open set in $\oplus_1^m \mathbb{C}\backslash \oplus_1^m \mathcal{R}$, and let $\oplus_1^m f$ be an $\oplus_1^m \mathcal{H}$-valued analytic function on $\oplus_1^m G$ such that $(T^{(m)} - \lambda I^{(m)}) \oplus_1^m f(\lambda) \equiv 0$ on $\oplus_1^m G$. Then

$$Y(T^{(m)} - \lambda I^{(m)}) \oplus_1^m f(\lambda) = (S^{(n)} - \lambda I^{(n)})Y \oplus_1^m f(\lambda) \equiv 0$$

on $\oplus_1^m G$. Since $S^{(n)}$ has the single valued extension property and Y is one-to-one, $f(\lambda) \equiv 0$ on G. Thus $T^{(m)}$ has the single valued extension property modulo $\oplus_1^n \mathcal{R}$. Now the remaining part for the proof is to show that T has the single valued extension property modulo $\mathcal{R}$. Let D be any open set in $\mathbb{C}\backslash\mathcal{R}$, and let h be an $\mathcal{H}$-valued analytic function on D such that $(T-\lambda)h(\lambda) \equiv 0$ on D. Then

$$(T^{(m)} - \lambda I^{(m)}) \oplus_1^m h(\lambda) \equiv 0.$$

Since $T^{(m)}$ has the single valued extension property modulo $\oplus_1^m \mathcal{R}$, $\oplus_1^m h(\lambda) \equiv 0$. Hence $h(\lambda) \equiv 0$. Thus T has the single valued extension property.

The converse implication is similar.

(ii) Since $S^{(n)}$ and $T^{(m)}$ are quasisimilar,

$$\begin{cases} (S^{(n)})^* X^* = X^* (T^{(m)})^* \quad \text{and} \\ Y^* (S^{(n)})^* = (T^{(m)})^* Y^*, \end{cases}$$

where X^* and Y^* are quasi-invertible. As some applications of the proof of (i), we can prove (ii). □

Recall that a conjugation on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be complex symmetric if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$ (see [12] and [13] for more details).

Lemma 3.2. *Assume that S and T are ampliation quasisimilar. If S is a complex symmetric operator with a conjugation C, then there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that*

$$\begin{cases} D(T^*)^{(m)} = T^{(m)} D \quad \textit{and} \\ (T^*)^{(m)} F = F T^{(m)}, \end{cases}$$

where D and F are antilinear and quasi-invertible. Also, the same relations hold for S and T^*.

Proof. If S and T are ampliation quasisimilar, then there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that $S^{(n)}$ and $T^{(m)}$ are quasisimilar, i.e.,

$$\begin{cases} X S^{(n)} = T^{(m)} X \quad \text{and} \\ S^{(n)} Y = Y T^{(m)}, \end{cases}$$

where X and Y are quasi-invertible. Since $CS^*C = S$, set $D = XC^{(n)}X^*$. Then D is antilinear and quasi-invertible. Also we have

$$\begin{aligned} T^{(m)} D = T^{(m)} X C^{(n)} X^* &= X S^{(n)} C^{(n)} X^* = X C^{(n)} (S^*)^{(n)} X^* \\ &= X C^{(n)} X^* (T^*)^{(m)} = D (T^*)^{(m)}. \end{aligned}$$

Similarly, set $F = Y^* C^{(m)} Y$. Then F is antilinear and quasi-invertible. We also get that $FT^{(m)} = (T^*)^{(m)} F$.

