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LINEAR AUTOMORPHISMS OF SMOOTH HYPERSURFACES GIVING GALOIS POINTS

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Abstract. Let X be a smooth hypersurface X of degree $d \geq 4$ in a projective space $\mathbb{P}^{n+1}$. We consider a projection of X from $p \in \mathbb{P}^{n+1}$ to a plane $H \cong \mathbb{P}^n$. This projection induces an extension of function fields $\mathbb{C}(X)/\mathbb{C}(\mathbb{P}^n)$. The point p is called a Galois point if the extension is Galois. In this paper, we will give necessary and sufficient conditions for X to have Galois points by using linear automorphisms.

1. Introduction

In this paper, we work over $\mathbb{C}$. For an irreducible variety Y, let $\mathbb{C}(Y)$ be the function field of Y. Let X be a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$, p be a point in $\mathbb{P}^{n+1}$, and $\pi_p : X \dashrightarrow H$ be a projection with center p where H is a hyperplane not containing p. We have an extension of function fields $\pi^* : \mathbb{C}(H) \to \mathbb{C}(X)$ such that $[\mathbb{C}(X) : \mathbb{C}(H)] = d-1$ (resp. d) if $p \in X$ (resp. $p \notin X$). The structure of this extension does not depend on the choice of H but on the point p. We write K_p instead of $\mathbb{C}(H)$. Since $H \cong \mathbb{P}^n$, $K_p \cong \mathbb{C}(\mathbb{P}^n)$ as a field.

Let Y be an irreducible variety Y. Let K be a non-trivial intermediate field between $\mathbb{C}(Y)$ and $\mathbb{C}$ such that K is a purely transcendental extension of $\mathbb{C}$ with the transcendence degree n. The field K is called a maximal rational subfield if there is not a non-trivial intermediate field L between $\mathbb{C}(Y)$ and K such that L is a purely transcendental extension of $\mathbb{C}$ with the transcendence degree n.

Let X be a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$. If $n = 1$, then the field K_p is a maximal rational subfield of $\mathbb{C}(X)$ ([17]). In the case where $n = 2$ and $d = 4$, if p is not an outer Galois point of X, then the field K_p is a maximal rational subfield. If $d \geq 5$, then K_p is always a maximal rational subfield. Please see [3, 20] for details.

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Definition 1.1 ([21–23])**.** The point $p \in \mathbb{P}^{n+1}$ is called a Galois point for X if the extension $\mathbb{C}(X)/K_p$ is Galois. Moreover, if $p \in X$ (resp. $p \notin X$), then we call p an inner (resp. outer) Galois point.

Pay attention that if $n = 1$ or $p \notin X$, then π_p is a morphism such that $\pi_p : X \to \mathbb{P}^n$ is a Galois cover of a variety.

Theorem 1.2 ([21–23])**.** *Let X be a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$, and $p \in \mathbb{P}^{n+1}$ be a Galois point of X. Then the Galois group of $\mathbb{C}(X)/K_p$ is induced by a linear automorphism of X. In addition, if p is an inner* (*resp. outer*) *Galois point, then the Galois group of $\mathbb{C}(X)/K_p$ is a cyclic group of $d-1$* (*resp. d*)

Definition 1.3. An automorphism g of X is called linear if there is an automorhism h of $\mathbb{P}^{n+1}$ such that $h(X) = X$ and $h_{|X} = g$.

If X is a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$ and $(n, d) \neq (2, 4)$, then the automorphism group $\mathrm{Aut}(X)$ of X is a finite subgroup of the group $\mathrm{PGL}(n+2, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^{n+1})$, for instance, see ([14]).

Definition 1.4. Let $p \in \mathbb{P}^{n+1}$ be a Galois point of X. An automorphism g of X is called an automorphism belonging to the Galois point p if g generates the Galois group of the Galois extension $\mathbb{C}(X)/K_p$.

Definition 1.5. Let g be a linear automorphism of X. A matrix A is called a representation matrix of g if $g = A$ in $\mathrm{PGL}(n+2, \mathbb{C})$.

A necessary and sufficient condition for a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ to have Galois points is given by the defining equation of X ([21–23]). For the case $n = 1$, there is a sufficient condition for a smooth plane X curve to have Galois points by the structure of the automorphism group $\mathrm{Aut}(X)$ as follows.

Theorem 1.6 ([1])**.** *Let X be a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$, and $g \in \mathrm{Aut}(X)$ be a linear automorphism of order $k(d-1)$* (*resp. kd*) *for $n, k \geq 1$. If $n = 1$ and $k \geq 2$, then X has an inner* (*resp. outer*) *Galois point p, and g^k is an automorphism belonging to the Galois point p.*

Smooth curves in $\mathbb{P}^2$ with Galois points are characterized by other methods as well [11–13]. There are smooth plane curves of degree d with a linear automorphism of order $d-1$ or d acting but without Galois points (see Examples 2.7 and 2.8). In addition, there is a smooth hypersurface X of degree d in $\mathbb{P}^4$ with a linear automorphism of order $(d-1)d$ acting but without Galois points (see Example 2.9). Therefore, Theorem 1.6 does not hold for all $n, k \geq 1$.

For $g \in \mathrm{Aut}(X)$, we set $\mathrm{Fix}(g) := \{x \in X \mid g(x) = x\}$, and we write the order of g as $\mathrm{ord}(g)$. Recall that if X is a smooth hypersurface and $(n, d) \neq (2, 4)$, then $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{PGL}(n+2, \mathbb{C})$, i.e., all automorphisms of X are linear. In this paper, by using $\mathrm{Fix}(g)$ and $\mathrm{ord}(g)$, we will study the case $k, n \geq 1$ of Theorem 1.6. Our main results are Theorems 1.7, 1.8, 1.9, and 1.10.

Theorem 1.7, is for $n = k = 1$.

Theorem 1.7. *Let X be a smooth plane curve degree $d \geq 4$, and g be a linear automorphism of X.*

(1) *If* $\mathrm{ord}(g) = d-1$, *then* $\sharp|\mathrm{Fix}(g)| \neq 2$ *if and only if X has an inner Galois point p, and g is an automorphism belonging to the Galois point p.*

(2) *If* $\mathrm{ord}(g) = d$, *then* $\mathrm{Fix}(g) \neq \emptyset$ *if and only if X has an outer Galois point p, and g is an automorphism belonging to the Galois point p.*

Theorem 1.8 is for $k = 1$, $n \geq 2$, and an inner Galois point.

Theorem 1.8. *Let X be a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$, and $g \in \mathrm{Aut}(X)$ be a linear automorphism of order $d-1$.*

(1) *If $n = 2$, then* $\mathrm{Fix}(g)$ *contains a curve C' which is not a smooth rational curve if and only if X has an inner Galois point p, and g is an automorphism belonging to the Galois point p.*

(2) *If $n \geq 3$, then* $\mathrm{Fix}(g)$ *has codimension* 1 *in X if and only if X has an inner Galois point p, and g is an automorphism belonging to the Galois point p.*

Theorem 1.9 is for $k = 1$, $n \geq 2$, and an outer Galois point.

