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REAL-VARIABLE CHARACTERIZATIONS OF VARIABLE HARDY SPACES ON LIPSCHITZ DOMAINS OF $\mathbb{R}^n$

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Abstract. Let Ω be a proper open subset of $\mathbb{R}^n$ and $p(\cdot) : \Omega \to (0, \infty)$ be a variable exponent function satisfying the globally log-Hölder continuous condition. In this article, the author introduces the "geometrical" variable Hardy spaces $H_r^{p(\cdot)}(\Omega)$ and $H_z^{p(\cdot)}(\Omega)$ on Ω, and then obtains the grand maximal function characterizations of $H_r^{p(\cdot)}(\Omega)$ and $H_z^{p(\cdot)}(\Omega)$ when Ω is a strongly Lipschitz domain of $\mathbb{R}^n$. Moreover, the author further introduces the "geometrical" variable local Hardy spaces $h_r^{p(\cdot)}(\Omega)$, and then establishes the atomic characterization of $h_r^{p(\cdot)}(\Omega)$ when Ω is a bounded Lipschitz domain of $\mathbb{R}^n$.

1. Introduction

The real-variable theory of Hardy spaces $H^p(\Omega)$ with $p \in (0, 1]$ on domains of $\mathbb{R}^n$ and their duals are well studied (see, for example, [14]) and have been playing an important and fundamental role in the boundary value problems for the Laplace equation. In recent years, there has been a lot of attention paid to the study of Hardy spaces on domains of $\mathbb{R}^n$, which has become a very active research topic in harmonic analysis (see, for instance, [1, 3–7, 13, 16]).

Originally, Chang et al. [7] introduced the Hardy spaces $H_r^p(\Omega)$ and $H_z^p(\Omega)$ on domains Ω of $\mathbb{R}^n$, respectively, by restricting arbitrary elements of $H^p(\mathbb{R}^n)$ to Ω, and restricting elements of $H^p(\mathbb{R}^n)$ which are zero outside $\overline{\Omega}$ to Ω, where and in what follows, $\overline{\Omega}$ denotes the closure of Ω in $\mathbb{R}^n$. For the Hardy spaces $H_r^p(\Omega)$ and $H_z^p(\Omega)$, atomic characterizations have been obtained in [7] when Ω is a special Lipschitz domain or a bounded Lipschitz domain of $\mathbb{R}^n$, and grand maximal function characterizations have been established in [8] when Ω is a strongly Lipschitz domain of $\mathbb{R}^n$. Moreover, Chang et al. [7] also introduced the local Hardy spaces $h_r^p(\Omega)$ and $h_z^p(\Omega)$ in a similar way and obtained atomic

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decompositions for these local Hardy spaces when Ω is a bounded Lipschitz domain of $\mathbb{R}^n$.

On the other hand, as a natural generalization of classical Hardy spaces, Nakai and Sawano [15] introduced the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$, established their atomic characterizations and investigated their dual spaces. Independently, Cruz-Uribe and Wang [10] also investigated the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with $p(\cdot)$ satisfying some conditions slightly weaker than those used in [15]. In [10], equivalent characterizations of $H^{p(\cdot)}(\mathbb{R}^n)$ by means of radial or non-tangential maximal functions or atoms were established. Moreover, Yang et al. [19, 22] established equivalent characterizations of variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ via Riesz transforms and intrinsic square functions. Furthermore, Tan [17] introduced the variable local Hardy space $h^{p(\cdot)}(\mathbb{R}^n)$ and established atomic characterizations for $h^{p(\cdot)}(\mathbb{R}^n)$ by using the discrete Littlewood-Paley-Stein theory.

As a more general class of function spaces including both Hardy spaces on Euclidean spaces with variable exponents $H^{p(\cdot)}(\mathbb{R}^n)$ and Hardy spaces on RD-spaces with constant exponents $H^p(\mathcal{X})$. Recently, Zhuo et al. [20] introduced the variable Hardy space $H^{*,\,p(\cdot)}(\mathcal{X})$ on the so-called RD-space with infinite measures via the grand maximal function, and then obtained its several real-variable characterizations, respectively, in terms of atoms and Littlewood–Paley functions.

Motivated by the above results, especially by the theory of the classical Hardy space on domains in [5, 7, 8] and the variable Hardy space in [10, 15, 20], it is the main target of this article to establish a real-variable theory of the "geometrical" variable (local) Hardy spaces on a proper open subset Ω in $\mathbb{R}^n$. Precisely, let Ω be a proper open subset of $\mathbb{R}^n$ and $p(\cdot) : \Omega \to (0, \infty)$ be a variable exponent function satisfying the globally log-Hölder continuous condition. In this article, the author introduces the "geometrical" variable Hardy spaces $H_r^{p(\cdot)}(\Omega)$ and $H_z^{p(\cdot)}(\Omega)$ on Ω, and then obtains the grand maximal function characterizations of $H_r^{p(\cdot)}(\Omega)$ and $H_z^{p(\cdot)}(\Omega)$ when Ω is a strongly Lipschitz domain of $\mathbb{R}^n$. Moreover, the author further introduces the "geometrical" variable local Hardy spaces $h_r^{p(\cdot)}(\Omega)$, and then establishes the atomic characterization of $h_r^{p(\cdot)}(\Omega)$ when Ω is a bounded Lipschitz domain of $\mathbb{R}^n$.

To state the main results of this article, we begin with recall some notation and notions. Let Ω be an open subset in $\mathbb{R}^n$. A measurable function $p(\cdot) : \Omega \to (0, \infty)$ is called a *variable exponent*. Moreover, for any given variable exponent $p(\cdot)$, let

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) \quad \text{and} \quad \underline{p} := \min\{p_-, 1\}. \tag{1.1}$$

Denote by $\mathcal{P}(\Omega)$ the set of all variable exponents $p(\cdot)$ on Ω satisfying $0 < p_- \le p_+ < \infty$.

