

**GLOBAL AXISYMMETRIC SOLUTIONS TO
THE 3D NAVIER–STOKES–POISSON–NERNST–PLANCK
SYSTEM IN THE EXTERIOR OF A CYLINDER**

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ABSTRACT. In this paper we prove global existence and uniqueness of axisymmetric strong solutions for the three dimensional electro-hydrodynamic model based on the coupled Navier–Stokes–Poisson–Nernst–Planck system in the exterior of a cylinder. The key ingredient is that we use the axisymmetry of functions to derive the L^p interpolation inequalities, which allows us to establish all kinds of a priori estimates for the velocity field and charged particles via several cancellation laws.

1. Introduction

In this paper, we study the following dissipative system of nonlinear and nonlocal equations modeling the flow of electro-hydrodynamics in a connected open set $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$:

$$(1) \quad \begin{cases} u_t + (u \cdot \nabla)u - \Delta u + \nabla P = \Delta \Psi \nabla \Psi, \\ \operatorname{div} u = 0, \\ n_t^- + (u \cdot \nabla)n^- = \nabla \cdot (\nabla n^- - n^- \nabla \Psi), \\ n_t^+ + (u \cdot \nabla)n^+ = \nabla \cdot (\nabla n^+ + n^+ \nabla \Psi), \\ \Delta \Psi = n^- - n^+, \end{cases}$$

where $u = (u_1, u_2, u_3)$ and P denote the unknown vector velocity and scalar pressure of fluid, respectively, n^- and n^+ denote the densities of binary diffuse negatively and positively charged particles, respectively, and Ψ is the electrostatic potential. All physical parameters in (1) have been taken to be 1 for simplicity of presentation. Furthermore, in the domain Ω where the fluid occupies, the system (1) is assumed to supplement with the following initial

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conditions:

$$(2) \quad (u, n^-, n^+)|_{t=0} = (u_0, n_0^-, n_0^+), \quad \operatorname{div} u_0 = 0, \quad n_0^- > 0, \quad n_0^+ > 0 \quad \text{in } \Omega,$$

and the velocity field equations is determined by the no slip boundary condition:

$$(3) \quad u = 0 \quad \text{on } \partial\Omega \times (0, T),$$

while the equations for the charged densities are determined by the pure Neumann boundary conditions:

$$(4) \quad \frac{\partial n^-}{\partial \nu} = 0, \quad \frac{\partial n^+}{\partial \nu} = 0, \quad \frac{\partial \Psi}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, T),$$

where ν denotes the unit outward normal vector of $\partial\Omega$.

The system (1)–(4) appears in the context as the Navier–Stokes–Poisson–Nernst–Planck system in electro-hydrodynamics, which was intended to account for the electro-diffusion phenomenon in an incompressible electrical fluid medium, for example, see [14,16]. Generally speaking, the self-consistent charge transport is described by the Poisson–Nernst–Planck equations, while the fluid motion is governed by the incompressible Navier–Stokes equations with forcing terms. We refer the readers to see [16] for the detailed mathematical description and physical background of this fluid-dynamical model.

If the flow is charge-free, i.e., $n^- = n^+ = \Psi = 0$, then the system (1) reduces to the following incompressible Navier–Stokes equations:

$$(5) \quad \begin{cases} u_t + (u \cdot \nabla)u - \Delta u + \nabla P = 0, \\ \operatorname{div} u = 0. \end{cases}$$

In their celebrated works, Leray [12] and Hopf [5] proved that the n -dimensional ($n \geq 2$) Navier–Stokes equations (5), subject to the initial data of L^2 -finite energy, admits at least a global Leray–Hopf weak solution. It is well-known that such a global weak solution is regular and unique in two dimensional case, but in three dimensional case, the regularity and uniqueness of such weak solutions still remains a challenging open problem in mathematical fluid dynamics. On the other hand, many efforts have been made to study various solutions with certain special structures, and the axisymmetric solution is such an important case. For the axisymmetric Navier–Stokes equations without swirl, Ladyzhenskaya [9] and Ukhovskii–Yudovich [17] independently proved global existence, uniqueness and regularity of axisymmetric weak solutions. Later on, Leonardi et al. [11] gave a refined proof, and Abidi [2] extended this global regularity result to certain initial data in critical space $\dot{H}^{\frac{1}{2}}$. For the axisymmetric Navier–Stokes equations with non-trivial swirl, Ladyzhenskaya [10] and Abe–Seregin [1] proved the global existence of unique axisymmetric strong solution in the exterior of a cylinder subject to the no slip and Navier boundary conditions. The crucial points for the analysis in [10] and [1] are the interpolation inequalities and the maximum principle.

To our knowledge, mathematical analysis of the system (1) was initiated by Jerome [6], where the author established a local well-posedness theory and stability under the inviscid limit based on the Kato’s semigroup framework. Subsequently, local existence with any initial data and global existence with small initial data in various critical functional spaces (e.g., Lebesgue and Besov spaces) were established by [13, 18–20]. On the other hand, the global existence, as well as regularity and uniqueness, of weak solutions for the system (1) adapted with various boundary conditions have been studied in two or three dimensions by [3, 4, 7, 8, 15, 16], some analytical results similar to the Navier–Stokes equations were obtained.

The main goal of this paper is to study global existence of axisymmetric strong solutions for the electro-hydrodynamic system (1) in the exterior of a cylinder subject to the initial boundary-value conditions (2)–(4). The key ingredient is that we shall make use of axisymmetry of functions to derive some L^p interpolation inequalities, which allow us to establish some crucial a priori estimates for the velocity field and charged particles via several important cancellation laws. Without loss of generality, we assume that

$$\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3, \sqrt{x_1^2 + x_2^2} = \delta > 0, x_3 \in \mathbb{R}\}.$$

Denote a point in Ω by x . Let us consider the cylindrical coordinates:

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3,$$

and denote the three standard basis vectors are as follows:

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1).$$

A function f or a vector function $u = (u^r, u^\theta, u^z)$ is said to be axisymmetric if f, u^r, u^θ and u^z are independent of the angular variable θ , i.e., f and u has the following forms:

$$f(x_1, x_2, x_3) = f(r, z), \quad u(x_1, x_2, x_3) = u^r(r, z)e_r + u^\theta(r, z)e_\theta + u^z(r, z)e_z.$$

Due to the uniqueness of strong solutions, it is clear that if the initial data (u_0, n_0^-, n_0^+) is axisymmetric, then the strong solution (u, n^-, n^+) to the problem (1)–(4) is also axisymmetric.