Theorem 1.9. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$, and $g \in \mathrm{Aut}(X)$ be a linear automorphism of order d. If $d \geq 2$, then* $\mathrm{Fix}(g)$ *has codimension* 1 *in X if and only if X has an outer Galois point p, and g is an automorphism belonging to the Galois point p.*

The following Theorem is for $n, k \geq 2$ and an inner Galois point.

Theorem 1.10. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$, and $g \in \mathrm{Aut}(X)$ be a linear automorphism of order $k(d-1)$ for $k \geq 2$.*

(1) *If $n = 2$ and* $\sharp|\mathrm{Fix}(g)| \geq 5$, *then X has an inner Galois point p, and g^k is an automorphism belonging to the Galois point p.*

(2) *If $n \geq 3$ and* $\mathrm{Fix}(g)$ *has codimension* 1 *or* 2 *in X, then X has an inner Galois point p, and g^k is an automorphism belonging to the Galois point p.*

Theorem 1.10 does not fold for an outer Galois point (see Example 3.9). For $n = 1$, the automorphism groups of curves with Galois points are classified ([1,8]). There are studies on automorphism groups of plane curves using Galois points ([1, 4, 9, 11, 12, 15, 16]). For the case $n \geq 2$, determining whether X has Galois points from the structure of $\mathrm{Aut}(X)$ may be an important issue.

Question 1.11. For $n \geq 1$, is there a group G_n satisfying the following condition? The condition: If the automorphism group $\mathrm{Aut}(X)$ of a smooth hypersurface X of degree $d \geq 4$ in $\mathbb{P}^{n+1}$ has a subgroup H which is isomorphic to G as a group, then X has a Galois point.

Theorem 1.6 is an answer to Question 1.11 for the case $n = 1$. However, our main theorems are not answers to Question 1.11, because they need the fixed points set. Section 2 is preliminary. We will explain the basic facts of Galois point. In Section 3, we will show Theorems 1.7, 1.8, 1.9, and 1.10.

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2. Preliminary

Let X be a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$. We denote the number of inner (resp. outer) Galois points of X by $\delta(X)$ (resp. $\delta'(X)$). Here $[s]$ represents the integer part of $s \in \mathbb{R}$.

Theorem 2.1 ([21–23]). *Let X be a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$. The following holds.*

(1) *If $n = 1$, then $\delta(X) = 0, 1,$ or 4, and $\delta'(X) = 0, 1,$ or 3. In particular, if $n = 1$ and $d \geq 5$, then $\delta(X) = 0$ or 1.*

(2) *If $n \geq 2$ and $d = 4$, then $\delta(X) \leq 4([\frac{n}{2}] + 1)$. In particular, if $n = 2$ and $d = 4$, then $\delta(X) = 0, 1, 2, 4,$ or 8.*

(3) *If $n \geq 2$ and $d \geq 5$, then $\delta(X) \leq [\frac{n}{2}] + 1$.*

(4) *If $n \geq 2$ and $d \geq 4$, then $\delta'(X) \leq n + 2$.*

The numbers of Galois points of normal hypersurfaces are investigated ([5, 19]).

The defining equations for smooth hypersurfaces with a Galois point are determined.

Theorem 2.2 ([21–23]). *Let X be a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$. The following holds.*

(1) *X has an inner Galois point p if and only if by replacing the local coordinate system if necessary, $p = [1 : 0 : \cdots : 0]$ and X is defined by*

$$X_1 X_0^{d-1} + F(X_1, \ldots, X_{n+1}) = 0.$$

(2) *X has an outer Galois point p if and only if by replacing the local coordinate system if necessary, $p = [1 : 0 : \cdots : 0]$ and X is defined by*

$$X_0^d + F(X_1, \ldots, X_{n+1}) = 0.$$

The definition equations with many Galois points are also studied (please see [23] for more detailed results).

For a positive integer l, let I_l be the identity matrix of size l, and e_l be a primitive l-th root of unity. Theorem 2.3 below is a rewrite of Theorem 2.2 from the viewpoint of a liner automorphism.

Theorem 2.3. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$, $g \in \mathrm{Aut}(X)$ be a linear automorphism of order $d-1$ (resp. d), and A be a representation matrix of g. There is a Galois point p of X such that g is an automorphism belonging to the Galois point p if and only if the matrix A is conjugate to a matrix*

$$\begin{pmatrix} a & 0 \\ 0 & bI_{n+1} \end{pmatrix}$$

such that $\frac{a}{b} = e_{d-1}$ (*resp.* e_d). *In particular, if* A *is conjugate to the above matrix, then the Galois point* p *is the eigenvector corresponding to the eigenvalue* a.

From Theorem 2.3, we see that the only if parts of Theorems 1.8 and 1.9 holds.

From here, we give examples of smooth hypersurfaces of degree d without Galois points which have a linear automorphism such that the order is a multiple of $d-1$ or d. As a corollary of Theorem 2.3, we give the following two lemmas.

Lemma 2.4. *Let* X *be a smooth hypersurface of degree* $d \geq 4$ *in* $\mathbb{P}^{n+1}$, $p \in \mathbb{P}^{n+1}$, *and* g *be an automorphism belonging to the Galois point* p. *For any linear automorphism* h *of* X, $h(p)$ *is also a Galois point of* X, *and* $h \circ g \circ h^{-1}$ *is an automorphism belonging to the Galois point* $h(p)$. *In particular, if* p *is an inner* (*resp. outer*) *Galois point, then* $h(p)$ *is also an inner* (*resp. outer*) *Galois point.*

Proof. By a linear automorphism $h \circ g \circ h^{-1}$ and Theorem 2.3, $h(p)$ is a Galois point of X, and $h \circ g \circ h^{-1}$ is an automorphism belonging to the Galois point $h(p)$. □

Lemma 2.5. *Let* X *be a smooth hypersurface of degree* $d \geq 4$ *in* $\mathbb{P}^{n+1}$, $p \in \mathbb{P}^{n+1}$, *and* g *be an automorphism belonging to the Galois point* p. *For a linear automorphism* k *of* X *such that* $k(p) = p$, *we get that* $k \circ g = g \circ k$.

Proof. By Lemma 2.4, $k \circ g \circ k^{-1}$ is an automorphism belonging to the Galois point p. By Theorem 2.3, $k \circ g \circ k^{-1} = g$. □

In Example 2.7, we give an example of a smooth plane curve of degree d with a linear automorphism of order $d-1$ but has no Galois points. Before that, we prepare a lemma.