Since $(FX)S^{(n)} = (T^*)^{(m)}(FX)$ and $S^{(n)}(YD) = (YD)(T^*)^{(m)}$, the same relations hold for S and T^*. □

Theorem 3.3. *Let S and T be ampliation quasisimilar. If S is a complex symmetric operator with a conjugation C, then, for a closed set $\mathcal{R} \subset \mathbb{C}$, the following statements are equivalent.*

(i) *S has the single valued extension property modulo $\mathcal{R}$.*

(ii) *T has the single valued extension property modulo $\mathcal{R}$.*

(iii) *S^* has the single valued extension property modulo $\mathcal{R}$.*

(iv) *T^* has the single valued extension property modulo $\mathcal{R}$.*

Proof. Since (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) hold from Lemma 3.1, it suffices to show that (ii) $\Leftrightarrow$ (iv) holds. Assume that (ii) holds. By Theorem 3.2, there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that

$$\begin{cases} D(T^*)^{(m)} = T^{(m)}D \quad \text{and} \\ (T^*)^{(m)}F = FT^{(m)}, \end{cases}$$

where D and F are antilinear and quasi-invertible. Let G be any open set in $\mathbb{C}\backslash\mathcal{R}$, and let f be an $\mathcal{H}$-valued analytic function on G such that $(T^* - \lambda)f(\lambda) \equiv 0$ on G. Then

$$(T^{(m)} - \overline{\lambda}I^{(m)})D \oplus_1^m f(\lambda) = D((T^*)^{(m)} - \lambda I^{(m)}) \oplus_1^m f(\lambda) \equiv 0$$

on $\oplus_1^m G$. Set $\oplus_{j=1}^m g_j(\overline{\lambda}) = D \oplus_1^m f(\lambda)$. Then every g_j is an $\mathcal{H}$-valued analytic function on $\oplus_1^m \overline{G} = \{\overline{\lambda} : \lambda \in G\}$. Since $T^{(m)}$ has the single valued extension property modulo $\oplus_1^m \mathcal{R}$, $\oplus_{j=1}^m g_j(\overline{\lambda}) = D \oplus_1^m f(\lambda) \equiv 0$ on $\oplus_1^m \overline{G}$. Thus $f(\lambda) \equiv 0$ on G. Hence T^* has the single valued extension property modulo $\mathcal{R}$.

Conversely, let O be any open set in the complex plane, and let h be an $\mathcal{H}$-valued analytic function on O such that $(T - \lambda)h(\lambda) \equiv 0$ on O. Then

$$((T^*)^{(m)} - \overline{\lambda}I^{(m)})F \oplus_1^m h(\lambda) = F(T^{(m)} - \lambda I^{(m)}) \oplus_1^m h(\lambda) \equiv 0$$

on $\oplus_1^m O$. By the similar method, we can show that $h(\lambda) \equiv 0$ on O. Hence T has the single valued extension property modulo $\mathcal{R}$. □

As some applications of Theorem 3.3, we get the following corollaries.

Corollary 3.4. *Let S and T be ampliation quasisimilar. If S is normal, then T and T^* are the single valued extension property.*

Proof. Since S is complex symmetric and has the single valued extension property, both T and T^* have the single valued extension property from Theorem 3.3. □

Corollary 3.5. *Assume that operators S and T in $\mathcal{L}(\mathcal{H})$ are ampliation quasisimilar where S is complex symmetric. If S has the single valued extension property, then $\sigma(T) = \sigma_{ap}(T) = \sigma_{su}(T) = \cup_{x \in \mathcal{H}} \sigma_T(x)$ where $\sigma_{su}(T)$ denotes the surjectivity spectrum.*

Proof. Since T and T^* have the single valued extension property from Theorem 3.3, the proof follows from [1]. □

In the following theorem, we show that the ampliation quasisimilarity of two normal operators implies their unitary equivalence.

Theorem 3.6. *Assume that operators S and T in $\mathcal{L}(\mathcal{H})$ are normal. If S and T are ampliation quasisimilar, then there exist reducing subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ such that $S|_{\mathcal{M}}$ and $T|_{\mathcal{N}}$ are unitarily equivalent. In particular, if $\mathcal{M} = \mathcal{N} = \mathcal{H}$, then S and T are unitarily equivalent.*