Before we state the main result, let us introduce the following notations. We denote by $L^p(\Omega)$, $1 < p < \infty$ (or $L^\infty(\Omega)$) the space of the usual scalar-valued or vector-valued functions defined on Ω with the p -th power absolutely integrable (or essentially bounded scalar-valued or vector-valued functions) for the Lebesgue measure. For $m \in \mathbb{N}$, $1 \leq p \leq \infty$, the Sobolev space $W^{m,p}(\Omega)$ is the space of functions in $L^p(\Omega)$ with derivatives of order less than or equal to m in $L^p(\Omega)$, i.e.,

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega), |\alpha| \leq m\}.$$

In particular, when $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$. Let $C_0^\infty(\Omega)$ be the space of C^∞ functions with compact support contained in Ω . The closure of

$C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$ ($H_0^m(\Omega)$ when $p = 2$). Moreover, let $C_{0,\sigma}^\infty(\Omega)$ be the space

$$C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega) : \operatorname{div} u = 0\}.$$

The closure of $C_{0,\sigma}^\infty(\Omega)$ in $H^1(\Omega)$ is denoted by $H_{0,\sigma}^1(\Omega)$.

Our main result is stated as follows.

Theorem 1.1. *Let $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3, \sqrt{x_1^2 + x_2^2} = \delta > 0, x_3 \in \mathbb{R}\}$. Assume that (u_0, n_0^-, n_0^+) is an axisymmetric initial data and satisfy the following regularity conditions:*

$$(6) \quad u_0 \in H^2(\Omega) \cap H_{0,\sigma}^1(\Omega), \quad n_0^-, n_0^+ \in H^2(\Omega) \cap L^1(\Omega), \quad n_0^- > 0, \quad n_0^+ > 0 \text{ in } \Omega.$$

Then for every $T > 0$, there exists a unique axisymmetric strong solution (u, n^-, n^+) to the initial-boundary value problem (1)–(4) with

$$u \in C([0, T], H_{0,\sigma}^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), \quad n^-, n^+ \in C([0, T], H^2(\Omega) \cap L^1(\Omega))$$

and

$$u_t, n_t^-, n_t^+ \in L^\infty(0, T; L^2(\Omega)), \quad \nabla u_t, \nabla n_t^-, \nabla n_t^+ \in L^2(0, T; L^2(\Omega)).$$

We shall prove Theorem 1.1 in the next section. Throughout the paper, we shall use the notation $\|\cdot\|_{L^p}$ instead of $\|\cdot\|_{L^p(\Omega)}$, and $\|\cdot\|_{H^m}$ instead of $\|\cdot\|_{H^m(\Omega)}$ for simplicity. We denote by C the harmless positive constant, which may depend on initial datum and its value may change from line to line, the special dependence will be pointed out explicitly in the text if necessary.

2. The proof of Theorem 1.1

We first prove the following crucial interpolation inequalities for axisymmetric functions in Lebesgue spaces.

Lemma 2.1. *Let $D = \{(r, z) : r \geq \delta > 0, z \in \mathbb{R}\}$. Then for any axisymmetric vector function $u = u^r(r, z)e_r + u^\theta(r, z)e_\theta + u^z(r, z)e_z$ satisfying $u^i \in H^1(D)$ for $i = r, \theta, z$, there exists a constant C depending only on p and δ such that for any $2 \leq p < \infty$, we have*

$$(7) \quad \|u\|_{L^p(\Omega)} \leq C(p, \delta) (\|u\|_{L^2(\Omega)}^{\frac{2}{p}} \|\nabla u\|_{L^2(\Omega)}^{1-\frac{2}{p}} + \|u\|_{L^2(\Omega)}),$$

where $\Omega = \{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} = \delta > 0, x_3 \in \mathbb{R}\}$ is the corresponding domain of D in Cartesian coordinates. We emphasize here that (7) also holds for any axisymmetric scalar function.

Proof. Notice that in the Cartesian coordinates,

$$u_1 = \frac{x_1}{r}u^r - \frac{x_2}{r}u^\theta, \quad u_2 = \frac{x_2}{r}u^r + \frac{x_1}{r}u^\theta, \quad u_3 = u^z.$$

Then we have

$$\|u\|_{L^p}^p = \int_\Omega |u|^p dx = \int_\Omega \left(\left(\frac{x_1}{r}u^r - \frac{x_2}{r}u^\theta \right)^2 + \left(\frac{x_2}{r}u^r + \frac{x_1}{r}u^\theta \right)^2 + (u^z)^2 \right)^{\frac{p}{2}} dx$$

$$\begin{aligned}
 &= \int_{\Omega} (|u^r|^2 + |u^\theta|^2 + |u^z|^2)^{\frac{p}{2}} dx \\
 &\leq C(p) \int_{\Omega} (|u^r|^p + |u^\theta|^p + |u^z|^p) dx.
 \end{aligned}$$

By using the Gagliardo–Nirenberg’s inequality in two dimensions, we know that for $i = r, \theta, z$,

$$\|u^i(r, z)\|_{L^p(D)}^p \leq C(p) \|u^i(r, z)\|_{L^2(D)}^2 \|\tilde{\nabla} u^i(r, z)\|_{L^2(D)}^{p-2},$$

where $\tilde{\nabla} := (\partial_r, \partial_z)$ is the two dimensional gradient operator. It follows that

$$\begin{aligned}
 \|u^i\|_{L^p(\Omega)}^p &= \int_{\Omega} |u^i|^p dx = 2\pi \int_D |u^i|^p r dr dz = 2\pi \int_D (|u^i| r^{\frac{1}{p}})^p dr dz \\
 &\leq C(p) \left(\int_D (|u^i| r^{\frac{1}{p}})^2 dr dz \right) \left(\int_D (|\partial_r u^i| r^{\frac{1}{p}})^2 + \frac{1}{p^2} (|u^i| r^{\frac{1}{p}-1})^2 + (|\partial_z u^i| r^{\frac{1}{p}})^2 dr dz \right)^{\frac{p}{2}-1} \\
 &\leq C(p) \left(\int_D (|u^i| r^{\frac{1}{p}})^2 dr dz \right) \left(\int_D (|\partial_r u^i| r^{\frac{1}{p}})^2 + (|\partial_z u^i| r^{\frac{1}{p}})^2 dr dz \right)^{\frac{p}{2}-1} \\
 &\quad + C(p) \left(\int_D (|u^i| r^{\frac{1}{p}})^2 dr dz \right) \left(\int_D (|u^i| r^{\frac{1}{p}-1})^2 dr dz \right)^{\frac{p}{2}-1}.
 \end{aligned}$$