Lemma 2.6. *Let* $A := (a_{ij})$ *be a diagonal* $m \times m$ *matrix such that* $a_{ii} \neq a_{jj}$ *for* $1 \leq i < j \leq m$. *For a* $m \times m$ *matrix* $B := (b_{ij})$, *if* $AB = BA$, *then* B *is a diagonal matrix.*

Proof. We assume that $AB = BA$. The (i,j)-th entry of the matrix AB is $a_{ii}b_{ij}$. The (i,j)-th entry of the matrix BA is $a_{jj}b_{ij}$. Since $a_{ii} \neq a_{jj}$ for $\leq i < j \leq m$, we get that $b_{ij} = 0$ for $\leq i < j \leq m$. Then the matrix B is a diagonal matrix. □

Example 2.7. Let d be an even number of 6 or more, and X be a smooth curve in $\mathbb{P}^2$ defined by

$$X_2^d + X_0^{d-1}X_2 + X_1^{d-1}X_2 + X_0^{\frac{d}{2}}X_1^{\frac{d}{2}} = 0.$$

The curve X has an automorphism g of order $d-1$ such that the following matrix A is a representation matrix of g:

$$A := \begin{pmatrix} e_{d-1}^{\frac{d}{2}} & 0 & 0 \\ 0 & e_{d-1}^{\frac{d}{2}-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $1 \le i < d-1$, we get that $\frac{d}{2}i \not\equiv 0 \pmod{d-1}$, $(\frac{d}{2}-1)i \not\equiv 0 \pmod{d-1}$, and $\frac{d}{2}i \not\equiv (\frac{d}{2}-1)i \pmod{d-1}$. We assume that X has a Galois point $p \in \mathbb{P}^2$. By Lemma 2.4, $g^j(p)$ is a Galois point for $1 \le j < d-1$. By Theorem 2.1, $\delta(X) \le 4$ and $\delta'(X) \le 3$. Since $d \ge 6$, $g^l(p) = p$ for some $1 \le l < d-1$. Let $h \in \mathrm{Aut}(X)$ be an automorphism belonging to the Galois point p. Since $g^l(p) = p$, the automorphism $g^l \circ h \circ g^{-l}$ is also an automorphism belonging to the Galois point p. Then $g^l \circ h \circ g^{-l} = h^i$ for some $1 \le i < d-1$. By Theorem 2.3, we can take a representation matrix B of h such that

$$CBC^{-1} = \begin{pmatrix} e_k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some a matrix C where if $p \in X$, then $k = d-1$, and if $p \notin X$, then $k = d$. By the equation $g^l \circ h \circ g^{-l} = h^i$, we get that $i = 1$, and $A^l B A^{-l} = B$. Since the diagonal entries of A^l are different from each other, Lemma 2.6, and $A^l B A^{-l} = B$, we get that B is a diagonal matrix. Since $h = B$ is an automorphism belonging to the Galois point p, and Theorem 2.3, we get that

$$p \in \{[1:0:0], [0:1:0], [0:0:1]\},$$

and the matrix B is one of the following matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \text{ and } \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix}$$

where if $p \in X$, then $\frac{a}{b} = e_{d-1}$, and if $p \notin X$, then $\frac{a}{b} = e_d$. The defining equation of X implies that $h = B$ is not an automorphism of X. This is a contradiction. Therefore, X does not have Galois points.

Below is an example of a smooth plane curve of degree d with a linear automorphism d but has no Galois points.

Example 2.8. Let d_1 and d_2 be integers greater than 4 such that $\gcd(d_1, d_2) = 1$. Let $d := d_1 d_2$, and X be a smooth curve in $\mathbb{P}^2$ defined by

$$X_0^d + X_1^d + X_2^d + X_0^{d_1} X_1^{d_2} X_2^{d-d_1-d_2} = 0.$$

The curve X has an automorphism g of order d such that the following matrix A is a representation matrix of g:

$$A := \begin{pmatrix} e_{d_1} & 0 & 0 \\ 0 & e_{d_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $1 \leq i < d$, the diagonal entries of A^i are different from each other. As like Example 2.7, we get that X does not have Galois points.

We give an example of a smooth surface X of degree $d \geq 4$ in $\mathbb{P}^3$ such that X has a linear automorphism g of order $(d-1)d$ but has no Galois points.

Example 2.9. Let $d_1 \geq 5$ be an odd integer, and $d := 2d_1 + 1$. Let X be a smooth surface of degree d in $\mathbb{P}^3$ defined by

$$X_0^d + X_0^{\frac{d+1}{2}} X_1^{\frac{d-1}{2}} + X_0 X_1^{d-1} + X_2^{d-1} X_3 + X_2 X_3^{d-1} = 0.$$

The surface X has an automorphism g of order $(d-1)d$ such that the following matrix A is a representation matrix of g

$$A := \begin{pmatrix} e^{1-d}_{\frac{(d-1)}{2}d} & 0 & 0 & \\ 0 & e_{\frac{(d-1)}{2}d} & 0 & \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In addition, the surface X has an automorphism h of order $(d-2)\frac{(d-1)}{2}d$ such that the following matrix B is a representation matrix of h

$$B := \begin{pmatrix} e^{1-d}_{\frac{(d-1)}{2}d} & 0 & 0 & \\ 0 & e_{\frac{(d-1)}{2}d} & 0 & \\ 0 & 0 & e_{d-2} & 0 \\ 0 & 0 & 0 & e^{-1}_{d-2} \end{pmatrix}.$$

For $1 \leq i < \frac{(d-1)}{2}d$, the diagonal entries of B^i are different from each other. By Theorem 2.1, $\delta(X) \leq 2$ and $\delta'(X) \leq 4$. Since $\frac{(d-1)}{2}d \geq 5$, if X has a Galois point, then there is a Galois point p of X such that $g^l(p) = p$ for some $1 \leq l < \frac{(d-1)}{2}d$. As like Example 2.7, this is a contradiction. Then X does not have Galois points.

From here, based on [1], we explain the orders of automorphisms of smooth plane curves of degree $d \geq 4$. Let X be a smooth plane curve of degree $d \geq 4$, and g be an automorphism of X. By replacing the local coordinate system if necessary, we may assume that g is defined by a diagonal matrix, i.e., $g = \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{smallmatrix}\right)$. Let

$$n(g) := \sharp|\mathrm{Fix}(g) \cap \{[1:0:0], [0:1:0], [0:0:1]\}|.$$

Since g is defined by a diagonal matrix, $n(g) = \sharp|X \cap \{[1:0:0],[0:1:0],[0:0:1]\}|$. Then $n(g) = 0, 1, 2$, or 3. The following Theorem 2.10 determines orders of cyclic groups acting on smooth plane curves. Theorem 1.7 is shown by Theorems 2.3 and 2.10.

For a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, orders of automorphisms of X and the structure of the group $\mathrm{Aut}(X)$ are studied for $n \geq 1$ ([2,6–8,18,24]). Also, as in [10,18], the structures of subgroups of $\mathrm{Aut}(X)$ are also investigated based on the way they act on X. In this paper, we examine automorphisms of X that give Galois points. At the end of this section, we classify abelian groups acting on smooth plane curves (Theorem 2.11).