Proof. If S and T are normal and are ampliation quasisimilar, then there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that $S^{(n)}$ and $T^{(m)}$ are normal and quasisimilar. By [5], $S^{(n)}$ and $T^{(m)}$ are unitarily equivalent. Hence there is a unitary operator $U = (U_{ij})_{n\times m}$ such that $U^*S^{(n)}U = T^{(m)}$, i.e., $SU_{ij} = U_{ij}T$ for all i, j. Set $\mathcal{M} = (\ker U_{ij})^{\perp}$ and $\mathcal{N} = \overline{ranU_{ij}}$. Then by [7], $S|_{\mathcal{M}}$ and $T|_{\mathcal{N}}$ are unitarily equivalent. In particular, if $\mathcal{M} = \mathcal{N} = \mathcal{H}$, then S and T are unitarily equivalent. □

If an operator S is an ampliation quasiaffine transform of an operator T, then it is known from [19] that $\sigma(S) \cap \sigma(T) \neq \emptyset$. For example, let S be denote the unilateral shift of multiplicity one. Then it is known from p. 14 in [4] that αS is an ampliation quasiaffine transform of S, i.e., $S^{(2)} \prec \alpha S$, where $0 < |\alpha| < 1$. Moreover, $\sigma(\alpha S) \subset \sigma(S)$. Let an operator S be an ampliation quasiaffine transform of an operator T. If T is an algebraic operator (i.e., satisfies some polynomial equation), then S is also algebraic. It was shown in [18] that if T is a nonalgebraic strict contraction having a cyclic vector, then S is an ampliation quasiaffine transform of T, i.e., $S^{(\infty)} \prec T$ where S denotes the unilateral shift of multiplicity one.

Theorem 3.7. *Let S and T be normal. Assume that an operator S is an ampliation quasiaffine transform of an operator T. If T is unitarily equivalent to $T|_{\mathcal{N}}$ for every infinite dimensional invariant subspace $\mathcal{N}$ for T, then $S^{(n)}$ is unitarily equivalent to $S^{(n)}|_{\oplus_1^n \mathcal{M}}$ for every infinite dimensional invariant subspace $\mathcal{M}$ for S and some positive integer n.*

Conversely, assume that an operator T is an ampliation quasiaffine transform of an operator S. If S is unitarily equivalent to $S|_{\mathcal{M}}$ for every infinite dimensional invariant subspace $\mathcal{M}$ for S, then $T^{(m)}$ is unitarily equivalent to $T^{(m)}|_{\oplus_1^m \mathcal{N}}$ for every infinite dimensional invariant subspace $\mathcal{N}$ for T and some positive integer m.

Proof. Assume that T is unitarily equivalent to $T|_{\mathcal{N}}$ for every infinite dimensional invariant subspace $\mathcal{N}$ for T. Since S is an ampliation quasiaffine transform of T, then there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that $S^{(n)}$ is a quasiaffine transform, i.e., $XS^{(n)} = T^{(m)}X$, where X is quasi-invertible. By [5], $S^{(n)}$ and $T^{(m)}$ are unitarily equivalent. Hence there is a unitary operator U such that $U^*T^{(m)}U = S^{(n)}$. For any infinite dimensional $\mathcal{M} \in Lat\ S$,

$$T^{(m)}U(\oplus_1^m \mathcal{M}) = US^{(n)}(\oplus_1^m \mathcal{M}) \subset U(\oplus_1^m \mathcal{M}).$$

Then $U(\oplus_1^m \mathcal{M}) \in Lat\ T^{(m)}$. Since $U(\oplus_1^m \mathcal{M})$ is infinite dimensional, by hypothesis there exists a unitary operator V such that $V^* T^{(m)} V = T^{(m)}|_{U(\oplus_1^m \mathcal{M})}$. Hence

$$\begin{aligned} S^{(n)}|_{\oplus_1^n \mathcal{M}} &= U^* T^{(m)} U|_{\oplus_1^n \mathcal{M}} = U^* T^{(m)}|_{U(\oplus_1^n \mathcal{M})} \\ &= U^* V^* T^{(m)} V = U^* V^* U (U^* T^{(m)} U) U^* V U U^* \\ &= (U^* V U)^* (U^* T^{(m)} U)(U^* V U) U^* \\ &= (U^* V U)^* S^{(n)} (U^* V U) U^*. \end{aligned}$$

Since $U^* \oplus_1^m \mathcal{H} = \oplus_1^n \mathcal{H}$,

$$S^{(n)}|_{\oplus_1^n \mathcal{M}} = (U^* V U)^* S^{(n)} (U^* V U)|_{\oplus_1^n \mathcal{H}}.$$

Hence $S^{(n)}$ is unitarily equivalent to $S^{(n)}|_{\oplus_1^n \mathcal{M}}$ for every infinite dimensional invariant subspace $\mathcal{M}$ for S and some positive integer n.