Since $2 \leq p < \infty$, $r \geq \delta > 0$, it is easily seen that

$$r^{\frac{2}{p}} = r r^{\frac{2}{p}-1} \leq r \delta^{\frac{2}{p}-1}, \quad r^{\frac{2}{p}-2} \leq r \delta^{\frac{2}{p}-3}.$$

This yields immediately that

$$\begin{aligned}
 \|u^i\|_{L^p(\Omega)}^p &\leq C(p, \delta) \left[\int_D |u^i|^2 r dr dz \left(\int_D (|\partial_r u^i|^2 + |\partial_z u^i|^2) r dr dz \right)^{\frac{p}{2}-1} \right. \\
 &\quad \left. + \left(\int_D |u^i|^2 r dr dz \right)^{\frac{p}{2}} \right] \\
 &\leq C(p, \delta) (\|u^i\|_{L^2(\Omega)}^2 \|\nabla u^i\|_{L^2(\Omega)}^{p-2} + \|u^i\|_{L^2(\Omega)}^p).
 \end{aligned}$$

The proof of Lemma 2.1 is achieved. □

Next we establish the following several crucial a priori estimates. Let us introduce two new functions $v := n^- + n^+$ and $w := n^- - n^+$, and then the problem (1) is reduced into the following system of equations:

$$(8) \quad \begin{cases} u_t + (u \cdot \nabla)u - \Delta u + \nabla P = w \nabla \Psi, \\ \operatorname{div} u = 0, \\ v_t + (u \cdot \nabla)v = \nabla \cdot (\nabla v - w \nabla \Psi), \\ w_t + (u \cdot \nabla)w = \nabla \cdot (\nabla w - v \nabla \Psi), \\ \Delta \Psi = w, \end{cases}$$

while initial conditions (2) and boundary conditions (3)–(4) are correspondingly changed into the following way:

$$(9) \quad (u, v, w)|_{t=0} = (u_0, v_0, w_0) = (u_0, n_0^- + n_0^+, n_0^- - n_0^+), \quad \operatorname{div} u_0 = 0 \quad \text{in } \Omega$$

and

$$(10) \quad u = 0, \quad \frac{\partial v}{\partial \nu} = 0, \quad \frac{\partial w}{\partial \nu} = 0, \quad \frac{\partial \Psi}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, T).$$

It is clear that if (u, n^-, n^+) is an axisymmetric strong solution of the problem (1)–(4) (P and Ψ can be determined by (u, n^-, n^+)), then (u, v, w) is an axisymmetric strong solution of the problem (8)–(10) (P and Ψ can be determined by (u, v, w)), and vice versa. Therefore, we aim at establishing some crucial a priori estimates of axisymmetric strong solutions for the problem (8)–(10).

Lemma 2.2. *Let the assumptions (6) be in force, and let (u, v, w) be the corresponding axisymmetric strong solution to the problem (8)–(10) on $[0, T]$ for any $0 < T \leq \infty$. Then we have*

$$(11) \quad \sup_{0 \leq t \leq T} \|(v(t), w(t))\|_{L^2}^2 + 2 \int_0^T \|(\nabla v(t), \nabla w(t))\|_{L^2}^2 dt \leq \|(v_0, w_0)\|_{L^2}^2;$$

$$(12) \quad \sup_{0 \leq t \leq T} \|(u(t), \nabla \Psi(t))\|_{L^2}^2 + 2 \int_0^T \|(\nabla u(t), \Delta \Psi(t))\|_{L^2}^2 dt \leq \|(u_0, \nabla \Psi(0))\|_{L^2}^2.$$

Proof. Multiplying the third equation of (8) by v and integrating over Ω , one has

$$(13) \quad \frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \int_{\Omega} v \nabla w \cdot \nabla \Psi dx + \int_{\Omega} v w^2 dx = 0,$$

where we have used integration by parts, the boundary conditions (10) and the divergence free condition $\operatorname{div} u = 0$ to yield

$$\int_{\Omega} (u \cdot \nabla) v dx = \frac{1}{2} \int_{\Omega} (u \cdot \nabla) v^2 dx = -\frac{1}{2} \int_{\Omega} (\nabla \cdot u) v^2 dx = 0.$$

Repeating the same steps for w , we have

$$(14) \quad \frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \int_{\Omega} w \nabla v \cdot \nabla \Psi dx + \int_{\Omega} v w^2 dx = 0.$$

After integration by parts, we obtain the following cancellation by consideration of the fifth equation of (8) and the boundary conditions (10):

$$\int_{\Omega} w \nabla v \cdot \nabla \Psi dx + \int_{\Omega} v \nabla w \cdot \nabla \Psi dx = - \int_{\Omega} v w^2 dx.$$

Therefore, adding up the above estimates (13) and (14) yields

$$(15) \quad \frac{1}{2} \frac{d}{dt} \|(v, w)\|_{L^2}^2 + \|(\nabla v, \nabla w)\|_{L^2}^2 + \int_{\Omega} v w^2 dx = 0.$$

Integrating (15) over $[0, t]$ for all $0 < t \leq T$ implies that

$$(16) \quad \sup_{0 \leq t \leq T} \|(v, w)\|_{L^2}^2 + 2 \int_0^T \|(\nabla v, \nabla w)\|_{L^2}^2 d\tau + 2 \int_0^T \int_{\Omega} vw^2 dx d\tau = \|(v_0, w_0)\|_{L^2}^2.$$

Since v is nonnegative, which can be ensured by the nonnegativity of n^- and n^+ (see for example [6, 16]), we get (11).