Let f be a measurable function on Ω and $p(\cdot) \in \mathcal{P}(\Omega)$. Then the *modular function* (for simplicity, the *modular*) $\varrho_{p(\cdot)}$, associated with $p(\cdot)$, is defined by setting

$$\varrho_{p(\cdot)}(f) := \int_\Omega |f(x)|^{p(x)} dx$$

and the *Luxemburg* (also called *Luxemburg–Nakano*) *quasi-norm* $\|f\|_{L^{p(\cdot)}(\Omega)}$ of f is defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda \in (0, \infty) : \varrho_{p(\cdot)}(f/\lambda) \le 1\right\}.$$

Furthermore, the *variable Lebesgue space* $L^{p(\cdot)}(\Omega)$ is defined to the set of all measurable functions f on Ω satisfying that $\varrho_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}(\Omega)}$.

A function $p(\cdot) \in \mathcal{P}(\Omega)$ is said to satisfy the *globally log-Hölder continuous condition*, denoted by $p(\cdot) \in C^{\log}(\Omega)$, if there exist positive constants $C_{\log}$, $C_\infty \in (0,\infty)$ and $p_\infty \in \mathbb{R}$, where $p_\infty := \lim_{x\to\infty} p(x)$, such that, for any x, $y \in \Omega$,

$$|p(x) - p(y)| \le \frac{C_{\log}}{\log(e + 1/|x-y|)} \quad \text{and} \quad |p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}.$$

In the whole article, we denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual space*. For $N \in \mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$, let

$$\mathcal{F}_N(\mathbb{R}^n) := \left\{\psi \in \mathcal{S}(\mathbb{R}^n) : \sum_{\alpha\in\mathbb{Z}_+^n,\, |\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1 + |x|)^N |\partial^\alpha \psi(x)| \le 1\right\}, \tag{1.2}$$

where, for any $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, the *grand maximal function* $M_\psi(f)$ of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$M_\psi(f)(x) := \sup\left\{|f * \psi_t(x)| : t \in (0, \infty) \text{ and } \psi \in \mathcal{F}_N(\mathbb{R}^n)\right\}, \tag{1.3}$$

where, for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, $\psi_t(x) := t^{-n}\psi(x/t)$.

We begin with recall the definition of the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$, which can be found in [15, Definition 1.1].

Definition 1.1. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $N \in (n/(p_-) + n + 1, \infty) \cap \mathbb{N}$, where p_- is as in (1.1). The *variable Hardy space* denoted by $H^{p(\cdot)}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $M_\psi(f) \in L^{p(\cdot)}(\mathbb{R}^n)$ with the quasi-norm

$$\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} := \|M_\psi(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $M_\psi(f)$ is as in (1.3).

For an open subset $\Omega \subset \mathbb{R}^n$, let $\mathcal{D}(\Omega)$ denote the *space of all infinitely differentiable functions with compact supports in* Ω equipped with the inductive topology and $\mathcal{D}'(\Omega)$ its *topological dual* equipped with the weak-$*$ topology, which is called the *space of distributions on* Ω. Then we introduce the "geometric" variable Hardy spaces $H_z^{p(\cdot)}(\Omega)$ and $H_r^{p(\cdot)}(\Omega)$ on proper open subset $\Omega \subset \mathbb{R}^n$ following the way in [7,8].

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ be a proper open subset and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the *variable Hardy space* $H_z^{p(\cdot)}(\Omega)$ is defined by setting

$$H_z^{p(\cdot)}(\Omega) := \left\{ f \in \mathcal{D}'(\mathbb{R}^n) : f \in H^{p(\cdot)}(\mathbb{R}^n), \ \operatorname{supp}(f) \subset \overline{\Omega} \right\}$$

equipped with the *quasi-norm* $\|f\|_{H_z^{p(\cdot)}(\Omega)} := \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}$, where $\overline{\Omega}$ denotes the closure of Ω in $\mathbb{R}^n$.

A distribution f on Ω is said to belong to the *variable Hardy space* $H_r^{p(\cdot)}(\Omega)$ if f is the restriction to Ω of a distribution $F \in H^{p(\cdot)}(\mathbb{R}^n)$, namely,

$$\begin{aligned} H_r^{p(\cdot)}(\Omega) : &= \left\{ f \in \mathcal{D}'(\Omega) : \text{there exists an } F \in H^{p(\cdot)}(\mathbb{R}^n) \text{ such that } F|_\Omega = f \right\} \\ &= H^{p(\cdot)}(\mathbb{R}^n) / \left\{ F \in H^{p(\cdot)}(\mathbb{R}^n) : F = 0 \text{ on } \Omega \right\}. \end{aligned}$$

Moreover, for any $f \in H_r^{p(\cdot)}(\Omega)$, the *quasi-norm* $\|f\|_{H_r^{p(\cdot)}(\Omega)}$ of f in $H_r^{p(\cdot)}(\Omega)$ is defined by setting

$$\|f\|_{H_r^{p(\cdot)}(\Omega)} := \inf \left\{ \|F\|_{H^{p(\cdot)}(\mathbb{R}^n)} : F \in H^{p(\cdot)}(\mathbb{R}^n) \text{ and } F|_\Omega = f \right\},$$

where the infimum is taken over all $F \in H^{p(\cdot)}(\mathbb{R}^n)$ satisfying $F = f$ on Ω.

