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MULTIPLICATIVE FUNCTIONS WHICH ARE ADDITIVE ON TRIANGULAR NUMBERS

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Abstract. Fix $k \geq 3$. If a multiplicative function f satisfies

$$f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$$

for arbitrary positive triangular numbers $x_1, x_2, \ldots, x_k$, then f is the identity function. This extends Chung and Phong's work for $k = 2$.

1. Introduction

Claudia Spiro's paper [6] in 1992 has inspired lots of mathematicians to produce many related papers. She showed that a multiplicative function f satisfying $f(p+q) = f(p) + f(q)$ for arbitrary prime numbers p and q is the identity function under some condition. Let E be a set of arithmetic functions and let S be a set of positive integers. Spiro dubbed S the *additive uniqueness set* for E if a function $f \in E$ is uniquely determined under the condition $f(a+b) = f(a) + f(b)$ for $a, b \in S$.

In 1999 Chung and Phong [2] showed that the set of positive triangular numbers and the set of positive tetrahedral numbers are new additive uniqueness sets for multiplicative functions. They also conjectured that the set

$$H_k = \left\{ \frac{n(n+1)\cdots(n+k-1)}{1 \cdot 2 \cdots k} \,\middle|\, n = 1, 2, 3, \ldots \right\}$$

is an additive uniqueness set for every $k \geq 4$.

In 2010 Fang [4] extended Spiro's work to the condition $f(p+q+r) = f(p) + f(q) + f(r)$ for arbitrary prime numbers p, q, r. His work was generalized by Dubickas and Šarka [3] to sums of arbitrary number of primes.

Let us consider the general condition *k-additivity*. That is, if a function $f \in E$ satisfying $f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k)$ for arbitrary $x_i \in S$ is uniquely determined, we call S a k-additive uniqueness set

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for E. We can say that the set of prime numbers is a k-additive uniqueness set with $k \geq 2$.

Here is an interesting example. The set of nonzero squares for the set of multiplicative functions is not a 2-additive uniqueness set [1], but is a k-additive uniqueness set for every $k \geq 3$ [5]. So it is natural to ask whether a 2-additive uniqueness set is also a k-additive uniqueness set or not for $k \geq 3$.

Let $\mathbb{T}$ be the set of triangular numbers $T_n = \frac{n(n+1)}{2}$ for $n \geq 1$. That is,

$$\mathbb{T} = \{1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \dots\}.$$

In this article, we show that $\mathbb{T}$ is a k-additive uniqueness set for multiplicative functions. This extends Chung and Phong's work for 2-additive uniqueness of $\mathbb{T}$.

The proof consists of three parts. The first is about the 3-additivity, the second is about the 4-additivity, and the last is about the k-additivity with $k \geq 5$. For convenience, we denote a triangular number by $\triangle$. If the triangular number is restricted to be positive, we use the symbol $\overset{+}{\triangle}$.

2. 3-additive uniqueness set

Theorem 2.1. *If a multiplicative function f satisfies*

$$f(a+b+c) = f(a) + f(b) + f(c)$$

for $a, b, c \in \mathbb{T}$, then f is the identity function.

Clearly, $f(1) = 1$ and $f(3) = 3$. Note that $f(5) = f(1+1+3) = 5$. The equalities

$$\begin{aligned} f(10) &= f(1+3+6) = 4 + 3f(2) \\ &= f(2) \cdot f(5) = 5f(2) \end{aligned}$$

yields $f(2) = 2$. Then, $f(6) = 6$ and $f(10) = 10$.

We use induction. Suppose that $f(n) = n$ for all $n < N$. Now let us show $f(N) = N$. We may assume that $N = p^r$ for some prime p by the multiplicity of f.