To prove (12), multiplying the first equations of (8) by u , after integration by parts, it can be easily seen that

$$(17) \quad \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = \int_{\Omega} w \nabla \Psi \cdot u dx.$$

On the other hand, multiplying the fourth equation of (8) by Ψ , after integration by parts and using the fifth equation of (8), one has

$$(18) \quad \frac{1}{2} \frac{d}{dt} \|\nabla \Psi\|_{L^2}^2 + \int_{\Omega} |\Delta \Psi|^2 dx + \int_{\Omega} v |\nabla \Psi|^2 dx + \int_{\Omega} u \cdot \nabla \Psi w dx = 0.$$

Adding up the above estimates (17) and (18) yields that

$$(19) \quad \frac{1}{2} \frac{d}{dt} \|(u, \nabla \Psi)\|_{L^2}^2 + \|(\nabla u, \Delta \Psi)\|_{L^2}^2 + \int_{\Omega} v |\nabla \Psi|^2 dx = 0.$$

Then (12) follows by integrating (19) in the time interval $[0, t]$ for any $0 < t \leq T$, and using v is nonnegative. The proof of Lemma 2.2 is achieved. □

Lemma 2.3. *Let the assumptions (6) be in force, and let (u, v, w) be the corresponding axisymmetric strong solution to the problem (8)–(10) on $[0, T]$ for any $0 < T < \infty$. Then we have*

$$(20) \quad \sup_{0 \leq t \leq T} \|(\nabla v(t), \nabla w(t))\|_{L^2}^2 + \int_0^T \|(\Delta v(t), \Delta w(t))\|_{L^2}^2 dt \leq C,$$

where $C = C(\|u_0\|_{L^2}, \|(v_0, w_0)\|_{H^1 \cap L^1}, T)$.

Proof. Multiplying the third equation of (8) by $-\Delta v$ and integrating over Ω , after integration by parts, one gets

$$(21) \quad \frac{1}{2} \frac{d}{dt} \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 = \int_{\Omega} (u \cdot \nabla) v \Delta v dx + \int_{\Omega} \nabla w \cdot \nabla \Psi \Delta v dx + \int_{\Omega} w^2 \Delta v dx.$$

The three terms on the right hand side of (21) can be bounded in the following way by using Hölder’s inequality, (11), (12), (7) with $p = 4$ and Young’s inequality,

$$\begin{aligned} & \int_{\Omega} (u \cdot \nabla) v \Delta v dx \\ & \leq \|u\|_{L^4} \|\nabla v\|_{L^4} \|\Delta v\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\leq C(\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2})(\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\Delta v\|_{L^2}^{\frac{1}{2}} + \|\nabla v\|_{L^2})\|\Delta v\|_{L^2} \\
&\leq C(1 + \|\nabla u\|_{L^2}^{\frac{1}{2}})(\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\Delta v\|_{L^2}^{\frac{1}{2}} + \|\nabla v\|_{L^2})\|\Delta v\|_{L^2} \\
&\leq C(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}} + \|\nabla v\|_{L^2}^{\frac{1}{2}})\|\Delta v\|_{L^2}^{\frac{3}{2}} + C(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2} + \|\nabla v\|_{L^2})\|\Delta v\|_{L^2} \\
&\leq \frac{1}{8}\|\Delta v\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^2)\|\nabla v\|_{L^2}^2;
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega} \nabla w \cdot \nabla \Psi \Delta v dx \\
&\leq \|\nabla w\|_{L^4} \|\nabla \Psi\|_{L^4} \|\Delta v\|_{L^2} \\
&\leq C(\|\nabla w\|_{L^2}^{\frac{1}{2}}\|\Delta w\|_{L^2}^{\frac{1}{2}} + \|\nabla w\|_{L^2})(\|\nabla \Psi\|_{L^2}^{\frac{1}{2}}\|\Delta \Psi\|_{L^2}^{\frac{1}{2}} + \|\nabla \Psi\|_{L^2})\|\Delta v\|_{L^2} \\
&\leq C(\|\nabla w\|_{L^2}^{\frac{1}{2}}\|\Delta w\|_{L^2}^{\frac{1}{2}} + \|\nabla w\|_{L^2})\|\Delta v\|_{L^2} \\
&\leq \frac{1}{8}\|\Delta v\|_{L^2}^2 + C(\|\nabla w\|_{L^2}\|\Delta w\|_{L^2} + \|\nabla w\|_{L^2}^2) \\
&\leq \frac{1}{8}\|\Delta v\|_{L^2}^2 + \frac{1}{8}\|\Delta w\|_{L^2}^2 + C\|\nabla w\|_{L^2}^2;
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega} w^2 \Delta v dx &\leq \|w\|_{L^4}^2 \|\Delta v\|_{L^2} \leq C(\|w\|_{L^2}\|\nabla w\|_{L^2} + \|w\|_{L^2}^2)\|\Delta v\|_{L^2} \\
&\leq \frac{1}{8}\|\Delta v\|_{L^2}^2 + C(\|\nabla w\|_{L^2}^2 + 1).
\end{aligned}$$

Taking the above three estimates into (21) yields

$$\begin{aligned}
(22) \quad &\frac{d}{dt}\|\nabla v\|_{L^2}^2 + \frac{5}{4}\|\Delta v\|_{L^2}^2 \\
&\leq \frac{1}{4}\|\Delta w\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^2)(1 + \|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2).
\end{aligned}$$

Repeating the same calculations for the equation of w , we get

$$\begin{aligned}
(23) \quad &\frac{d}{dt}\|\nabla w\|_{L^2}^2 + \frac{5}{4}\|\Delta w\|_{L^2}^2 \\
&\leq \frac{1}{4}\|\Delta v\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^2)(1 + \|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2).
\end{aligned}$$

Adding up (22) and (23) provides

$$\begin{aligned}
(24) \quad &\frac{d}{dt}\|(\nabla v, \nabla w)\|_{L^2}^2 + \|(\Delta v, \Delta w)\|_{L^2}^2 \\
&\leq C(1 + \|\nabla u\|_{L^2}^2)(1 + \|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2).
\end{aligned}$$