The converse implication is similar. □

In the following results, we consider some connections among local spectra and local spectral subspaces.

Lemma 3.8. *Assume that an operator S in $\mathcal{L}(\mathcal{H})$ is an ampliation quasiaffine transform of an operator T in $\mathcal{L}(\mathcal{H})$. Then $\cup_{j=1}^m \sigma_T(y_j) \subset \sigma_S(x)$ for all $x \in \mathcal{H}$ where $\oplus_{j=1}^m y_j = X(\oplus_1^n x)$ and X is quasi-invertible and*

$$X(\oplus_1^n \mathcal{H}_S(F)) \subset \oplus_1^m \mathcal{H}_T(F)$$

for any subset F of $\mathbb{C}$.

Proof. If S and T are ampliation quasisimilar, then there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that $XS^{(n)} = T^{(m)} X$ where X is quasi-invertible. If $\lambda_0 \in \rho_S(x)$, then there is an $\mathcal{H}$-valued analytic function $f(\lambda)$ in a neighborhood D of λ_0 such that $(S - \lambda) f(\lambda) = x$ for every $\lambda \in D$. Hence $(S^{(n)} - \lambda I^{(n)}) \oplus_1^n f(\lambda) = \oplus_1^n x$. Multiplying both sides by X,

$$(T^{(m)} - \lambda I^{(m)}) X \oplus_1^n f(\lambda) = X(S^{(n)} - \lambda I^{(n)}) \oplus_1^n f(\lambda) = X \oplus_1^n x.$$

Set $\oplus_{j=1}^m g_j(\lambda) := X \oplus_1^n f(\lambda)$ and $\oplus_{j=1}^m y_j := X \oplus_1^n x$. Then

$$(T^{(m)} - \lambda I^{(m)}) \oplus_{j=1}^m g_j(\lambda) = \oplus_{j=1}^m y_j.$$

Hence $(T - \lambda) g_j(\lambda) = y_j$ for $j = 1, 2, \ldots, m$. Since $g_j(\lambda)$ is an $\mathcal{H}$-valued analytic function in a neighborhood D of λ_0, $\lambda_0 \in \rho_T(y_j)$ for $j = 1, 2, \ldots, m$. Thus $\sigma_T(y_j) \subset \sigma_S(x)$ for all $x \in \mathcal{H}$ and $j = 1, 2, \ldots, m$ where $\oplus_{j=1}^m y_j = X(\oplus_1^n x)$. Therefore, $\cup_{j=1}^m \sigma_T(y_j) \subset \sigma_S(x)$ for all $x \in \mathcal{H}$ where $\oplus_{j=1}^m y_j = X(\oplus_1^n x)$.

If $x \in \mathcal{H}_S(F)$ for any subset F of $\mathbb{C}$, then $\sigma_T(y_j) \subset \sigma_S(x) \subset F$ for all $x \in \mathcal{H}$ and $j = 1, 2, \ldots, m$. Hence $y_j \in \mathcal{H}_T(F)$ for $j = 1, 2, \ldots, m$ and

$$X(\oplus_1^n x) = \oplus_{j=1}^m y_j \in \oplus_1^m \mathcal{H}_T(F).$$

Thus $X(\oplus_1^n \mathcal{H}_S(F)) \subset \oplus_1^m \mathcal{H}_T(F)$. □

Theorem 3.9. *Assume that an operator S in $\mathcal{L}(\mathcal{H})$ is an ampliation quasiaffine transform of an operator T in $\mathcal{L}(\mathcal{H})$. If T has the Bishop's property (β), then $\sigma(T) \subset \sigma(S)$.*