Theorem 2.10 ([1]). *Let X be a smooth curve of degree $d \geq 4$ in $\mathbb{P}^2$, and g be an automorphism of X. By replacing the local coordinate system if necessary, the order of g and a representation matrix of g are one of Table 1.*

TABLE 1. Cyclic groups of smooth plane curves of degree $d \geq 4$

No.	$n(g)$	Order l of g	Representation matrix of g
1	0	l divides d	$\begin{pmatrix} e_l^s & 0 & 0 \\ 0 & e_l^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$
2	1	l divides $d-1$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
3	1	l divides $(d-1)d$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
4	2	l divides $d-1$	$\begin{pmatrix} e_l^s & 0 & 0 \\ 0 & e_l^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$
5	2	l divides $(d-1)^2$	$\begin{pmatrix} e_l^{1-d} & 0 & 0 \\ 0 & e_l & 0 \\ 0 & 0 & 1 \end{pmatrix}$
6	2	l divides $(d-2)d$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{1-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
7	3	l divides $d-1$	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
8	3	l divides d^2-3d+3	$\begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l^{d-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Theorem 2.11. *Let X be a smooth plane curve of degree $d \geq 4$, and G be an abelian subgroup of* $\mathrm{Aut}(X)$. *If G is not a cyclic group, then G is isomorphic to a subgroup of $\mathbb{Z}/d\mathbb{Z}^{\oplus 2}$ as a group.*

Proof. Since $d \geq 4$, G is a finite subgroup of $\mathrm{PGL}(3, \mathbb{C})$. Let

$$l := \max\{\mathrm{ord}(k) \mid k \in G\}.$$

We take an element $g \in G$ such that $\mathrm{ord}(g) = l$. By replacing the local coordinate system if necessary, we may assume that g is defined by a diagonal matrix.

First, we assume that $g = \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{smallmatrix}\right)$, where $\alpha, \beta \in \mathbb{C}^*$. For simplicity, we may assume that $\alpha = e_l$ and $\beta = 1$. Let h be an element of G such that $h \notin \langle g \rangle$, and $A := (a_{ij})_{1 \leq i,j \leq 3}$ be a representation matrix of h. Since $g \circ h = h \circ g$, we get that

$$\begin{aligned} &\begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} e_l & 0 & 0 \\ 0 & e_l & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

and hence

$$\begin{pmatrix} ue_l a_{11} & ue_l a_{12} & ue_l a_{13} \\ ue_l a_{21} & ue_l a_{22} & ue_l a_{23} \\ ua_{31} & ua_{32} & ua_{33} \end{pmatrix} = \begin{pmatrix} e_l a_{11} & e_l a_{12} & a_{13} \\ e_l a_{21} & e_l a_{22} & a_{23} \\ e_l a_{31} & e_l a_{32} & a_{33} \end{pmatrix}$$

for $u \in \mathbb{C}^*$. If $u \neq 1$, then $a_{11} = a_{12} = a_{21} = a_{22} = a_{33} = 0$. Since A is invertible, this is a contradiction. Therefore, $u = 1$. Then $a_{13} = a_{23} = a_{31} = a_{32} = 0$. This means that there is an injective homomorphism

$$\vartheta : G \ni k \mapsto C \in \mathrm{GL}(2, \mathbb{C}) \ \text{ such that } \ k = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\vartheta(G)$ is an abelian but not cyclic subgroup of $\mathrm{GL}(2, \mathbb{C})$, there are two matrices $S_1, S_2 \in \mathrm{GL}(2, \mathbb{C})$ such that $\vartheta(G) = \langle S_1 \rangle \oplus \langle S_2 \rangle$. In order to show $G \subset \mathbb{Z}/d\mathbb{Z}^{\oplus 2}$, we only show that $\mathrm{ord}(g')$ is a divisor of d for any $g' \in G$. Since $G \cong \vartheta(G) = \langle S_1 \rangle \oplus \langle S_2 \rangle$, by replacing the local coordinate system if necessary, we may assume that G is generated by two diagonal matrices. We assume that $p := [1 : 0 : 0] \in X$. Since G is generated by diagonal matrices, we get that $p \in \mathrm{Fix}(g)$ for any $g \in G$. Since $\dim X = 1$, and X is smooth, we get that G is a cyclic group. This contradicts that G is not a cyclic group. Therefore, we get that $[1 : 0 : 0] \notin X$. Similarly, we get that $[0 : 1 : 0], [0 : 0 : 1] \notin X$. Since $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1] \notin X$, X is defined by

$$aX^d + bY^d + cZ^d + \sum_{i=0}^{d-1} F_{d-i}(Y, Z) X^i = 0,$$

where $abc \neq 0$, $F_{d-i}(Y,Z)$ is a homogeneous polynomial of degree $d-i$ for $0 \leq i \leq d-1$, and $F_0(Y,Z)$ has no Y^d and Z^d terms. Then since G is generated by diagonal matrices, we get that $\mathrm{ord}(g')$ is a divisor of d for any $g' \in G$. Therefore, G is a subgroup of $\mathbb{Z}/d\mathbb{Z}^{\oplus 2}$.

Next, we assume that there is not an element $g' \in G$ such that a representation matrix of g' is conjugate to $\begin{pmatrix} \alpha' & 0 & 0 \\ 0 & \alpha' & 0 \\ 0 & 0 & \beta' \end{pmatrix}$, where $\alpha', \beta' \in \mathbb{C}^*$. Then we may assume that $g = \begin{pmatrix} e_l^s & 0 & 0 \\ 0 & e_l^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $e_l^s \neq e_l^t$, $e_l^s \neq 1$, and $e_l^t \neq 1$. Let h be an element of G such that $h \notin \langle g \rangle$, and $A := (a_{ij})_{1 \leq i,j \leq 3}$ be a representation matrix of h. Since $g \circ h = h \circ g$,

$$\begin{pmatrix} ue_l^s a_{11} & ue_l^s a_{12} & ue_l^s a_{13} \\ ue_l^t a_{21} & ue_l^t a_{22} & ue_l^t a_{23} \\ ua_{31} & ua_{32} & ua_{33} \end{pmatrix} = \begin{pmatrix} e_l^s a_{11} & e_l^t a_{12} & a_{13} \\ e_l^s a_{21} & e_l^t a_{22} & a_{23} \\ e_l^s a_{31} & e_l^t a_{32} & a_{33} \end{pmatrix}$$

for $u \in \mathbb{C}^*$. If $a_{ii} \neq 0$ for some $1 \leq i \leq 3$, then $u = 1$. Since $e_l^s \neq e_l^t$, $e_l^s \neq 1$, and $e_l^t \neq 1$, we get that $a_{ij} = 0$ for $i \neq j$, i.e., A is a diagonal matrix. Since $\mathrm{ord}(h)$ divides l, and g and h are defined by diagonal matrices, we get that $\langle g, h \rangle$ contains an automoprhism k such that a representation matrix of k is conjugate to $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$, where $\alpha, \beta \in \mathbb{C}^*$. This contradicts the assumption for G. Therefore, $a_{ii} = 0$ for any $i = 1, 2, 3$. Since A is invertible, $a_{12} \neq 0$ or $a_{13} \neq 0$. We assume that $a_{12}a_{13} \neq 0$. Then $ue_l^s = e_l^t$ and $ue_l^s = 1$, and hence we get that $e_l^t = 1$. This contradicts the assumption that $e_l^t \neq 1$. Therefore, $a_{12}a_{13} = 0$ and $(a_{12}, a_{13}) \neq (0,0)$. In the same way, $a_{21}a_{23} = a_{31}a_{32} = 0$, $(a_{21}, a_{23}) \neq (0,0)$, and $(a_{31}, a_{32}) \neq (0,0)$. Since A is invertible,