Notice that $\|\nabla \Psi(0)\|_{L^2}$ can be controlled by $\|w_0\|_{L^2 \cap L^1}$. Therefore, applying the Gronwall's inequality to (24) yields that

$$(25) \quad \sup_{0 \leq t \leq T} \|(\nabla v, \nabla w)\|_{L^2}^2 + \int_0^T \|(\Delta v, \Delta w)\|_{L^2}^2 dt \leq C$$

for some $C = C(\|u_0\|_{L^2}, \|(v_0, w_0)\|_{H^1 \cap L^1}, T)$. The proof of Lemma 2.3 is achieved. \square

Lemma 2.4. *Let the assumptions (6) be in force, and let (u, v, w) be the corresponding axisymmetric strong solution to the problem (8)–(10) on $[0, T]$ for any $0 < T < \infty$. Then we have*

$$(26) \quad \sup_{0 \leq t \leq T} \|\nabla u(t)\|_{L^2}^2 + \int_0^T \|\Delta u(t)\|_{L^2}^2 dt \leq C(\|u_0\|_{H^1}, \|(v_0, w_0)\|_{L^2 \cap L^1}, T).$$

Proof. Multiplying the first equations of (8) by $-\Delta u$ and integrating over Ω , after integration by parts, we get

$$(27) \quad \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \int_{\Omega} (u \cdot \nabla) u \Delta u dx - \int_{\Omega} w \nabla \Psi \Delta u dx.$$

Similarly, we can estimate two terms on the right hand side of (27) based on the facts (11), (12) and (20):

$$\begin{aligned} & \int_{\Omega} (u \cdot \nabla) u \Delta u dx \\ & \leq \|u\|_{L^4} \|\nabla u\|_{L^4} \|\Delta u\|_{L^2} \\ & \leq C(\|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}) (\|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} + \|\nabla u\|_{L^2}) \|\Delta u\|_{L^2} \\ & \leq C(\|\nabla u\|_{L^2}^{\frac{1}{2}} + \|\nabla u\|_{L^2}) \|\Delta u\|_{L^2}^{\frac{3}{2}} + C(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{3}{2}}) \|\Delta u\|_{L^2} \\ & \leq \frac{1}{4} \|\Delta u\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^2}^2) \|\nabla u\|_{L^2}^2; \\ & - \int_{\Omega} w \nabla \Psi \Delta u dx \\ & \leq \|w\|_{L^4} \|\nabla \Psi\|_{L^4} \|\Delta u\|_{L^2} \\ & \leq C(\|w\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2}^{\frac{1}{2}} + \|w\|_{L^2}) (\|\nabla \Psi\|_{L^2}^{\frac{1}{2}} \|\Delta \Psi\|_{L^2}^{\frac{1}{2}} + \|\nabla \Psi\|_{L^2}) \|\Delta u\|_{L^2} \\ & \leq \frac{1}{4} \|\Delta u\|_{L^2}^2 + C(1 + \|\nabla w\|_{L^2}^2). \end{aligned}$$

From the above two estimates, it follows easily from (27) that

$$(28) \quad \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \leq C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2)(1 + \|\nabla u\|_{L^2}^2),$$

which yields (26) by applying Gronwall’s inequality, (11) and (12). The proof of Lemma 2.4 is achieved. \square

Lemma 2.5. *Let the assumptions (6) be in force, and let (u, v, w) be the corresponding axisymmetric strong solution to the problem (8)–(10) on $[0, T]$ for any $0 < T < \infty$. Then we have*

$$(29) \quad \sup_{0 \leq t \leq T} \|(u_t, v_t, w_t, \nabla \Psi_t)\|_{L^2}^2 + \int_0^T \|(\nabla u_t, \nabla v_t, \nabla w_t)\|_{L^2}^2 dt$$

$$\leq C(\|u_0\|_{H^2}, \|(v_0, w_0)\|_{H^2 \cap L^1}, T).$$

Proof. Taking the derivative to the system (8) with respect to t , we see that

$$(30) \quad \begin{cases} u_{tt} + (u_t \cdot \nabla)u + (u \cdot \nabla)u_t - \Delta u_t + \nabla P_t = w_t \nabla \Psi + w \nabla \Psi_t, \\ v_{tt} + (u_t \cdot \nabla)v + (u \cdot \nabla)v_t = \nabla \cdot (\nabla v_t - w_t \nabla \Psi - w \nabla \Psi_t), \\ w_{tt} + (u_t \cdot \nabla)w + (u \cdot \nabla)w_t = \nabla \cdot (\nabla w_t - v_t \nabla \Psi - v \nabla \Psi_t), \\ \Delta \Psi_t = w_t. \end{cases}$$

We first consider the estimates for v_t and w_t . Multiplying the second equation of (30) by v_t , the third equation of (30) by w_t , and integrating over Ω , respectively, then adding up the resultant two equalities, we get

$$(31) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(v_t, w_t)\|_{L^2}^2 + \|(\nabla v_t, \nabla w_t)\|_{L^2}^2 \\ &= - \int_{\Omega} (u_t \cdot \nabla)v \cdot v_t + (u_t \cdot \nabla)w \cdot w_t dx \\ & \quad - \int_{\Omega} \nabla \cdot (w_t \nabla \Psi)v_t + \nabla \cdot (v_t \nabla \Psi)w_t dx \\ & \quad - \int_{\Omega} \nabla \cdot (w \nabla \Psi_t)v_t + \nabla \cdot (v \nabla \Psi_t)w_t dx \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Applying Hölder's inequality, interpolation inequality (7) and Young's inequality, and using (11), (12), (20) and (26), the right hand side of (31) can be estimated as follows:

$$\begin{aligned} I_1 &\leq \|\nabla v\|_{L^2} \|u_t\|_{L^4} \|v_t\|_{L^4} + \|\nabla w\|_{L^2} \|u_t\|_{L^4} \|w_t\|_{L^4} \\ &\leq C(\|u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} + \|u_t\|_{L^2}) (\|(v_t, w_t)\|_{L^2}^{\frac{1}{2}} \|(\nabla v_t, \nabla w_t)\|_{L^2}^{\frac{1}{2}} + \|(v_t, w_t)\|_{L^2}) \\ &\leq \frac{1}{4} \|\nabla u_t\|_{L^2}^2 + \frac{1}{8} \|(\nabla v_t, \nabla w_t)\|_{L^2}^2 + C(\|u_t\|_{L^2}^2 + \|(v_t, w_t)\|_{L^2}^2); \\ I_2 &= - \int_{\Omega} w v_t w_t dx \leq \|w\|_{L^2} \|v_t\|_{L^4} \|w_t\|_{L^4} \\ &\leq C(\|v_t\|_{L^2}^{\frac{1}{2}} \|\nabla v_t\|_{L^2}^{\frac{1}{2}} + \|v_t\|_{L^2}) (\|w_t\|_{L^2}^{\frac{1}{2}} \|\nabla w_t\|_{L^2}^{\frac{1}{2}} + \|w_t\|_{L^2}) \\ &\leq \frac{1}{8} \|(\nabla v_t, \nabla w_t)\|_{L^2}^2 + C\|(v_t, w_t)\|_{L^2}^2; \\ I_3 &= \int_{\Omega} (w \nabla \Psi_t \cdot \nabla v_t + v \nabla \Psi_t \cdot \nabla w_t) dx \\ &\leq \|w\|_{L^4} \|\nabla \Psi_t\|_{L^4} \|\nabla v_t\|_{L^2} + \|v\|_{L^4} \|\nabla \Psi_t\|_{L^4} \|\nabla w_t\|_{L^2} \\ &\leq C(\|\nabla \Psi_t\|_{L^2}^{\frac{1}{2}} \|\Delta \Psi_t\|_{L^2}^{\frac{1}{2}} + \|\nabla \Psi_t\|_{L^2}) (\|\nabla v_t\|_{L^2} + \|\nabla w_t\|_{L^2}) \\ &\leq \frac{1}{8} \|(\nabla v_t, \nabla w_t)\|_{L^2}^2 + \frac{1}{4} \|\Delta \Psi_t\|_{L^2}^2 + C(\|w_t\|_{L^2}^2 + \|\nabla \Psi_t\|_{L^2}^2), \end{aligned}$$

where we have used (20) and (26) to bound $\|(v, w)\|_{L^4}$ in the derivation of I_3 :

$$\|(v, w)\|_{L^4} \leq C(\|(v, w)\|_{L^2}^{\frac{1}{2}}\|(\nabla v, \nabla w)\|_{L^2}^{\frac{1}{2}} + \|(v, w)\|_{L^2}) \leq C.$$

Taking all above estimates into (31), we obtain

$$(32) \quad \begin{aligned} & \frac{d}{dt}\|(v_t, w_t)\|_{L^2}^2 + \frac{5}{4}\|(\nabla v_t, \nabla w_t)\|_{L^2}^2 \\ & \leq \frac{1}{2}\|(\nabla u_t, \Delta \Psi_t)\|_{L^2}^2 + C\|(u_t, \nabla \Psi_t, v_t, w_t)\|_{L^2}^2. \end{aligned}$$

Next we derive the estimates for u_t and $\nabla \Psi_t$. Adding up the first equations of (30) and the third equation of (30) tested with u_t and Ψ_t , respectively, using the relation $\Delta \Psi_t = w_t$, after integration by parts, we see that

$$(33) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt}\|(u_t, \nabla \Psi_t)\|_{L^2}^2 + \|(\nabla u_t, \Delta \Psi_t)\|_{L^2}^2 \\ & = - \int_{\Omega} (u_t \cdot \nabla) u \cdot u_t dx + \int_{\Omega} w_t \nabla \Psi \cdot u_t dx \\ & \quad + \int_{\Omega} (u \cdot \nabla w_t) \cdot \Psi_t dx + \int_{\Omega} \nabla \cdot (v_t \nabla \Psi + v \nabla \Psi_t) \Psi_t dx \\ & := I_4 + I_5 + I_6 + I_7, \end{aligned}$$

where we have used the cancellation relation

$$\int_{\Omega} w \nabla \Psi_t \cdot u_t dx + \int_{\Omega} (u_t \cdot \nabla) w \Psi_t dx = 0.$$

Applying (11), (12), (20) and (26) again, we can bound I_i ($i = 4, 5, 6, 7$) one by one as follows:

$$\begin{aligned} I_4 & \leq \|\nabla u\|_{L^2} \|u_t\|_{L^4}^2 \leq C(\|u_t\|_{L^2} \|\nabla u_t\|_{L^2} + \|u_t\|_{L^2}^2) \\ & \leq \frac{1}{8} \|\nabla u_t\|_{L^2}^2 + C\|u_t\|_{L^2}^2; \\ I_5 & \leq \|w_t\|_{L^2} \|\nabla \Psi\|_{L^4} \|u_t\|_{L^4} \\ & \leq C\|w_t\|_{L^2} (\|\nabla \Psi\|_{L^2}^{\frac{1}{2}} \|\Delta \Psi\|_{L^2}^{\frac{1}{2}} + \|\nabla \Psi\|_{L^2}) (\|u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} + \|u_t\|_{L^2}) \\ & \leq \frac{1}{8} \|\nabla u_t\|_{L^2}^2 + C(\|u_t\|_{L^2}^2 + \|w_t\|_{L^2}^2); \\ I_6 & = - \int_{\Omega} w_t \nabla \Psi_t \cdot u dx \leq \|u\|_{L^2} \|w_t\|_{L^4} \|\nabla \Psi_t\|_{L^4} \\ & \leq C(\|w_t\|_{L^2}^{\frac{1}{2}} \|\nabla w_t\|_{L^2}^{\frac{1}{2}} + \|w_t\|_{L^2}) (\|\nabla \Psi_t\|_{L^2}^{\frac{1}{2}} \|\Delta \Psi_t\|_{L^2}^{\frac{1}{2}} + \|\nabla \Psi_t\|_{L^2}) \\ & \leq \frac{1}{8} \|\nabla w_t\|_{L^2}^2 + \frac{1}{8} \|\Delta \Psi_t\|_{L^2}^2 + C(\|w_t\|_{L^2}^2 + \|\nabla \Psi_t\|_{L^2}^2); \\ I_7 & = \int_{\Omega} (v_t \nabla \Psi \cdot \nabla \Psi_t + v |\nabla \Psi_t|^2) dx \\ & \leq \|\nabla \Psi\|_{L^2} \|v_t\|_{L^4} \|\nabla \Psi_t\|_{L^4} + \|v\|_{L^2} \|\nabla \Psi_t\|_{L^4}^2 \end{aligned}$$

$$\leq \frac{1}{8} \|\nabla v_t\|_{L^2}^2 + \frac{1}{8} \|\Delta \Psi_t\|_{L^2}^2 + C(\|v_t\|_{L^2}^2 + \|\nabla \Psi_t\|_{L^2}^2).$$