Proof. If S is an ampliation quasiaffine transform of T, then there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that $XS^{(n)} = T^{(m)}X$ for some quasi-invertible X. Since T has the Bishop's property (β), so does $T^{(m)}$. If $\lambda_0 \in \sigma(T) \backslash \sigma(S)$, then $d = dist(\lambda_0, \sigma(S)) > 0$. Set $F = \{\lambda : |\lambda - \lambda_0| \geq \frac{d}{3}\}$. Then $\sigma(S) \subset F$. Since $\sigma_S(x) \subset \sigma(S) \subset F$ for any $x \in \mathcal{H}$, $\mathcal{H} = \mathcal{H}_S(\sigma(S)) = \mathcal{H}_S(F)$. By Lemma 3.8,

$$X(\oplus_1^n \mathcal{H}) = X(\oplus_1^n \mathcal{H}_S(F)) \subset \oplus_1^m \mathcal{H}_T(F).$$

Since X has dense range, $\oplus_1^m \mathcal{H} = \overline{X(\oplus_1^n \mathcal{H})} \subset \overline{\oplus_1^m \mathcal{H}_T(F)}$. Since T has the Bishop's property (β), $\oplus_1^m \mathcal{H}_T(F)$ is closed. Hence $\oplus_1^m \mathcal{H} \subset \oplus_1^m \mathcal{H}_T(F)$. Thus $\oplus_1^m \mathcal{H} = \oplus_1^m \mathcal{H}_T(F)$, and hence $\mathcal{H} = \mathcal{H}_T(F)$. From [6], we know that $\sigma(T) = \sigma(T|_{\mathcal{H}_T(F)}) \subset \sigma(T) \cap F \subset F$. Since $\lambda_0 \in \mathbb{C} \backslash F$, $\lambda_0 \notin \sigma(T)$. So we have a contradiction. Thus $\sigma(T) \subset \sigma(S)$. □

The following example shows that the ampliation quasiaffine transform does not preserve the hyponormality.

Example 3.10. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of $\mathcal{H}$ and let T be the unilateral shift. Define a weighted shift S by $Se_0 = e_1$, $Se_1 = \sqrt{2}e_2$, and $Se_n = e_{n+1}$ for all $n \geq 2$. Then S is an ampliation quasiaffine transform of T such that $YS^{(n)} = T^{(n)}Y$ where $Y = \oplus_1^n X$ and X is a quasi-invertible operator defined by $Xe_0 = e_0$, $Xe_1 = e_1$, and $Xe_n = \frac{1}{\sqrt{2}}e_n$ for all $n \geq 2$. But S is not hyponormal.

Corollary 3.11. *Assume that an operator S in $\mathcal{L}(\mathcal{H})$ is an ampliation quasiaffine transform of an operator T in $\mathcal{L}(\mathcal{H})$. If T is hyponormal, then $\sigma(T) \subset \sigma(S)$. In addition, if S is quasinilpotent, then T is the zero operator.*

Proof. Since T has the Bishop's property (β), the proof follows from Theorem 3.9. In addition, if S is quasinilpotent, then T is quasinilpotent and hyponormal, and hence is the zero operator. Thus S is the zero operator. □

Corollary 3.12. *Let an operator S be an ampliation quasiaffine transform of an operator T. Assume that T has the Bishop's property (β). Then the following statements hold.*

(i) *If S is isoloid (i.e., iso $\sigma(S) \subset \sigma_p(S)$), then T is isoloid.*

(ii) *If S is normal, then $\sigma(S) = \sigma(T)$.*

Proof. If S is an ampliation quasiaffine transform of T, then there exist cardinal numbers m, $n \in \mathbb{N} \cup \{\aleph_0\}$ such that $XS^{(n)} = T^{(m)}X$ for some quasi-invertible X.