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}.$$

If A is the former, then $ue_l^s = e_l^t$, $ue_l^t = 1$, and $u = e_l^s$. Therefore, we get that ${e_l^s}^3 = {e_l^t}^3 = u^3 = 1$. In the same way, for the latter case, we get that ${e_l^s}^3 = {e_l^t}^3 = u^3 = 1$. Therefore, we may assume that $g = \begin{pmatrix} e_3^2 & 0 & 0 \\ 0 & e_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and for an automorphism $k \in G \backslash \langle g \rangle$, k is defined by a matrix of the form:

$$\begin{pmatrix} 0 & b_{12} & 0 \\ 0 & 0 & b_{23} \\ b_{31} & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & b_{13} \\ b_{21} & 0 & 0 \\ 0 & b_{32} & 0 \end{pmatrix}.$$

Note that the square of the former (resp. latter) form of the matrix is of the latter (resp. former) form of the matrix. From here, we show that $G \cong \mathbb{Z}/3\mathbb{Z}^{\oplus 3}$, and the degree d of X is a multiple of 3. We assume that there are two automorphisms $h_1, h_2 \in G$ such that

$$h_1 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & a' & 0 \\ 0 & 0 & b' \\ c' & 0 & 0 \end{pmatrix},$$

and $h_1 \notin \langle h_2 \rangle$. Then

$$h_1^2 \circ h_2 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a' & 0 \\ 0 & 0 & b' \\ c' & 0 & 0 \end{pmatrix} = \begin{pmatrix} abc' & 0 & 0 \\ 0 & a'bc & 0 \\ 0 & 0 & abc' \end{pmatrix}.$$

Since G is abelian, and $\mathrm{ord}(h_i) = 3$ for $i = 1, 2$, we get that $\mathrm{ord}(h_1^2 \circ h_2) = 3$. Since $\mathrm{ord}(g) = 3$, and the assumption for G, we get that $h_1^2 \circ h_2 \in \langle g \rangle$. Therefore, $G = \langle g, h \rangle \cong \mathbb{Z}/3\mathbb{Z}^{\oplus 3}$ where

$$g = \begin{pmatrix} e_3^2 & 0 & 0 \\ 0 & e_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}.$$

Since $h([1:0:0]) = [0:0:1]$ and $h^2([1:0:0]) = [0:1:0]$, if $\{[1:0:0], [0:1:0], [0:0:1]\} \cap X \neq \emptyset$, then $\{[1:0:0], [0:1:0], [0:0:1]\} \subset X$, i.e., $n(g) = 3$. By Table 1 and a representation matrix of g, we get that 3 divides d. Then G is a subgroup of $\mathbb{Z}/d\mathbb{Z}^{\oplus 2}$. We assume that $\{[1:0:0], [0:1:0], [0:0:1]\} \cap X = \emptyset$. By Table 1 and a representation matrix of g, we get that $\mathrm{ord}(g) = 3$ divides $d^2 - 3d + 3$, and hence 3 divides d. Therefore, G is a subgroup of $\mathbb{Z}/d\mathbb{Z}^{\oplus 2}$. □

3. Proof of main theorems

First, we will show Theorem 1.7 (Theorem 3.1). Theorem 1.7 is immediately followed by Theorems 2.3 and 2.10.

Theorem 3.1. *Let X be a smooth plane curve degree $d \geq 4$, and g be an automorphism of X.*

(1) *If $\mathrm{ord}(g) = d - 1$ and $\sharp|\mathrm{Fix}(g)| \neq 2$, then X has an inner Galois point p, and g is an automorphism belonging to the Galois point p.*

(2) *If $\mathrm{ord}(g) = d$ and $\mathrm{Fix}(g) \neq \emptyset$, then X has an outer Galois point p, and g is an automorphism belonging to the Galois point p.*

Proof. Since $d \geq 4$, $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{PGL}(3, \mathbb{C})$. We will show (1) of this theorem. Since $\mathrm{ord}(g) = d - 1$, by replacing the local coordinate system if necessary, we may assume that g is defined by a diagonal matrix A such that A is one of no. 2, 3, 4, 5, and 7 of Table 1. By Theorem 2.3, if A is one of no. 2, 3, 5, and 7 of Table 1, then X has an inner Galois point p, and g is an automorphism belonging to the Galois point p. We assume that A is no.4 of Table 1, i.e., $A = \begin{pmatrix} e_{d-1}^s & 0 & 0 \\ 0 & e_{d-1}^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $1 \leq s, t < d - 1$. Then $X \cap \{[1:0:0], [0:1:0], [0:0:1]\} \subset \mathrm{Fix}(g)$ and $\sharp|X \cap \{[1:0:0], [0:1:0], [0:0:1]\}| = 2$. Since $\sharp|\mathrm{Fix}(g)| \neq 2$, $\sharp|\mathrm{Fix}(g)| \geq 3$. Then we get that $s = t$,

$s = 1$, or $t = 1$. By Theorem 2.3, X has an inner Galois point p, and g is an automorphism belonging to the Galois point p. In the same way, we get (2) of this theorem. $\square$

Let X be a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$, p be a point in $\mathbb{P}^{n+1}$. Recall that $\pi_p : X \dashrightarrow H$ is a projection with center p where H is a hyperplane not containing p.

The following result is obtained for an inner Galois point ([21]).

Theorem 3.2 ([21])**.** *Let X be a smooth plane curve degree $d \geq 4$, and $\mathbb{C}(X)$ be the function field of X, and $k \subset \mathbb{C}(X)$ be a subfield such that k is isomorphic to $\mathbb{C}(\mathbb{P}^1)$ as a field. If $\mathbb{C}(X)/k$ is a Galois extension of degree $d-1$, then X has an inner Galois point p, and the Galois extension $\mathbb{C}(X)/k$ is induced by $\pi_p : X \to \mathbb{P}^1$, i.e., $k = \pi_p^*(\mathbb{C}(\mathbb{P}^1))$.*

In the case of the outer Galois point, by Example 3.3, we see that the same result as in Theorem 3.2 does not hold.

Example 3.3. Let X be a smooth curve of degree 4 in $\mathbb{P}^2$ defined by

$$X_0^4 + X_1^4 + X_2^4 = 0$$

which is called the Fermat curve of degree 4. The X has two automorphism g_1 and g_2 of order 2 such that the following matrices A_1 and A_2 are representation matrices of g_1 and g_2, respectively

$$A_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let G be the subgroup of $\mathrm{Aut}(X)$ generated by g_1 and g_2, and $g_3 := g_1 \circ g_2 \in G$. Then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, and $\sharp|\mathrm{Fix}(g_i)| = 4$ for $i = 1, 2, 3$.