Our first main result is the grand maximal function characterizations of the variable Hardy spaces $H_z^{p(\cdot)}(\Omega)$ and $H_r^{p(\cdot)}(\Omega)$ on a strongly Lipschitz domain Ω of $\mathbb{R}^n$. To this end, we recall the following definition of grand maximal functions (see, [8]). In what follows, for any $q \in [1, \infty]$, we denote by q' its *conjugate index*, namely, $1/q + 1/q' = 1$.

For any $x \in \mathbb{R}^n$ and $\frac{n}{n+1} < p_- \le p_+ \le 1$, where p_- and p_+ are as in (1.1), denote by $F_x(\Omega)$ the collection of all $\phi \in \mathcal{D}(\mathbb{R}^n)$, for which there exists a cube Q such that $\operatorname{supp}(\phi) \subset Q$, $x \in Q$, $c_Q \in \Omega$ and

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} + \ell(Q) \|\nabla \phi\|_{L^\infty(\mathbb{R}^n)} \le |Q|^{-1},$$

here and hereafter, c_Q denotes the *center* of the cube Q and $\ell(Q)$ its *sidelength*. For each $x \in \Omega$, let

$$G_x(\Omega) := \{\phi \in F_x(\Omega) : \phi = 0 \text{ on } \partial\Omega\}.$$

For each $f \in \mathcal{D}'(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$, let

$$M_z(f)(x) := \sup_{\phi \in F_x(\Omega)} |\langle f, \phi \rangle|.$$

Let $p^* := \frac{np_-}{n-p_-}$, where p_- is as in (1.1), $q := \frac{p^*}{p^*-1}$ and $W_0^{1,q}(\Omega)$ denote the *Sobolev space with zero boundary values* on Ω. For each bounded linear functional f on $W_0^{1,q}(\Omega)$, for any $x \in \Omega$, let

$$M_r(f)(x) := \sup_{\phi \in G_x(\Omega)} |\langle f, \phi \mathbf{1}_\Omega \rangle|,$$

where $\mathbf{1}_\Omega$ denotes the *characteristic function* of Ω. From the fact that $\phi \in G_x(\Omega)$, it follows that $\phi \mathbf{1}_\Omega \in W_0^{1,q}(\Omega)$, which implies that $M_r(f)$ is well defined.

For any $x \in \mathbb{R}^n$, $\frac{n}{n+1} < p_- \le p_+ \le 1$, where p_- and p_+ are as in (1.1), and $\frac{p^*}{p^*-1} < q < \infty$, we denote by $F_x^q(\Omega)$ the collection of all $\phi \in \mathcal{D}(\mathbb{R}^n)$, which satisfy $\operatorname{supp}(\phi) \subset Q$, $x \in Q$, $c_Q \in \Omega$ and

$$\|\phi\|_{L^q(\mathbb{R}^n)} + \ell(Q)\|\nabla\phi\|_{L^q(\mathbb{R}^n)} \le |Q|^{-1/q'}.$$

Similarly, for each $x \in \Omega$, we let

$$G_x^q(\Omega) := \{\phi \in F_x^q(\Omega) : \phi = 0 \text{ on } \partial\Omega\}.$$

We then let, for $f \in \mathcal{D}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_z^{(q)}(f)(x) := \sup_{\phi \in F_x^q(\Omega)} |\langle f,\, \phi\rangle|.$$

For each bounded linear functional f on $W_0^{1,\,q}(\Omega)$ and for any $x \in \Omega$, let

$$M_r^{(q)}(f)(x) := \sup_{\phi \in G_x^q(\Omega)} |\langle f,\, \phi\mathbf{1}_\Omega\rangle|.$$

From the fact that $\phi \in G_x^q(\Omega)$, it follows that $\phi\mathbf{1}_\Omega \in W_0^{1,\,q}(\Omega)$, which implies that $M_r^{(q)}(f)$ is well defined.

A domain Ω of $\mathbb{R}^n$ $(n \ge 2)$ is said to be *strongly Lipschitz* if it is a Lipschitz domain and its boundary $\partial\Omega$ is a finite union of parts of rotated graphs of Lipschitz maps and, at most one of these parts possibly unbounded. Moreover, a domain $\Omega \subset \mathbb{R}^n$ is said to be a *special Lipschitz domain*, i.e., $\Omega := \{(x',\, x_n) : x_n > \lambda(x')\}$. Here $\lambda : \mathbb{R}^{n-1} \to \mathbb{R}$ is a function which satisfies the Lipschitz condition $|\lambda(x') - \lambda(y')| \le A\,|x' - y'|$ for all $x', y' \in \mathbb{R}^{n-1}$. It is well known that strongly Lipschitz domains include special Lipschitz domains, bounded Lipschitz domains and exterior domains (see, for example, [1, 2, 7, 11]). Then we have the following grand maximal function characterizations of $H_z^{p(\cdot)}(\Omega)$ and $H_r^{p(\cdot)}(\Omega)$ on a strongly Lipschitz domain Ω of $\mathbb{R}^n$.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a strongly Lipschitz domain, $p(\cdot) \in C^{\log}(\Omega)$ and $\frac{n}{n+1} < p_- \le p_+ \le 1$, where p_- and p_+ are as in* (1.1).