(i) If S is isoloid, then iso $\sigma(S) \subset \sigma_p(S)$. If $\lambda \in$ iso $\sigma(T)$, then by Theorem 3.9 $\lambda \in$ iso $\sigma(S)$, and hence $\lambda \in \sigma_p(S)$. There exists a nonzero $x \in \mathcal{H}$ such that

$(S-\lambda)x=0$. Hence

$$(T^{(m)}-\lambda I^{(m)})X\oplus_1^n x = X(S^{(n)}-\lambda I^{(n)})\oplus_1^n x = 0.$$

Set $\oplus_{j=1}^m y_j := X\oplus_1^n x$. Since $\oplus_{j=1}^m y_j := X\oplus_1^n x \neq 0$, there exists j_0, $1\leq j_0\leq m$, such that $y_{j_0}\neq 0$ and $(T-\lambda)y_{j_0}=0$. Then $\lambda\in\sigma_p(T)$. Thus T is isoloid.

(ii) If S is normal, then $\sigma(S)\subset\sigma(T)$ from [3]. Hence the proof follows from Theorem 3.9. □

Proposition 3.13. *Assume that an operator S in $\mathcal{L}(\mathcal{H})$ is an ampliation quasiaffine transform of an operator T in $\mathcal{L}(\mathcal{H})$. If T has finite ascent (i.e., $\ker T^k = \ker T^{k+1}$ for some positive integer k), then S has finite ascent.*

Proof. Since S is an ampliation quasiaffine transform of T, then there exist cardinal numbers m, $n\in\mathbb{N}\cup\{\aleph_0\}$ such that $XS^{(n)}=T^{(m)}X$ for some quasi-invertible X. If $x\in\ker S^{k+1}$, then $S^{k+1}x=0$. Since $(S^{(n)})^{k+1} = (S^{k+1})^{(n)}$, $X(S^{k+1})^{(n)}\oplus_1^n x = X(S^{(n)})^{k+1}\oplus_1^n x = 0$. Hence $(T^{(m)})^{k+1}X\oplus_1^n x = (T^{k+1})^{(m)}X\oplus_1^n x = 0$. Set $\oplus_{j=1}^m y_j := X\oplus_1^n x$. Thus $T^{k+1}y_j = 0$ for $j=1,2,\ldots,m$. Since $\ker T^{k+1}=\ker T^k$, $T^k y_j=0$ for $j=1,2,\ldots,m$. Hence $(T^k)^{(m)}\oplus_{j=1}^m y_j = (T^{(m)})^k X\oplus_1^n x = 0$. Since $XS^{(n)}=T^{(m)}X$, $X(S^{(n)})^k\oplus_1^n x = 0$. Since X is quasi-invertible, $S^k x=0$. Hence $\ker S^{k+1}\subseteq\ker S^k$. So we complete the proof. □

Theorem 3.14. *Assume that an operator S in $\mathcal{L}(\mathcal{H})$ is an ampliation quasiaffine transform of a hyponormal operator T in $\mathcal{L}(\mathcal{H})$ where $S\neq\mathbb{C}I$ and that $\cup_{j=1}^m\sigma_T(y_j)=\sigma_S(x)$ for all $x\in\mathcal{H}$ where $\oplus_{j=1}^m y_j = X(\oplus_1^n x)$ and X is quasi-invertible. If there exists a nonzero vector $z\in\mathcal{H}$ such that $\sigma_S(z)\subsetneq\sigma(S)$, then S has a nontrivial hyperinvariant subspace.*