If $N = 3^r$, then from the equalities

$$\begin{aligned} f(3T_{3^{r-1}}) = 3f(T_{3^{r-1}}) &= 3f\left(\frac{3^{r-1}(3^{r-1}+1)}{2}\right) = 3f(3^{r-1}) \cdot f\left(\frac{3^{r-1}+1}{2}\right) \\ &= f\left(3^r \frac{3^{r-1}+1}{2}\right) = f(3^r) \cdot f\left(\frac{3^{r-1}+1}{2}\right) \end{aligned}$$

we conclude that $f(3^r) = 3^r$ since $f(3^{r-1}) = 3^{r-1}$ and $f\left(\frac{3^{r-1}+1}{2}\right) = \frac{3^{r-1}+1}{2}$ by induction hypothesis.

Now, assume that $N = p^r = 3s - 1$ with odd prime p. Note that $f(T_{s-1}) = T_{s-1}$ and $f(T_s) = T_s$ by induction hypothesis since T_s can be factored into integers smaller than N. Since

$$f(T_{s-1}+T_{s-1}+T_s)=\frac{s(s-1)}{2}+\frac{s(s-1)}{2}+\frac{s(s+1)}{2}=\frac{s(3s-1)}{2}=\frac{sp^r}{2}$$

and also

$$f(T_{s-1}+T_{s-1}+T_s)=f\left(\frac{s(3s-1)}{2}\right)=f\left(\frac{s}{2}\right)\cdot f(3s-1)=\frac{s}{2}f(p^r),$$

we know that $f(p^r)=p^r$.

If $N=p^r=3s+1$ with odd prime p, then the equalities

$$f(T_{s-1}+T_s+T_s)=\frac{s(s-1)}{2}+\frac{s(s+1)}{2}+\frac{s(s+1)}{2}=\frac{s(3s+1)}{2}=\frac{sp^r}{2}$$

and

$$f(T_{s-1}+T_s+T_s)=f\left(\frac{s(3s+1)}{2}\right)=f\left(\frac{s}{2}\right)\cdot f(3s+1)=\frac{s}{2}f(p^r)$$

show that $f(p^r)=p^r$.

Now, we consider the last case $N=2^r$. Let $2^{r+1}=3s\pm 1$. Then, the following two equalities

$$f(T_{s-1}+T_{s-1}+T_s)=\frac{s(s-1)}{2}+\frac{s(s-1)}{2}+\frac{s(s+1)}{2}=\frac{s(3s-1)}{2}=s2^r$$

and

$$f(T_{s-1}+T_{s-1}+T_s)=f\left(\frac{s(3s-1)}{2}\right)=f(s)\cdot f\left(\frac{3s-1}{2}\right)=sf(2^r)$$

give that $f(2^r)=2^r$ when $2^r=3s-1$. Also, the following two equalities

$$f(T_{s-1}+T_s+T_s)=\frac{s(s-1)}{2}+\frac{s(s+1)}{2}+\frac{s(s+1)}{2}=\frac{s(3s+1)}{2}=s2^r$$

and

$$f(T_{s-1}+T_s+T_s)=f\left(\frac{s(3s+1)}{2}\right)=f(s)\cdot f\left(\frac{3s+1}{2}\right)=sf(2^r).$$

give that $f(2^r)=2^r$ when $2^r=3s+1$.

3. 4-additive uniqueness set

Theorem 3.1. *If a multiplicative function f satisfies*

$$f(a+b+c+d)=f(a)+f(b)+f(c)+f(d)$$

for $a,b,c,d\in\mathbb{T}$, then f is the identity function.

Lemma 3.2. *Let $\mathbb{T}_k$ be the set of sums of k numbers $\triangle^+$. If $k\geq 4$, then $\mathbb{T}_k$ is the set of all positive integers except for $1,2,\ldots,k-1,k+1,k+3$.*

Proof. Gauss' theorem guarantees that every positive integer can be written as $\triangle + \triangle + \triangle$, some of which possibly vanish. Thus, if $n > 21$ is given, then $n - 21$ is $\triangle^{+}$, $\triangle^{+} + \triangle^{+}$, or $\triangle^{+} + \triangle^{+} + \triangle^{+}$. Since $21 \in \mathbb{T}$ and

$$n = (n-21) + 21 = (n-21) + 6 + 15 = (n-21) + 3 + 3 + 15,$$

every integer > 21 can be written as a sum of four $\triangle^{+}$.