(i) *If Ω is bounded, then $f \in H_z^{p(\cdot)}(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp}(f) \subset \overline{\Omega}$, $M_z(f) \in L^{p(\cdot)}(\Omega)$ and $\langle f,\, \phi\rangle = 0$ for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω. Moreover, there exists a positive constant C independent of f, such that*

$$C^{-1}\,\|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \le \|f\|_{H_z^{p(\cdot)}(\Omega)} \le C\,\|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

(ii) *If Ω is unbounded, then $f \in H_z^{p(\cdot)}(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp}(f) \subset \overline{\Omega}$ and $M_z(f) \in L^{p(\cdot)}(\Omega)$. Moreover, there exists a positive constant C independent of f, such that*

$$C^{-1}\,\|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \le \|f\|_{H_z^{p(\cdot)}(\Omega)} \le C\,\|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

(iii) *Assume that* $\frac{p^*}{p^*-1} < q \le \infty$. *If* Ω *is bounded, then* $f \in H_z^{p(\cdot)}(\Omega)$ *if and only if* $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp}(f) \subset \overline{\Omega}$, $M_z^{(q)}(f) \in L^{p(\cdot)}(\Omega)$ *and* $\langle f, \phi\rangle = 0$ *for each* $\phi \in \mathcal{D}(\mathbb{R}^n)$ *with* $\phi \equiv 1$ *on* Ω. *Moreover, there exists a positive constant* C *independent of* f, *such that*

$$C^{-1}\left\|M_z^{(q)}(f)\right\|_{L^{p(\cdot)}(\Omega)} \le \|f\|_{H_z^{p(\cdot)}(\Omega)} \le C\left\|M_z^{(q)}(f)\right\|_{L^{p(\cdot)}(\Omega)}.$$

(iv) *Assume that* $\frac{p^*}{p^*-1} < q \le \infty$. *If* Ω *is unbounded, then* $f \in H_z^{p(\cdot)}(\Omega)$ *if and only if* $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp}(f) \subset \overline{\Omega}$ *and* $M_z^{(q)}(f) \in L^{p(\cdot)}(\Omega)$. *Moreover, there exists a positive constant* C *independent of* f, *such that*

$$C^{-1}\left\|M_z^{(q)}(f)\right\|_{L^{p(\cdot)}(\Omega)} \le \|f\|_{H_z^{p(\cdot)}(\Omega)} \le C\left\|M_z^{(q)}(f)\right\|_{L^{p(\cdot)}(\Omega)}.$$

Theorem 1.4. *Let* $\Omega \subset \mathbb{R}^n$ *be a strongly Lipschitz domain,* $p(\cdot) \in C^{\log}(\Omega)$ *and* $\frac{n}{n+1} < p_- \le p_+ \le 1$, *where* p_- *and* p_+ *are as in* (1.1).

(i) *Assume that* $q = (p^*)'$ *and* Ω *is bounded. Then* $f \in H_r^{p(\cdot)}(\Omega)$ *if and only if* f *is a bounded linear functional on* $W_0^{1,q}(\Omega)$ *and* $M_r(f) \in L^{p(\cdot)}(\Omega)$. *Moreover, there exists a positive constant* C *independent of* f, *such that*

$$C^{-1}\|M_r(f)\|_{L^{p(\cdot)}(\Omega)} \le \|f\|_{H_r^{p(\cdot)}(\Omega)} \le C\|M_r(f)\|_{L^{p(\cdot)}(\Omega)}.$$

(ii) *Assume that* $\frac{p^*}{p^*-1} < q \le \infty$. *Then* $f \in H_r^{p(\cdot)}(\Omega)$ *if and only if* f *is a bounded linear functional on* $W_0^{1,q}(\Omega)$ *and* $M_r^{(q)}(f) \in L^{p(\cdot)}(\Omega)$. *Moreover, there exists a positive constant* C *independent of* f, *such that*

$$C^{-1}\left\|M_r^{(q)}(f)\right\|_{L^{p(\cdot)}(\Omega)} \le \|f\|_{H_r^{p(\cdot)}(\Omega)} \le C\left\|M_r^{(q)}(f)\right\|_{L^{p(\cdot)}(\Omega)}.$$

Remark 1.5. (i) Let $p \in (\frac{n}{n+1}, 1]$ be a given constant. We point out that, if $p(\cdot) := p$, then Theorems 1.3 and 1.4 were established by Chen et al. in [8].

(ii) It is worth pointing out that in the process of the proofs of (i) and (ii) of Theorem 1.3 are composed by three steps: we first deal with the case Ω being special Lipschitz, and then the case Ω being bounded, finally Ω being unbounded (see Section 2 below). Based on Theorems 1.3 and 1.4, [8, Corollaries 2.13 and 2.14] are also true under the setting of variable exponent function spaces $H_r^{p(\cdot)}(\Omega)$ and $H_z^{p(\cdot)}(\Omega)$.

Our second main result concerning the atomic characterization of the variable local Hardy spaces $h_r^{p(\cdot)}(\Omega)$. We first introduce the notions of $h^{p(\cdot)}(\mathbb{R}^n)$ and $h_r^{p(\cdot)}(\Omega)$ as follows. The following definition was introduced by Tan in [17, Theorem 1.3].

Definition 1.6. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Denote by $M_{\rm loc}(f)$ the grand maximal function given by

$$M_{\rm loc}(f)(x) := \sup\{|\phi_t * f(x)| : 0 < t < 1,\, \phi \in \mathcal{F}_N(\mathbb{R}^n)\}$$

for any fixed large integer N, where $\mathcal{F}_N(\mathbb{R}^n)$ is as in (1.2). A distribution f on $\mathbb{R}^n$ is in the variable local Hardy spaces $h^{p(\cdot)}(\mathbb{R}^n)$ if and only if the grand maximal function $M_{\rm loc}(f)$ lies in $L^{p(\cdot)}(\mathbb{R}^n)$, i.e., for all $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\|f\|_{h^{p(\cdot)}(\mathbb{R}^n)} \sim \|M_{\rm loc}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Definition 1.7. Let $\Omega \subset \mathbb{R}^n$ be a proper open subset and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A distribution f on Ω is said to belong to the *variable local Hardy space* $h_r^{p(\cdot)}(\Omega)$ if f is the restriction to Ω of a distribution $F \in h^{p(\cdot)}(\mathbb{R}^n)$, namely,

$$\begin{aligned} h_r^{p(\cdot)}(\Omega) :&= \left\{f \in \mathcal{D}'(\Omega) : \text{there exists an } F \in h^{p(\cdot)}(\mathbb{R}^n) \text{ such that } F|_\Omega = f\right\} \\ &= h^{p(\cdot)}(\mathbb{R}^n) / \left\{F \in h^{p(\cdot)}(\mathbb{R}^n) : F = 0 \text{ on } \Omega\right\}. \end{aligned}$$