Proof. If there exists a nonzero vector $z\in\mathcal{H}$ such that $\sigma_S(z)\subsetneq\sigma(S)$, set $\mathcal{M}:=\mathcal{H}_S(\sigma_S(z))$. Since T is hyponormal, it has the Dunford's property (C). Since S is an ampliation quasiaffine transform of T, then there exist cardinal numbers m, $n\in\mathbb{N}\cup\{\aleph_0\}$ such that $XS^{(n)}=T^{(m)}X$ for some quasi-invertible X. It is clear that S has the single valued extension property from Lemma 3.1. First, we want to show that $\mathcal{H}_S(F)$ is closed for any closed set F. If $x\in\overline{\mathcal{H}_S(F)}$, then there exists a sequence $\{x_k\}$ in $\mathcal{H}_S(F)$ such that $x_k\to x$ as $k\to\infty$. Then $\sigma_S(x_k)\subset F$ and from Lemma 3.8, $X\oplus_1^n x_k\in\oplus_1^m\mathcal{H}_T(F)$. Since $X\oplus_1^n x_k\to X\oplus_1^n x$ and $\oplus_1^m\mathcal{H}_T(F)$ is closed, $\oplus_{j=1}^m y_j = X\oplus_1^n x\in\oplus_1^m\mathcal{H}_T(F)$. Hence $y_j\in\mathcal{H}_T(F)$ for $j=1,2,\ldots,m$. Thus $\sigma_S(x)=\cup_{j=1}^m\sigma_T(y_j)\subset F$, and hence $x\in\mathcal{H}_S(F)$. So $\mathcal{H}_S(F)$ is closed for any closed set F. That means that S has the Dunford's property (C). Hence $\mathcal{M}$ is an S-hyperinvariant subspace from [6]. Since $z\in\mathcal{M}$, $\mathcal{M}\neq(0)$. Assume that $\mathcal{M}=\mathcal{H}$. Since S has the single valued extension property,

$$\sigma(S)=\cup_{y\in\mathcal{H}}\sigma_S(y)\subset\sigma_S(z)\subsetneq\sigma(S).$$

However, this is a contradiction. Hence $\mathcal{M}\neq\mathcal{H}$. So $\mathcal{M}$ is a nontrivial S-hyperinvariant subspace. □

Corollary 3.15. *Assume that an operator S in $\mathcal{L}(\mathcal{H})$ is an ampliation quasi-affine transform of a hyponormal operator T in $\mathcal{L}(\mathcal{H})$ where $S \neq \mathbb{C}I$ and that $\cup_{j=1}^{m}\sigma_T(y_j) = \sigma_S(x)$ for all $x \in \mathcal{H}$ where $\oplus_{j=1}^{m} y_j = X(\oplus_1^n x)$ and X is quasi-invertible. If there exists a nonzero vector $z \in \mathcal{H}$ such that $\|S^k z\| \leq Cr^k$ for all positive integer k, where $C > 0$ and $0 < r < r(S)$ are constants, then S has a nontrivial hyperinvariant subspace.*

Proof. Set $f(\lambda) := -\sum_{k=0}^{\infty} \lambda^{-(k+1)} S^k z$, which is analytic for $|\lambda| > r$. Since

$$(S-\lambda)f(\lambda) = -\sum_{k=0}^{\infty} \lambda^{-(k+1)} S^{k+1} z + \sum_{k=0}^{\infty} \lambda^{-k} S^k z = z$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| > r$, we have $\rho_S(z) \supset \{\lambda \in \mathbb{C} : |\lambda| > r\}$, i.e., $\sigma_S(z) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$. Since $r < r(S)$, $\sigma_S(z) \subsetneq \sigma(S)$. By Theorem 3.14, S has a nontrivial hyperinvariant subspace. □

Corollary 3.16. *Assume that an operator S in $\mathcal{L}(\mathcal{H})$ is an ampliation quasi-affine transform of a hyponormal operator T in $\mathcal{L}(\mathcal{H})$ where $S \neq \mathbb{C}I$ and that $\cup_{j=1}^{m}\sigma_T(y_j) = \sigma_S(x)$ for all $x \in \mathcal{H}$ where $\oplus_{j=1}^{m} y_j = X(\oplus_1^n x)$ and X is quasi-invertible. If S has a nonzero invariant subspace $\mathcal{M}$ such that $\sigma(T|_{\mathcal{M}}) \subsetneq \sigma(S)$, then S has a nontrivial hyperinvariant subspace.*

Proof. For any nonzero $z \in \mathcal{M}$, we have

$$\sigma_S(z) \subseteq \sigma_{T|_{\mathcal{M}}}(z) \subseteq \sigma(S|_{\mathcal{M}}) \subsetneq \sigma(S).$$

Hence S has a nontrivial hyperinvariant subspace from Theorem 3.9. □

Proposition 3.17. *Assume that an operator S is an ampliation quasiaffine transform of T. Then the following statements hold.*

(i) *If $\lambda \in \sigma_{ap}(S)$ and $\overline{\mu} \in \sigma_{ap}(T^*)$ with $\lambda \neq \mu$, then there exist $\{x_k\}$ and $\{y_k\}$ with $\|x_k\| = \|y_k\| = 1$ such that*

$$\lim_{k\to\infty} \langle X \oplus_1^n x_k, \oplus_1^m y_k\rangle = \lim_{k\to\infty} \sum_{j=1}^{m} \langle z_j, y_k\rangle = 0,$$

where $\oplus_{j=1}^{m} z_{k,j} = X \oplus_1^n x_k$.