It can be easily verified that every positive integer ≤ 21 is a sum of four $\triangle^{+}$ except for $1, 2, 3, 5$, and 7. Hence, we can conclude that every positive integer ≥ 8 can be written as a sum of four $\triangle^{+}$.

Now, consider the general cases. It is clear that the sum of k $\triangle^{+}$ can represent k and $k+2$ but cannot represent any number from 1 through $k-1$. It is also easily checked that the sum cannot represent $k+1$ and $k+3$. Since sums of four $\triangle^{+}$ represent all integers ≥ 8, the sum

$$\underbrace{1 + \cdots + 1}_{k-4 \text{ summands}} + \triangle^{+} + \triangle^{+} + \triangle^{+} + \triangle^{+}$$

represents all integers $\geq k+4$. □

Now let us prove Theorem 3.1. Note that $f(1) = 1$ and $f(4) = 4$. Then

$$\begin{aligned} f(6) &= f(1+1+1+3) = 3 + f(3) \\ &= f(2) \cdot f(3), \\ f(10) &= f(1+3+3+3) = 1 + 3f(3) \\ &= f(2) \cdot f(5), \\ f(15) &= f(3+3+3+6) = 3f(3) + f(2) \cdot f(3) \\ &= f(3) \cdot f(5). \end{aligned}$$

For convenience, let $x = f(2)$, $y = f(3)$, and $z = f(5)$. The above equations can be rewritten:

$$\begin{cases} 3 + y = xy, \\ 1 + 3y = xz, \\ 3y + xy = yz. \end{cases}$$

Note that $y = \frac{3}{x-1} \neq 0$ from the first equation. So, the third equation becomes $3 + x = z$. Then, the second equation becomes

$$1 + 3 \cdot \frac{3}{x-1} = x(3+x)$$

or

$$x^3 + 2x^2 - 3x - x - 8 = (x-2)(x+2)^2 = 0.$$

Thus, we obtain the two solutions:

$$f(2) = -2, f(3) = -1, f(5) = 1 \quad \text{or} \quad f(2) = 2, f(3) = 3, f(5) = 5.$$

First case yields $f(9) = f(1+1+1+6) = 3 + f(2)\cdot f(3) = 5$. But, this would make a contradiction:

$$\begin{aligned} f(18) &= f(1+1+1+15) = 3 + f(3)\cdot f(5) = 2 \\ &= f(2)\cdot f(9) = -10. \end{aligned}$$

Thus, we can conclude that $f(2) = 2$, $f(3) = 3$, and $f(5) = 5$. Then, $f(14) = f(1+1+6+6) = f(2)\cdot f(7)$ gives $f(7) = 7$. So $f(n) = n$ for $n \le 7$.

By Lemma 3.2 every integer ≥ 8 can be written as a sum of four $\triangle^{+}$. Thus f must be the identity function by induction.

4. k-additive uniqueness set

Let $k \ge 5$. Note that

$$\begin{aligned} (k-2) + 16 &= (k-2)\cdot 1 + 6 + 10 \\ &= (k-2)\cdot 1 + 1 + 15, \\ (k-3) + 12 &= (k-3)\cdot 1 + 3 + 3 + 6 \\ &= (k-3)\cdot 1 + 1 + 1 + 10, \\ (k-4) + 19 &= (k-4)\cdot 1 + 1 + 6 + 6 + 6 \\ &= (k-4)\cdot 1 + 3 + 3 + 3 + 10. \end{aligned}$$

Thus, the equalities give rise to the system of equations

$$\begin{cases} f(2)\cdot f(3) + f(2)\cdot f(5) = 1 + f(3)\cdot f(5), \\ 2f(3) + f(2)\cdot f(3) = 2 + f(2)\cdot f(5), \\ 1 + 3f(2)\cdot f(3) = 3f(3) + f(2)\cdot f(5). \end{cases}$$

The solutions are

$$\begin{aligned} &f(2) = \frac{1}{4}, \quad f(3) = \frac{2}{3}, \quad f(5) = -2; \\ &f(2) = f(3) = f(5) = 1; \\ &f(2) = 2, \quad f(3) = 3, \quad f(5) = 5. \end{aligned}$$

Note that $f(k+2) = k - 1 + f(3)$ and $f(k+4) = k - 2 + 2f(3)$.