Moreover, for any $f \in h_r^{p(\cdot)}(\Omega)$, the *quasi-norm* $\|f\|_{h_r^{p(\cdot)}(\Omega)}$ of f in $h_r^{p(\cdot)}(\Omega)$ is defined by setting

$$\|f\|_{h_r^{p(\cdot)}(\Omega)} := \inf\left\{\|F\|_{h^{p(\cdot)}(\mathbb{R}^n)} : F \in h^{p(\cdot)}(\mathbb{R}^n) \text{ and } F|_\Omega = f\right\},$$

where the infimum is taken over all $F \in h^{p(\cdot)}(\mathbb{R}^n)$ satisfying $F = f$ on Ω.

In what follows, to establish the atomic characterization of the variable local Hardy space $h_r^{p(\cdot)}(\Omega)$, we introduce the notion of $(p(\cdot), q)_\Omega$-atoms.

Definition 1.8. Let Ω be an open subset of $\mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$ and $q \in (1, \infty]$. Assume that p_- and $\underline{p}$ are as in (1.1).

(i) A cube $Q \subset \mathbb{R}^n$ is said to be of *type* (a) *cube* if $4Q \subset \Omega$ with $\ell(Q) < 1$; a cube $\widetilde{Q} \subset \mathbb{R}^n$ is said to be of *type* (b) *cube* if either $\ell(Q) \geq 1$ or $2\widetilde{Q} \cap \Omega^{\complement} = \emptyset$ and $4\widetilde{Q} \cap \Omega^{\complement} \neq \emptyset$.

(ii) A measurable function a on Ω is called a *type* (a) $(p(\cdot), q)_\Omega$*-atom* if

(ii)$_1$ ${\rm supp}\,(a) \subset Q$, where ${\rm supp}\,(a) := \overline{\{x \in \mathbb{R}^n : a(x) \neq 0\}}$ and Q is a type (a) cube;

(ii)$_2$ $\|a\|_{L^q(\Omega)} \leq |Q|^{1/q}\|\mathbf{1}_Q\|^{-1}_{L^{p(\cdot)}(\Omega)}$;

(ii)$_3$ there exists an integer $s \geq d_{p(\cdot)}$, where $d \geq d_{p(\cdot)} := \min\{d \in \mathbb{Z}_+ : p_-(n + d + 1) > n\}$, such that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$, $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$.

Moreover, a measurable function b on Ω is called a *type* (b) $(p(\cdot), q)_\Omega$*-atom* if ${\rm supp}\,(b) \subset \widetilde{Q}$ with $\widetilde{Q}$ being a type (b) cube and $\|b\|_{L^q(\Omega)} \leq |\widetilde{Q}|^{1/q}\|\mathbf{1}_{\widetilde{Q}}\|^{-1}_{L^{p(\cdot)}(\Omega)}$. Furthermore, a measurable function a on $\mathbb{R}^n$ is called a $(p(\cdot), q)$*-atom*, if it satisfies the conditions (ii)$_2$, (ii)$_3$ above and ${\rm supp}\,(a) \subset Q \subset \mathbb{R}^n$.

For a sequence $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and a cubes sequence $\{Q_j\}_{j=1}^\infty$ of the supports of atoms, define that

$$\mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\right) := \left\| \left\{ \sum_{j=1}^\infty \left(\frac{|\lambda_j| \mathbf{1}_{Q_j}}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\Omega)}} \right)^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\Omega)}.$$

Then we have the atomic characterization of the variable local Hardy space $h_r^{p(\cdot)}(\Omega)$ as follows.

Theorem 1.9. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Let $p(\cdot) \in C^{\log}(\Omega)$ with $\frac{n}{n+1} < p_- \le p_+ \le 1$ and $q \in (1, \infty]$, where p_- and p_+ are as in* (1.1)*. Then, $f \in h_r^{p(\cdot)}(\Omega)$ if and only if there exist sequences $\{\lambda_j\}_{j=1}^\infty$, $\{\kappa_j\}_{j=1}^\infty \subset \mathbb{C}$, type* (a) *$(p(\cdot), q)_\Omega$-atoms $\{a_j\}_{j=1}^\infty$, and type* (b) *$(p(\cdot), q)_\Omega$-atoms $\{b_j\}_{j=1}^\infty$ such that $f = \sum_{j=1}^\infty \lambda_j a_j + \sum_{j=1}^\infty \kappa_j b_j$ in $\mathcal{D}'(\mathbb{R}^n)$, and*

$$\mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\right) + \mathcal{A}\left(\{\kappa_j\}_{j=1}^\infty, \{\widetilde{Q}_j\}_{j=1}^\infty\right) < \infty,$$

where $\{Q_j\}_{j=1}^\infty$ and $\{\widetilde{Q}_j\}_{j=1}^\infty$, respectively, denote the supports of $\{a_j\}_{j=1}^\infty$ and $\{b_j\}_{j=1}^\infty$. Moreover, for any given $f \in h_r^{p(\cdot)}(\Omega)$, there exists a positive constant C independent of f, such that

$$C^{-1}\|f\|_{h_r^{p(\cdot)}(\Omega)} \le \mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\right) + \mathcal{A}\left(\{\kappa_j\}_{j=1}^\infty, \{\widetilde{Q}_j\}_{j=1}^\infty\right) \le C\|f\|_{h_r^{p(\cdot)}(\Omega)}.$$

Remark 1.10. When $p(\cdot) := p \in (\frac{n}{n+1}, 1]$, then Theorem 1.9 is reduced to [7, Theorem 2.7].