(ii) *If $\lambda \in \sigma_p(S)$ and $\overline{\mu} \in \sigma_p(T^*)$ with $\lambda \neq \mu$, then there exist nonzero x and y such that*

$$\langle X \oplus_1^n x, \oplus_1^m y\rangle = \sum_{j=1}^{m} \langle z_j, y\rangle = 0,$$

where $\oplus_{j=1}^{m} z_j = X \oplus_1^n x$.

Proof. It suffices to show that (i) holds. Assume that S is an ampliation quasiaffine transform of T. Then there exist cardinal numbers $m, n \in \mathbb{N} \cup \{\aleph_0\}$ such that $XS^{(n)} = T^{(m)}X$ for some quasi-invertible X. Since $\lambda \in \sigma_{ap}(S)$ and $\overline{\mu} \in \sigma_{ap}(T^*)$ with $\lambda \neq \mu$, there exist $\{x_k\}$ and $\{y_k\}$ with $\|x_k\| = \|y_k\| = 1$ such that $\lim_{k\to\infty} \|(S-\lambda)x_k\| = 0$ and $\lim_{k\to\infty} \|(T^* - \overline{\mu})y_k\| = 0$. If there exist

$\{x_k\}$ and $\{y_k\}$ with $\|x_k\| = \|y_k\| = 1$ such that $\lim_{k\to\infty} \|(S-\lambda)x_k\| = 0$ and $\lim_{k\to\infty} \|(T^* - \overline{\mu})y_k\| = 0$, then

$$\lim_{k\to\infty} \|(S^{(n)} - \lambda I^{(n)}) \oplus_1^n x_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|(T^{*(m)} - \overline{\mu} I^{(m)}) \oplus_1^m y_k\| = 0.$$

Since $XS^{(n)} = T^{(m)}X$, we get that

$$\begin{aligned}
&|(\lambda - \mu)\langle X \oplus_1^n x_k, \oplus_1^m y_k\rangle| \\
&= |\langle \lambda \oplus_1^n x_k, X^* \oplus_1^m y_k\rangle - \langle X \oplus_1^n x_k, \overline{\mu} \oplus_1^m y_k\rangle| \\
&= |\langle (\lambda I^{(n)} - S^{(n)}) \oplus_1^n x_k, X^* \oplus_1^m y_k\rangle + \langle S^{(n)} \oplus_1^n x_k, X^* \oplus_1^m y_k\rangle \\
&\quad - \langle X \oplus_1^n x_k, (\overline{\mu} I^{(m)} - T^{*(m)}) \oplus_1^m y_k\rangle - \langle X \oplus_1^n x_k, T^{*(m)} \oplus_1^m y_k\rangle| \\
&\le \|(\lambda I^{(n)} - S^{(n)}) \oplus_1^n x_k\| \|X^* \oplus_1^m y_k\| \\
&\quad + \|X \oplus_1^n x_k\| \|(\overline{\mu} I^{(m)} - T^{*(m)}) \oplus_1^m y_k\| \to 0
\end{aligned}$$

as $k \to \infty$. Since $\lambda \ne \mu$,

$$\lim_{k\to\infty} \langle X \oplus_1^n x_k, \oplus_1^m y_k\rangle = \lim_{k\to\infty} \sum_{j=1}^{m} \langle z_{k,j}, y_k\rangle = 0,$$

where $\oplus_{j=1}^m z_{k,j} = X \oplus_1^n x_k$. So we complete the proof. □

Acknowledgments. The author wishes to thank the referee for a careful reading and valuable comments and suggestions for the original draft.

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