If $3 \nmid (k+2)$, then the equalities

$$\begin{aligned} f(3(k+2)) &= f(\underbrace{3 + \cdots + 3}_{k-2 \text{ summands}} + 6 + 6) = f(3)(k-2) + 2f(2)\cdot f(3) \\ &= f(3)\cdot f(k+2) = f(3)(k-1+f(3)) \end{aligned}$$

exclude the first solution set $f(2) = \frac{1}{4}, f(3) = \frac{2}{3}, f(5) = -2$.

If $3 \mid (k+2)$, then we consider

$$\begin{aligned} f(3(k+4)) &= f(\underbrace{3 + \cdots + 3}_{k-1 \text{ summands}} + 15) = f(3)(k-1) + f(3)\cdot f(5) \\ &= f(3)\cdot f(k+4) = f(3)(k-2+2f(3)), \end{aligned}$$

which exclude the first solution set.

Now, consider the second solution set $f(2) = f(3) = f(5) = 1$. Then, $f(T_1) = f(T_2) = f(T_3) = f(T_4) = f(T_5) = 1$. By Lemma 3.2 we have that every T_n with $n \geq 4$ can be written as a sum of four $\overset{+}{\triangle}$. From the equality

$$(*) \qquad \begin{aligned} &(k-5) + 1 + 1 + 1 + 3 + T_s \\ &= (k-5) + 6 + T_a + T_b + T_c + T_d \quad \text{with } a, b, c, d < s \end{aligned}$$

we conclude that $f(T_s) = 1$ for all $s \geq 6$ inductively.

But, if s is sufficiently large, T_s can be written as a sum of k numbers $\overset{+}{\triangle}$ by Lemma 3.2. So $f(T_s) = k$, which is a contradiction.

Thus, we can conclude that $f(2) = 2$, $f(3) = 3$, and $f(5) = 5$. Also, the above equality $(*)$ yields $f(T_s) = T_s$ for every s.

If N is a sum of k $\overset{+}{\triangle}$, then, clearly $f(N) = N$. Otherwise, we choose an integer M such that $M > k + 3$ and $\gcd(M, N) = 1$. Then, since M and MN can be written as sums of k $\overset{+}{\triangle}$, $Mf(N) = f(M) \cdot f(N) = f(MN) = MN$. Thus, $f(N) = N$. The proof is completed.

References

[1] P. V. Chung, *Multiplicative functions satisfying the equation* $f(m^2+n^2) = f(m^2)+f(n^2)$, Math. Slovaca **46** (1996), no. 2-3, 165–171.

[2] P. V. Chung and B. M. Phong, *Additive uniqueness sets for multiplicative functions*, Publ. Math. Debrecen **55** (1999), no. 3-4, 237–243.

[3] A. Dubickas and P. Šarka, *On multiplicative functions which are additive on sums of primes*, Aequationes Math. **86** (2013), no. 1-2, 81–89. https://doi.org/10.1007/s00010-012-0156-8

[4] J.-H. Fang, *A characterization of the identity function with equation* $f(p+q+r) = f(p)+f(q)+f(r)$, Combinatorica **31** (2011), no. 6, 697–701. https://doi.org/10.1007/s00493-011-2739-8

[5] P.-S. Park, *On k-additive uniqueness of the set of squares for multiplicative functions*, Aequationes Math. **92** (2018), no. 3, 487–495. https://doi.org/10.1007/s00010-017-0517-4

[6] C. A. Spiro, *Additive uniqueness sets for arithmetic functions*, J. Number Theory **42** (1992), no. 2, 232–246. https://doi.org/10.1016/0022-314X(92)90022-H

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