The layout of this article is as follows. Section 2 is devoted to the proofs of Theorems 1.3 and 1.4. In Section 3, we give the proof of Theorem 1.9.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Throughout the whole article, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, then we write $f \sim g$. Denote by $Q(c_Q, \ell(Q))$ the cube in $\mathbb{R}^n$ with *center* $c_Q \in \mathbb{R}^n$ and *sidelength* $\ell(Q) \in (0, \infty)$, and $\alpha \in (0, \infty)$, let $\alpha Q := Q(c_Q, \alpha\ell(Q))$. For any measurable subset E of $\mathbb{R}^n$, we denote the *set* $\mathbb{R}^n \backslash E$ by $E^{\complement}$ and its *characteristic function* by $\mathbf{1}_E$. Moreover, denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and $\mathcal{S}'(\mathbb{R}^n)$ its *dual space* (namely, the *space of all tempered distributions*). For any sets E, $F \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$, let $\operatorname{dist}(E, F) := \inf\{|x - y| : x \in E, y \in F\}$ and $\operatorname{dist}(z, E) := \inf\{|z - x| : x \in E\}$. Finally, we also use $W_0^{1,q}(\Omega)$ to denote the collection of elements in Sobolev spaces $W^{1,q}(\Omega)$ with zero boundary values. For $q \in (0, n)$, let q^* be its *Sobolev conjugate index* $\frac{nq}{n-q}$. For any given $p \in [1, \infty]$, we denote by p' its *conjugate exponent*, namely, $1/p + 1/p' = 1$.

2. Proofs of Theorems 1.3 and 1.4

In this section, we give the proofs of Theorems 1.3 and 1.4. We begin with recall some basic properties of variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$. The following Lemmas 2.1 and 2.2 come from [15, Lemma 2.2] and [9, 18], respectively.

Lemma 2.1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $0 < p_- \le p_+ < \infty$, where p_- and p_+ are as in* (1.1).

(i) *For all cubes $Q = Q(c_Q, \ell(Q))$ with $c_Q \in \mathbb{R}^n$ and $\ell(Q) \le 1$, we have $|Q|^{1/p_-(Q)} \lesssim |Q|^{1/p_+(Q)}$. In particular, we have*

$$|Q|^{1/p_-(Q)} \sim |Q|^{1/p_+(Q)} \sim |Q|^{1/p(c_Q)} \sim \|\mathbf{1}_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)};$$

(ii) *For all cubes $Q = Q(c_Q, \ell(Q))$ with $c_Q \in \mathbb{R}^n$ and $\ell(Q) \ge 1$, we have $\|\mathbf{1}_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim |Q|^{1/p_\infty}$.*

Here the implicit constants in $\sim$ do not depend on c_Q and $\ell(Q) > 0$.

Lemma 2.2. *Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then, for any $s \in (0, \infty)$, $\lambda \in \mathbb{C}$, and $f \in L^{p(\cdot)}(\Omega)$,*

$$\||f|^s\|_{L^{p(\cdot)}(\Omega)} = \|f\|^s_{L^{sp(\cdot)}(\Omega)} \quad \textit{and} \quad \|\lambda f\|_{L^{p(\cdot)}(\Omega)} = |\lambda|\|f\|_{L^{p(\cdot)}(\Omega)}.$$

Recall that, for any $f \in L^1_{\mathrm{loc}}(\Omega)$ and $x \in \Omega$, the *Hardy–Littlewood maximal function* $M(f)$ is defined by setting

$$M(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls $B \subset \Omega$ satisfying $B \ni x$.

Lemma 2.3. *Let Ω be an open subset of $\mathbb{R}^n$ and $r \in (1, \infty]$. Assume that $p(\cdot) \in C^{\log}(\Omega)$ satisfies $1 < p_- \le p_+ < \infty$, where p_- and p_+ are as in* (1.1). *Then there exists a positive constant C such that, for any sequence $\{f_k\}_{k\in\mathbb{N}}$ of measurable functions on Ω,*

$$\left\| \left\{ \sum_{k\in\mathbb{N}} [M(f_k)]^r \right\}^{1/r} \right\|_{L^{p(\cdot)}(\Omega)} \le C \left\| \left(\sum_{k\in\mathbb{N}} |f_k|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\Omega)}$$

with the usual modification made when $r = \infty$.

We point out that in the case of metric measurable spaces of homogeneous type, Lemma 2.3 was established in [20, Theorem 2.7]. Moreover, the proof of [20, Theorem 2.7] is also valid in the case of Lemma 2.3 and we omit the details here. The following remark is just [21, Remark 2.8].

Remark 2.4. Let $k \in \mathbb{N}$ and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then, from Lemmas 2.2 and 2.3, and the fact that, for any cubes Q of $\mathbb{R}^n$, $r \in (0, \underline{p})$, $\mathbf{1}_{2^kQ} \lesssim 2^{kn/r}[M(\mathbf{1}_Q)]^{1/r}$, where $\underline{p}$ is as in (1.1) and M denotes the Hardy–Littlewood maximal function, we deduce that there exists a positive constant C such that, for any $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ and cubes $\{Q_j\}_{j=1}^{\infty}$ of $\mathbb{R}^n$,

$$\left\| \left\{ \sum_{j=1}^{\infty} \left[\frac{|\lambda_j| \mathbf{1}_{2^k Q_j}}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C 2^{kn/r} \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}\right).$$

From the proof of [12, Theorem 1.12(i)] or see the proof of [8, lemma 2.3] with regular modification, it follows that the variable Hardy space $H_r^{p(\cdot)}(\Omega)$ admits the following atomic decomposition.