Taking all above estimates into (33), we have

$$(34) \quad \frac{d}{dt} \|(u_t, \nabla \Psi_t)\|_{L^2}^2 + \frac{3}{2} \|(\nabla u_t, \Delta \Psi_t)\|_{L^2}^2 \leq \frac{1}{4} \|(\nabla v_t, \nabla w_t)\|_{L^2}^2 + C\|(u_t, \nabla \Psi_t, v_t, w_t)\|_{L^2}^2.$$

Adding up (32) and (34) and using the fourth equation of (30), we finally obtain

$$(35) \quad \frac{d}{dt} \|(u_t, v_t, w_t, \nabla \Psi_t)\|_{L^2}^2 + \|(\nabla u_t, \nabla v_t, \nabla w_t)\|_{L^2}^2 \leq C\|(u_t, \nabla \Psi_t, v_t, w_t)\|_{L^2}^2.$$

Applying Gronwall’s inequality yields (29). The proof of Lemma 2.5 is achieved. \square

Based on Lemmas 2.2–2.5, one can apply the Galerkin approximation and the Aubin–Lions compactness principle to prove that for every $T > 0$, there exists a global axisymmetric strong solution (u, v, w) to the problem (8)–(10) satisfying

$$(36) \quad u \in C([0, T], H_{0,\sigma}^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), v, w \in C([0, T], H^2(\Omega) \cap L^1(\Omega))$$

and

$$(37) \quad u_t, v_t, w_t \in L^\infty(0, T; L^2(\Omega)), \quad \nabla u_t, \nabla v_t, \nabla w_t \in L^2(0, T; L^2(\Omega)).$$

Finally, let us prove the uniqueness of the axisymmetric strong solutions. Let (u_1, v_1, w_1) and (u_2, v_2, w_2) be two solutions of the problem (8)–(10) with the same initial data and satisfy (36) and (37). Denoting $\delta u = u_1 - u_2, \delta v = v_1 - v_2, \delta w = w_1 - w_2, \delta P = P_1 - P_2, \delta \Psi = \Psi_1 - \Psi_2$. Then we have

$$(38) \quad \begin{cases} (\delta u)_t + (\delta u \cdot \nabla)u_1 + (u_2 \cdot \nabla)\delta u - \Delta \delta u + \nabla \delta P = \delta w \nabla \Psi_1 + w_2 \nabla \delta \Psi, \\ \nabla \cdot \delta u = 0, \\ (\delta v)_t + (\delta u \cdot \nabla)v_1 + (u_2 \cdot \nabla)\delta v = \nabla \cdot (\nabla \delta v - \delta w \nabla \Psi_1 - w_2 \nabla \delta \Psi), \\ (\delta w)_t + (\delta u \cdot \nabla)w_1 + (u_2 \cdot \nabla)\delta w = \nabla \cdot (\nabla \delta w - \delta v \nabla \Psi_1 - v_2 \nabla \delta \Psi), \\ \Delta \delta \Psi = \delta w \end{cases}$$

with the initial condition

$$(\delta u, \delta v, \delta w)|_{t=0} = (0, 0, 0)$$

and the boundary conditions

$$\delta u = 0, \quad \frac{\partial(\delta v)}{\partial \nu} = 0, \quad \frac{\partial(\delta w)}{\partial \nu} = 0, \quad \frac{\partial(\delta \Psi)}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, T).$$

Taking the L^2 -inner product of the third equation of (38) with δv , the fourth equation of (38) with δw , and adding up two resultant equalities, one has

$$(39) \quad \frac{1}{2} \frac{d}{dt} \|(\delta v, \delta w)\|_{L^2}^2 + \|(\nabla \delta v, \nabla \delta w)\|_{L^2}^2$$

$$\begin{aligned}
 &= - \int_{\Omega} (\delta u \cdot \nabla) v_1 \delta v + (\delta u \cdot \nabla) w_1 \delta w dx \\
 &\quad - \int_{\Omega} \nabla \cdot (\delta w \nabla \Psi_1) \delta v + \nabla \cdot (\delta v \nabla \Psi_1) \delta w dx \\
 &\quad - \int_{\Omega} \nabla \cdot (w_2 \nabla \delta \Psi) \delta v + \nabla \cdot (v_2 \nabla \delta \Psi) \delta w dx \\
 &:= J_1 + J_2 + J_3.
 \end{aligned}$$

For J_i ($i = 1, 2, 3$), we can derive that

$$\begin{aligned}
 J_1 &\leq C \|(v_1, w_1)\|_{L^2} \|\delta u\|_{L^4} \|(\delta v, \delta w)\|_{L^4} \\
 &\leq \frac{1}{8} \|\nabla \delta u\|_{L^2}^2 + \frac{1}{8} \|(\nabla \delta v, \nabla \delta w)\|_{L^2}^2 + C \|(v_1, w_1)\|_{L^2}^2 \|(\delta u, \delta v, \delta w)\|_{L^2}^2;
 \end{aligned}$$

$$\begin{aligned}
 J_2 &= - \int_{\Omega} w_1 \delta v \delta w dx \leq C \|w_1\|_{L^2} \|\delta v\|_{L^4} \|\delta w\|_{L^4} \\
 &\leq \frac{1}{8} \|(\nabla \delta v, \nabla \delta w)\|_{L^2}^2 + C \|w_1\|_{L^2}^2 \|(\delta v, \delta w)\|_{L^2}^2;
 \end{aligned}$$

$$\begin{aligned}
 J_3 &\leq C \|(v_2, w_2)\|_{L^4} \|\nabla \delta \Psi\|_{L^4} \|(\nabla \delta v, \nabla \delta w)\|_{L^2} \\
 &\leq \frac{1}{8} \|(\nabla \delta v, \nabla \delta w)\|_{L^2}^2 + \frac{1}{8} \|\Delta \delta \Psi\|_{L^2}^2 + C \|(v_2, w_2)\|_{H^1}^2 \|\nabla \delta \Psi\|_{L^2}^2.
 \end{aligned}$$