Let $G_x := \{g \in G : g(x) = x\}$. For a smooth curve C, we write the genus of C as $g(C)$. By the Riemann-Hurwitz formula,

$$2 - 2g(X) + \sum_{x \in X} (\sharp|G_x| - 1) = \sharp|G|(2 - 2g(X/G)).$$

Since X is a smooth plane curve of degree 4, we get that $2-2g(X) = 4(3-4) = -4$. Then

$$2 - 2g(X) + \sum_{x \in X} (\sharp|G_x| - 1) = -4 + 12 = 8.$$

Since $\sharp|G| = 4$, and the Riemann-Hurwitz formula, we get that $g(X/G) = 0$, and hence $X/G \cong \mathbb{P}^1$. Let $p : X \to X/G$ be the quotient morphism. Since G is not cyclic group, the Galois extension $\mathbb{C}(X)/p^*\mathbb{C}(\mathbb{P}^1)$ is not induced by a Galois point of X.

The following theorem shows that similar results hold for an outer Galois point under the assumption of a cyclic extension.

Theorem 3.4. *Let X be a smooth plane curve degree $d \geq 4$, and $\mathbb{C}(X)$ be the function field of X, and $k \subset \mathbb{C}(X)$ be a subfield such that k is isomorphic to $\mathbb{C}(\mathbb{P}^1)$ as a field. If $\mathbb{C}(X)/k$ is a cyclic extension of degree d, then X has an outer Galois point p, and the cyclic extension $\mathbb{C}(X)/k$ is induced by $\pi_p : X \to \mathbb{P}^1$, i.e., $k = \pi_p^*(\mathbb{C}(\mathbb{P}^1))$.*

Proof. Since X is a smooth curve, there is a cyclic subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{P}^1$, and $k = p^*\mathbb{C}(\mathbb{P}^1)$ where $p : X \to X/G$ is the quotient morphism. Since $d \geq 4$, G is a subgroup of $\mathrm{PGL}(3, \mathbb{C})$. Let g be a generator of G. By replacing the local coordinate system if necessary, we assume that there is a diagonal matrix A such that A is a representation matrix of g. Since $\mathrm{ord}(g) = d$ and Theorem 3.1, we only show that $\mathrm{Fix}(g) \neq \emptyset$.

We assume that $\mathrm{Fix}(g) = \emptyset$. By Theorem 2.3, that is, by the no. 1 of Table 1, we may assume that $A = \begin{pmatrix} e_d^s & 0 & 0 \\ 0 & e_d^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since $\mathrm{Fix}(g) = \emptyset$, $X \cap \{[1:0:0], [0:1:0], [0:0:1]\} = \emptyset$. Then if $\mathrm{Fix}(g^i) \neq \emptyset$ for some $1 < i < d$, then $\sharp|\mathrm{Fix}(g^i)| = d$. By the Riemann-Hurwitz formula and $C/G \cong \mathbb{P}^1$,

$$2 - 2g(X) + \sum_{x \in X} (\sharp|G_x| - 1) = 2\sharp|G| = 2d.$$

Since X is a smooth plane curve of degree d, we get that $2 - 2g(X) = d(3 - d)$, and hence By the matrix A, we get that $\mathrm{Fix}(g^i) \backslash \{[1:0:0], [0:1:0]\} \neq \emptyset$ if and only if $(e_{d-1}^{si} - e_{d-1}^{ti})(e_{d-1}^{si} - 1)(e_{d-1}^{ti} - 1) = 0$ for $1 < i < d$. We define subgroups G_1, G_2, and G_3 of G as follows:

$$G_1 := \{g \in G \,|\, \text{a representation matrix of } g \text{ is } \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for some } \alpha \in \mathbb{C}^*\}.$$

$$G_2 := \{g \in G \,|\, \text{a representation matrix of } g \text{ is } \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for some } \beta \in \mathbb{C}^*\}.$$

$$G_3 := \{g \in G \,|\, \text{a representation matrix of } g \text{ is } \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for some } \gamma \in \mathbb{C}^*\}.$$

We set $a := \sharp|G_1|$, $b := \sharp|G_2|$, and $c := \sharp|G_3|$. Then $G_i \cap G_j = \{\mathrm{id}_X\}$ for $1 \leq i < j \leq 3$, and $\mathrm{Fix}(g^i) \neq \emptyset$ if and only if $g^i \in \bigcup_{j=1}^3 G_j$ for $1 < i < d$. Then

$$(d-1)d = \sum_{x \in C} (\sharp|G_x| - 1) = d(a + b + c - 3).$$

Therefore,

$$d + 2 = a + b + c.$$

For simplicity, we may assume that $a \leq b \leq c$. Since $d + 2 = a + b + c$, $1 < c$. Since $G_2 \cap G_3 = \{\mathrm{id}_X\}$ and $\sharp|G| = d$, we get that $bc \mid d$. By the equation $d+2 = a+b+c$, we get that $bc+2 \leq a+b+c \leq b+2c$, and hence $(b-2)(c-1) \leq 0$. Since $1 < c$, $b \leq 2$. If $b = 2$, then by the equation $bc + 2 \leq a + b + c$, we get that $a = b = c$. Since $G_i \cap G_i = \{\mathrm{id}_X\}$ for $1 \leq i < j \leq 3$, we get that $\mathbb{Z}_2^{\oplus 2} \cong \langle G_i, G_j \rangle \subset G$ where $1 \leq i < j \leq 3$, and $\langle G_i, G_j \rangle$ is the subgroup of G generated by G_i and G_j. This contradicts that G is a cyclic group. If $b = 1$, then $a = 1$ and $c = d$. This implies that $G = \langle g \rangle = G_3$. This contradicts that $G = \langle g \rangle$ and $\mathrm{Fix}(g) = \emptyset$. Therefore, $\mathrm{Fix}(g) \neq \emptyset$. By Theorem 3.1, X has an outer Galois point p, and g is an automorphism belonging to the Galois point p. $\square$

From here, we will study $X \subset \mathbb{P}^{n+1}$ for $n \geq 2$. First, we give Examples 3.5 and 3.6 which imply that Corollary 3.4 does not hold for $n = 2$.

Example 3.5. Let X be a smooth surface of degree 4 in $\mathbb{P}^3$ defined by

$$X_0^3 X_2 + X_1^3 X_3 + X_2^4 + X_3^4 = 0.$$

The surface X has an automorphism g of order 3 such that

$$g = \begin{pmatrix} e_3 & 0 & 0 & 0 \\ 0 & e_3^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathrm{Fix}(g)$ contains a smooth rational curve. Since the degree of X is 4, X is a $K3$ surface. Since $\mathrm{Fix}(g)$ contains a curve, g is a non-symplectic automorphism of order 3. Then the quotient space $Y := X/\langle g \rangle$ is rational. Let $q : X \to Y$ be the quotient morphism. Since Y is rational $k := q^*\mathbb{C}(Y) \cong \mathbb{C}(\mathbb{P}^2)$ as a field. However, by Theorem 2.3, there is no a Galois point p of X such that g is an automorphism belonging to the Galois point p. In other words, there is no a Galois point p of X such that $k = \pi_p^*\mathbb{C}(\mathbb{P}^2)$. Pay attention that X has Galois points, and $\delta(X) = 8$ ([22]).