Lemma 2.5. *Let $p(\cdot) \in C^{\log}(\Omega)$, $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ and $q \in (1, \infty]$, where p_- and p_+ are as in* (1.1). *Assume that $\Omega \subset \mathbb{R}^n$ is a strongly Lipschitz domain. Then, for each $f \in H_r^{p(\cdot)}(\Omega)$, there exist type (a) $(p(\cdot), q)_\Omega$-atoms $\{a_j\}_{j=1}^{\infty}$ and type (b) $(p(\cdot), q)_\Omega$-atoms $\{b_j\}_{j=1}^{\infty}$ such that $f = \sum_{j=1}^{\infty} \lambda_j a_j + \sum_{j=1}^{\infty} \kappa_j b_j$ in $\mathcal{D}'(\mathbb{R}^n)$, and*

$$\mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}\right) + \mathcal{A}\left(\{\kappa_j\}_{j=1}^{\infty}, \{\widetilde{Q}_j\}_{j=1}^{\infty}\right) \leq C\|f\|_{H_r^{p(\cdot)}(\Omega)},$$

where $\{Q_j\}_{j=1}^{\infty}$ and $\{\widetilde{Q}_j\}_{j=1}^{\infty}$, respectively, denote the supports of $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$, and the positive constant C is independent of f.

Lemma 2.6. *Let Ω be a strongly Lipschitz domain, $p(\cdot) \in C^{\log}(\Omega)$, $\frac{n}{n+1} < p_- \leq p_+ \leq 1$ and $q = \frac{p^*}{p^*-1}$, where p_- and p_+ are as in* (1.1). *Then each $f \in H_r^{p(\cdot)}(\Omega)$ induces a bounded linear functional on $W_0^{1,q}(\Omega)$ and there exists a positive constant C such that, for all $f \in H_r^{p(\cdot)}(\Omega)$ and $g \in W_0^{1,q}(\Omega)$,*

$$|\langle f, g\rangle| \leq C\|f\|_{H_r^{p(\cdot)}(\Omega)}\|g\|_{W_0^{1,q}(\Omega)}.$$

Proof. Since $\mathcal{D}(\Omega)$ is dense in $W_0^{1,q}(\Omega)$, we only need to show that Lemma 2.6 holds true for each $\phi \in \mathcal{D}(\Omega)$.

If $p_- = p_+ = 1$, then $q = (p^*)' = n$. For each $f \in H_r^{p(\cdot)}(\Omega)$, there exists $F \in H^1(\mathbb{R}^n)$ such that $F|_\Omega = f$ and $\|F\|_{H^1(\mathbb{R}^n)} \leq 2\|f\|_{H_r^1(\Omega)}$. By the duality of $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, the embedding of $W^{1,n}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ and similar to that of (2.2) in the proof of [8, Lemma 2.4], we know that

$$\left|\int_\Omega f(x)\phi(x)dx\right| \lesssim \|f\|_{H_r^1(\Omega)}\|\phi\|_{W^{1,n}(\Omega)}. \tag{2.1}$$

If $\frac{n}{n+1} < p_- \leq p_+ < 1$, then $q = (p^*)' > n$. For each $f \in H_r^{p(\cdot)}(\Omega)$, from Lemma 2.5, we deduce that there exist two sequences $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ and $\{\kappa_j\}_{j=1}^{\infty} \subset \mathbb{C}$, a sequence $\{a_j\}_{j=1}^{\infty}$ of type (a) $(p(\cdot), p^*)_\Omega$-atoms and a sequence $\{b_j\}_{j=1}^{\infty}$ of type (b) $(p(\cdot), p^*)_\Omega$-atoms such that $f = \sum_{j=1}^{\infty} \lambda_j a_j + \sum_{j=1}^{\infty} \kappa_j b_j$ in

$\mathcal{D}'(\mathbb{R}^n)$. Moreover, by [20, Lemma 5.9], we know that

$$\sum_{j=1}^{\infty}|\lambda_j|+\sum_{j=1}^{\infty}|\kappa_j|\lesssim \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty},\{Q_j\}_{j=1}^{\infty}\right)+\mathcal{A}\left(\{\kappa_j\}_{j=1}^{\infty},\{\widetilde{Q}_j\}_{j=1}^{\infty}\right). \tag{2.2}$$

Then for each type (a) $(p(\cdot),p^*)_\Omega$-atom a_j, from the moment condition of a_j, $\operatorname{supp}(a_j)\subset Q_j$, $\|a_j\|_{L^{p^*}(\Omega)}\le |Q_j|^{1/p^*}\|\mathbf{1}_{Q_j}\|^{-1}_{L^{p(\cdot)}(\Omega)}$, Lemma 2.1(i), $p^*=\frac{np_-}{n-p_-}$, and repeating the proof of [8, (2.3)], we deduce that

$$\begin{aligned}\left|\int_\Omega a_j(x)\phi(x)dx\right| &\lesssim \ell(Q_j)\int_0^1 \|a_j\|_{L^{p^*}(\Omega)}\left[\int_\Omega|\nabla\phi(x)|^q\,dx\right]^{1/q}t^{-n/q}dt\\ &\lesssim \ell(Q_j)\,|Q_j|^{1/p^*}\left\|\mathbf{1}_{Q_j}\right\|^{-1}_{L^{p(\cdot)}(\Omega)}\int_0^1 t^{-n/q}dt\,\|\nabla\phi\|_{L^q(\Omega)}\\ &\lesssim [\ell(Q_j)]^{1+n/p^*}\,|Q_j|^{-1/p_-}\,\|\phi\|_{W^{1,q}(\Omega)}\sim\|\phi\|_{W^{1,q}(\Omega)}.\end{aligned} \tag{2.3}$$