Taking the above estimates into (39), we get

$$\begin{aligned}
 (40) \quad &\frac{d}{dt} \|(\delta v, \delta w)\|_{L^2}^2 + \frac{5}{4} \|(\nabla \delta v, \nabla \delta w)\|_{L^2}^2 \\
 &\leq \frac{1}{4} \|\nabla \delta u\|_{L^2}^2 + \frac{1}{4} \|\Delta \delta \Psi\|_{L^2}^2 + C \|(v_1, w_1, v_2, w_2)\|_{H^1}^2 \|(\delta u, \delta v, \delta w, \nabla \delta \Psi)\|_{L^2}^2.
 \end{aligned}$$

Notice that we can rewrite the fourth equation of (38) as

$$(41) \quad (\delta w)_t + (u_1 \cdot \nabla) \delta w + (\delta u \cdot \nabla) w_2 = \nabla \cdot (\nabla \delta w - \delta v \nabla \Psi_1 - v_2 \nabla \delta \Psi).$$

Therefore, taking the L^2 -inner product to the first equations of (38) with δu , (41) with $\delta \Psi$, then adding up two resultant equalities together, and taking account of $\Delta \delta \Psi = \delta w$, we see that

$$\begin{aligned}
 (42) \quad &\frac{1}{2} \frac{d}{dt} \|(\delta u, \nabla \delta \Psi)\|_{L^2}^2 + \|(\nabla \delta u, \Delta \delta \Psi)\|_{L^2}^2 \\
 &= - \int_{\mathbb{R}^3} (\delta u \cdot \nabla u_1) \cdot \delta u dx + \int_{\Omega} \delta w \nabla \Psi_1 \delta u dx + \int_{\Omega} \nabla \cdot (\delta v \nabla \Psi_1 + v_2 \nabla \delta \Psi) \delta \Psi dx \\
 &:= J_4 + J_5 + J_6,
 \end{aligned}$$

where we have used the facts by consideration of integration by parts and $\Delta \delta \Psi = \delta w$:

$$\int_{\Omega} (u_1 \cdot \nabla) \delta w \delta \Psi dx = - \int_{\Omega} (u_1 \cdot \nabla) \frac{(\nabla \delta \Psi)^2}{2} dx = 0$$

and

$$\int_{\Omega} w_2 \nabla \delta \Psi \cdot \delta u + (\delta u \cdot \nabla) w_2 \delta \Psi dx = 0.$$

Similarly, for J_i ($i = 4, 5, 6$), we obtain

$$J_4 \leq C \|\nabla u_1\|_{L^2} \|\delta u\|_{L^4}^2 \leq \frac{1}{8} \|\nabla \delta u\|_{L^2}^2 + C \|\nabla u_1\|_{L^2}^2 \|\delta u\|_{L^2}^2;$$

$$J_5 \leq C \|\nabla \Psi_1\|_{L^2} \|\delta u\|_{L^4} \|\delta w\|_{L^4} \leq \frac{1}{8} \|\nabla \delta u\|_{L^2}^2 + \frac{1}{8} \|\nabla \delta w\|_{L^2}^2 \\ + C \|\nabla \Psi_1\|_{L^2}^2 \|(\delta u, \delta w)\|_{L^2}^2;$$

$$J_6 \leq C \|(\nabla \Psi_1, v_2)\|_{L^2} \|(\delta v, \nabla \delta \Psi)\|_{L^4} \|\nabla \delta \Psi\|_{L^4} \\ \leq \frac{1}{8} \|\nabla \delta v\|_{L^2}^2 + \frac{1}{8} \|\Delta \delta \Psi\|_{L^2}^2 + C \|(\nabla u_1, \nabla \Psi_1, v_2)\|_{L^2}^2 \|(\delta v, \nabla \delta \Psi)\|_{L^2}^2.$$

Taking all above estimates into (42) yields that

$$(43) \quad \frac{d}{dt} \|(\delta u, \nabla \delta \Psi)\|_{L^2}^2 + \frac{5}{4} \|(\nabla \delta u, \Delta \delta \Psi)\|_{L^2}^2 \\ \leq \frac{1}{4} \|(\nabla \delta v, \nabla \delta w)\|_{L^2}^2 + C \|(\nabla \Psi_1, v_2)\|_{L^2}^2 \|(\delta u, \delta v, \delta w, \nabla \delta \Psi)\|_{L^2}^2.$$

Summing up (40) and (43), we conclude that

$$(44) \quad \frac{d}{dt} \|(\delta u, \delta v, \delta w, \nabla \delta \Psi)\|_{L^2}^2 + \|(\nabla \delta u, \nabla \delta v, \nabla \delta w)\|_{L^2}^2 \\ \leq C \mathcal{Y}(t) \|(\delta u, \delta v, \delta w, \nabla \delta \Psi)\|_{L^2}^2,$$

where $\mathcal{Y}(t)$ is defined by

$$\mathcal{Y}(t) := \sum_{i=1}^2 \|(u_i, v_i, w_i, \nabla \Psi_i)\|_{H^1}^2.$$

Since $\mathcal{Y}(t)$ is integrable for the time interval $[0, T]$ for any $0 < T < \infty$, and Lebesgue dominated convergence theorem ensures that $\int_0^t \mathcal{Y}(\tau) d\tau$ is a continuous nondecreasing function which vanishes at zero. Hence, $(\delta u, \delta v, \delta w) \equiv (0, 0)$ on time interval $[0, t]$ for small enough t . Finally, because the function $t \rightarrow \|(\delta u, \delta v, \delta w)\|_{L^2}$ is also continuous, a standard connectivity argument enables us to conclude that $(\delta u, \delta v, \delta w) \equiv (0, 0)$ on $\Omega \times [0, T]$. We complete the proof of Theorem 1.1.

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