Example 3.6. Let X be a smooth surface in $\mathbb{P}^3$ defined by

$$X_0^6 + X_1^6 + X_2^6 + X_3^6 + X_0^2 X_1^3 X_2 + X_2^3 X_3^3 = 0.$$

The surface X has an automorphism g of order 6 such that

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathrm{Fix}(g^3) = \{X_0 = 0\} \cap X := H_1$ and $\mathrm{Fix}(g^2) = \{X_1 = 0\} \cap X := H_2$ are smooth curves, and $\mathrm{Fix}(g) = H_1 \cap H_2$. Then the quotient space $Y := X/\langle g \rangle$ is smooth. Let $p : X \to Y$ be the quotient morphism, and $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^3}(1)$ be the ample line bundle. By the ramification formula, $K_X = p^*K_Y + H_1 + 2H_2$, and hence $p^*K_Y = K_X - H_1 - 2H_2$. Since $K_X = \mathcal{O}_X(2)$, and $\mathcal{O}_X(H_i) = \mathcal{O}_X(1)$ for $i = 1, 2$, we get that $p^*\mathcal{O}_Y(-K_Y) = \mathcal{O}_X(1)$ is ample. Since the morphism $p : X \to Y$ is finite, $-K_Y$ is ample. Since Y is a smooth surface, Y is rational, and hence $k := q^*\mathbb{C}(Y) \cong \mathbb{C}(\mathbb{P}^2)$ as a field. However, by Theorem 2.3, there is no a Galois point p of X such that $k = \pi_p^*\mathbb{C}(\mathbb{P}^2)$.

We will show Theorems 1.8 and 1.9 (Theorem 3.7). Recall that for a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 4$, if $(n, d) \neq (2, 4)$, then all automorphisms of X are linear.

Theorem 3.7. *Let X be a smooth hypersurface of degree $d \geq 4$ in $\mathbb{P}^{n+1}$, and g be a linear automorphism of X.*

(1) *If $n = 2$, $\mathrm{ord}(g) = d - 1$, and $\mathrm{Fix}(g)$ contains a curve C' which is not a smooth rational curve, then X has an inner Galois point p, and g is an automorphism belonging to the Galois point p.*

(2) *If $n \geq 3$, $\mathrm{ord}(g) = d - 1$, and $\mathrm{Fix}(g)$ has codimension 1 in X, then X has an inner Galois point p, and g is an automorphism belonging to the Galois point p.*

(3) *If $n \geq 2$, $\mathrm{ord}(g) = d$, and $\mathrm{Fix}(g)$ has codimension 1 in X, then X has an outer Galois point p, and g is an automorphism belonging to the Galois point p.*

Proof. By replacing the local coordinate system if necessary, we may assume that

$$g = \begin{pmatrix} a_{i_1} I_{i_1} & & \\ & \ddots & \\ & & a_{i_m} I_{i_m} \end{pmatrix}$$

where I_{i_j} is the identity matrix of size i_j, $a_{i_j} \in \mathbb{C}^*$, $a_{i_j} \neq a_{i_k}$ for $1 \leq i_j, i_k \leq m$, and $\sum_{j=1}^{m} i_j = n + 2$. We assume that $\mathrm{Fix}(g)$ contains a hypersurface H in X. Since $\dim H = n - 1$, $i_j \geq n - 1$ for some $1 \leq j \leq m$. Then we may assume that

$$g = \begin{pmatrix} a & 0 \\ 0 & I_{n+1} \end{pmatrix} \text{ or } \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & I_n \end{pmatrix}.$$

If g is defined by the former matrix, then by Theorem 2.3 X has a Galois point p, and g is an automorphism belonging to the Galois point p.

From here, we will show that if g is defined by the latter matrix, then $n = 2$, $\mathrm{ord}(g) \neq d$, and curves contained in $\mathrm{Fix}(g)$ are smooth rational curves. Let $F(X_0, \ldots, X_{n+2})$ be the defining equation of X. We assume that $n \geq 3$. By $\dim H = n - 1$ and the representation matrix of g, $H = \{X_0 = 0\} \cap \{X_1 = 0\}$. Then

$$\begin{aligned} F(X_0, \ldots, X_{n+2}) = {} & F_{1,0}(X_2, \ldots, X_{n+2})X_0 + F_{0,1}(X_2, \ldots, X_{n+2})X_1 \\ & + \sum_{2 \leq i+j \leq d} F_{i,j}(X_2, \ldots, X_{n+2}) X_0^i X_1^j, \end{aligned}$$

where $F_{1,0}(X_2, \ldots, X_{n+2}) \neq 0$ and $F_{0,1}(X_2, \ldots, X_{n+2}) \neq 0$. Since X is smooth, $\{F_{1,0}(X_2, \ldots, X_{n+2}) = 0\} \cap \{F_{0,1}(X_2, \ldots, X_{n+2}) = 0\} \cap \{X_0 = 0\} \cap \{X_1 = 0\} = \emptyset$. This contradicts $n \geq 3$. Therefore, $n = 2$. We assume that $a \neq b$. By $\dim H = 1$ and the representation matrix of g, $H = \{X_0 = 0\} \cap \{X_1 = 0\}$.

Then

$$F(X_0,\dots,X_3) = F_{1,0}(X_2,X_3)X_0 + F_{0,1}(X_2,X_3)X_1 + \sum_{2\le i+j\le d} F_{i,j}(X_2,X_3)X_0^iX_1^j,$$

where $F_{1,0}(X_2,X_3) \neq 0$ and $F_{0,1}(X_2,X_3) \neq 0$. Then by the representation matrix of g, $a = b$. This is a contradiction, and hence $a = b$. Then $H = \{X_0 = 0\} \cap \{X_1 = 0\}$ or $H = \{X_2 = 0\} \cap \{X_3 = 0\}$. Curves contained in $\mathrm{Fix}(g)$ are smooth rational curves. By the representation matrix of g, we may assume that $H = \{X_0 = 0\} \cap \{X_1 = 0\}$ by replacing X_0 and X_1 with X_2 and X_3 if necessary. Then $F(X_0,X_1,X_2,X_3)$ is expressed as above. We assume that $\mathrm{ord}(g) = d$. Then $a = e_d$, and the defining equation of X is as follows.

$$F(X_0,X_1,X_2,X_3) = F_{1,0}(X_2,X_3)X_0 + F_{0,1}(X_2,X_3)X_1.$$

Points $[1:0:0:0]$ and $[0:1:0:0]$ are singular points of X. This contradicts that X is smooth. Therefore, $\mathrm{ord}(g) \neq d$. □

In the same way, we get Theorem 1.10 (Theorem 3.8).

Theorem 3.8. *Let X be a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$, $g \in \mathrm{Aut}(X)$ be a linear automorphism of order $k(d-1)$ for $k \ge 2$.*

(1) *If $n = 2$ and $\sharp|\mathrm{Fix}(g)| \ge 5$, then X has an inner Galois point p, and g^k is an automorphism belonging to the Galois point p.*

(2) *If $n \ge 3$, and the dimension of $\mathrm{Fix}(g)$ is $n-2$, then X has an inner Galois point p, and g^k is an automorphism belonging to the Galois point p.*

Proof. As like the proof of Theorem 3.7, we may assume that

$$g = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & I_n \end{pmatrix} \text{ or } \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & I_{n-1} \end{pmatrix}$$

where a, b, c, and 1 are different numbers from each other.

First, we will show that if g is defined by the former matrix, then X has an inner Galois point p, and g^k is an automorphism belonging to the Galois point p.

Let $F(X_0,\dots,X_{n+2})$ be the defining equation of X. Since $\dim\mathrm{Fix}(g) = n-2$,

$$F(X_0,\dots,X_{n+2}) = \sum_{1\le i+j\le d} F_{i,j}(X_2,\dots,X_{n+2})X_0^iX_1^j + G(X_2,\dots,X_{n+2}),$$

where $G(X_2,\dots,X_{n+2}) \neq 0$. Let $n(g) := \sharp|\{[1:0:\cdots:0],[0:1:0:\cdots:0]\} \cap X|$.

If $n(g) = 0$, then $\sum_{1\le i+j\le d} F_{i,j}(X_2,\dots,X_{n+2})X_0^iX_1^j$ has X_0^d and X_1^d terms. Since $G(X_2,\dots,X_{n+2}) \neq 0$, $a^d = b^d = 1$. This contradicts that $\mathrm{ord}(g) > d$.

If $n(g)=1$, then we may assume that $\sum_{1\le i+j\le d} F_{i,j}(X_2,\dots,X_{n+2})X_0^iX_1^j$ has (i) X_0^d and $X_iX_1^{d-1}$ terms, or (ii) X_0^d and $X_0X_1^{d-1}$ terms where $2\le i\le n+2$. The case (i) implies that $a^d=b^{d-1}=1$. By Theorem 2.3, there is an inner Galois point p of X, and g^k is an automorphism belonging to the Galois point p. The case (ii) implies that $a^d=ab^{d-1}=1$. Same as above, X has an inner Galois point p, and g^k is an automorphism belonging to the Galois point p.

If $n(g)=2$, then we may assume that $\sum_{1\le i+j\le d} F_{i,j}(X_2,\dots,X_{n+2})X_0^iX_1^j$ has (iii) $X_iX_0^d$ and $X_iX_1^{d-1}$ terms, (iv) $X_iX_0^d$ and $X_0X_1^{d-1}$, or (v) $X_1X_0^{d-1}$ and $X_0X_1^{d-1}$ terms where $2\le i,j\le n+2$. The case (iii) implies that $a^{d-1}=b^{d-1}=1$. This contradicts that $\mathrm{ord}(g)>d-1$. As like the case $n(g)=1$, if the case is (iv), then by Theorem 2.3, there is an inner Galois point p of X, and g^k is an automorphism belonging to the Galois point p. The case (v) implies that $a^{d-1}b=ab^{d-1}=1$. Then $\mathrm{ord}(g)$ divides $(d-2)d$. This contradicts that $\mathrm{ord}(g)=k(d-1)$.

From here, we study the latter case, i.e., $g=\begin{pmatrix} a&0&0&0\\ 0&b&0&0\\ 0&0&c&0\\ 0&0&0&I_{n-1}\end{pmatrix}$. As like the proof of Theorem 3.7, we get that $n\le 3$. We assume that $n=3$. Let $F(X_0,\dots,X_5)$ be the defining equation of X. Since the dimension of $\mathrm{Fix}(g)$ is $n-2$,

$$F(X_0,\dots,X_{n+2})=\sum_{i=0}^{2}F_i(X_3,\dots,X_{n+2})X_i + \sum_{2\le i+j+k\le d}F_{i,j,k}(X_3,\dots,X_{n+2})X_0^iX_1^jX_2^k.$$

Since X is smooth, $F_i(X_3,\dots,X_{n+2})\ne 0$ for $i=0,1,2$. Then $a=b=c$. This contradicts that $\mathrm{ord}(g)=k(d-1)$ for $k\ge 2$. Then $n=2$, and hence

$$g=\begin{pmatrix} a&0&0&0\\ 0&b&0&0\\ 0&0&c&0\\ 0&0&0&1\end{pmatrix}.$$

Since a, b, c, and 1 are different numbers from each other, $\sharp|\mathrm{Fix}(g)|\le 4$. From the above, we get this theorem. $\square$

The following example shows that Theorem 3.8 does not hold for an outer Galois point.

Example 3.9. Let $d_1\ge 7$ be an odd integer, and $d:=2d_1+1$. Let X be a smooth hypersurface of degree d in $\mathbb{P}^4$ defined by

$$X_0^d+X_0^{\frac{d+1}{2}}X_1^{\frac{d-1}{2}}+X_0X_1^{d-1}+X_2^{d-1}X_4+X_2X_3^{d-1}+X_3X_4^{d-1}=0.$$

The X has an automorphism g of order $\frac{(d-1)}{2}d$ such that the following matrix A is a representation matrix of g:

$$A := \begin{pmatrix} e^{1-d}_{\frac{(d-1)}{2}d} & 0 & 0 & 0 & 0 \\ 0 & e_{\frac{(d-1)}{2}d} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the dimension of $\mathrm{Fix}(g^{d^2-3d+3})$ is 1. In addition, X has an automorphism h such that the following matrix B is a representation matrix of h:

$$B := \begin{pmatrix} e^{1-d}_{\frac{(d-1)}{2}d} & 0 & 0 & 0 & 0 \\ 0 & e_{\frac{(d-1)}{2}d} & 0 & 0 & 0 \\ 0 & 0 & e_{d^2-3d+3} & 0 & 0 \\ 0 & 0 & 0 & e^{d-1}_{d^2-3d+3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If 3 divides d, then $\mathrm{ord}(h) = \frac{(d-1)}{6}d(d^2-3d+3)$, and if 3 does not divide d, then $\mathrm{ord}(h) = \frac{(d-1)}{2}d(d^2-3d+3)$. For $1 \le i < \frac{d-1}{2}$, the diagonal entries of B^i are different from each other. By Theorem 2.1, $\delta(X) \le 2$ and $\delta'(X) \le 5$. Since $\frac{d-1}{2} \ge 7$, if X has a Galois point, then there is a Galois point p of X such that $g^l(p) = p$ for some $1 \le l < \frac{(d-1)}{2}d$. As like Example 2.7, this is a contradiction. Then X does not have Galois points.

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