For each type (b) $(p(\cdot),p^*)_\Omega$-atom b_j, by $\operatorname{supp}(b_j)\subset\widetilde{Q}_j$,

$$\|b_j\|_{L^{p^*}(\Omega)}\le|\widetilde{Q}_j|^{1/p^*}\|\mathbf{1}_{\widetilde{Q}_j}\|^{-1}_{L^{p(\cdot)}(\Omega)},$$

Lemma 2.1(ii), $p^*=\frac{np_-}{n-p_-}$, and a proof similar to those of [8, (2.4)], we know that

$$\begin{aligned}\left|\int_\Omega b_j(x)\phi(x)dx\right| &\lesssim \ell(\widetilde{Q}_j)\int_0^1\|b_j\|_{L^{p^*}(\Omega)}\,\|\nabla\phi\|_{L^q(\Omega)}t^{-n/q}dt\\ &\lesssim \ell(\widetilde{Q}_j)\left|\widetilde{Q}_j\right|^{1/p^*}\left\|\mathbf{1}_{\widetilde{Q}_j}\right\|^{-1}_{L^{p(\cdot)}(\Omega)}\int_0^1 t^{-n/q}dt\,\|\nabla\phi\|_{L^q(\Omega)}\\ &\lesssim \left[\ell(\widetilde{Q}_j)\right]^{1+n/p^*}|Q_j|^{-1/p_\infty}\,\|\phi\|_{W^{1,q}(\Omega)}\\ &\lesssim \left[\ell(\widetilde{Q}_j)\right]^{n/p_--n/p_\infty}\|\phi\|_{W^{1,q}(\Omega)}\lesssim\|\phi\|_{W^{1,q}(\Omega)}.\end{aligned} \tag{2.4}$$

Combining (2.2), (2.3), (2.4), and Lemma 2.5, we obtain

$$\begin{aligned}|\langle f,\,\phi\rangle| &\le\sum_{j=1}^{\infty}|\langle\lambda_j a_j,\,\phi\rangle|+\sum_{j=1}^{\infty}|\langle\kappa_j b_j,\,\phi\rangle|\\ &\lesssim\left(\sum_{j=1}^{\infty}|\lambda_j|+\sum_{j=1}^{\infty}|\kappa_j|\right)\|\phi\|_{W^{1,q}(\Omega)}\lesssim\|f\|_{H_r^{p(\cdot)}(\Omega)}\|\phi\|_{W^{1,q}(\Omega)},\end{aligned}$$

which, together with (2.1) and the density of $\mathcal{D}(\Omega)$ in $W_0^{1,q}(\Omega)$, finishes the proof of Lemma 2.6. □

We first prove a weaker version of Theorem 1.3(i) as follows.

Lemma 2.7. *Let Ω be a strongly Lipschitz domain, $p(\cdot)\in C^{\log}(\Omega)$ and $\frac{n}{n+1}<p_-\le p_+\le 1$, where p_- and p_+ are as in* (1.1). *Then $f\in H_z^{p(\cdot)}(\Omega)$ if and only*

if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp}(f) \subset \overline{\Omega}$ *and* $M_z(f) \in L^{p(\cdot)}(\mathbb{R}^n)$*. Moreover, there exists a positive constant* C *independent of* f*, such that*

$$C^{-1} \|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{H_z^{p(\cdot)}(\Omega)} \le C \|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp}(f) \subset \overline{\Omega}$ and $M_z(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. Suppose that $\psi \in \mathcal{D}(\mathbb{R}^n)$ is a radial function such that $\operatorname{supp}(\psi) \subset Q := Q(\vec{0}_n, 1)$ and $\int_{\mathbb{R}^n} \psi(x)dx = 1$, where and in what follows, $\vec{0}_n$ denotes the origin of $\mathbb{R}^n$. Proceeding as in the proof of [8, Proposition 2.6], we know that

$$\sup_{t\in(0,\infty)} |f * \psi_t(x)| \lesssim M_z(f)(x),$$

which, together with (1.3) and Definition 1.1, implies that $f \in H^{p(\cdot)}(\mathbb{R}^n)$. Notice that $\operatorname{supp}(f) \subset \overline{\Omega}$, this implies that $f \in H_z^{p(\cdot)}(\Omega)$ and $\|f\|_{H_z^{p(\cdot)}(\Omega)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

On the contrary, let $f \in H_z^{p(\cdot)}(\Omega)$. By Definition 1.2 and [15, Theorem 4.6 and Definition 1.5], we conclude that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ is an atomic decomposition in $H^{p(\cdot)}(\mathbb{R}^n)$, where $\{a_j\}_{j=1}^{\infty}$ are $(p(\cdot), \infty)$-atoms and $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ satisfy

$$\|f\|_{H_z^{p(\cdot)}(\Omega)} \sim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}\right).$$

Thus, we have

$$\begin{aligned} &\|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le \left\|\sum_{j=1}^{\infty} |\lambda_j| M_z(a_j) \mathbf{1}_{4Q_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\|\sum_{j=1}^{\infty} |\lambda_j| M_z(a_j) \mathbf{1}_{(4Q_j)^{\complement}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &=: \mathrm{I} + \mathrm{II}. \end{aligned} \tag{2.5}$$

For the term I, by $x \in 4Q_j$ and a $(p(\cdot), \infty)$-atom a_j with $\operatorname{supp}(a_j) \subset Q_j$, we have

$$\begin{aligned} M_z(a_j)(x) &= \sup_{\phi\in F_x(\Omega)} \left|\int_{\mathbb{R}^n} a_j(y)\phi(y)dy\right| \le \sup_{\phi\in F_x(\Omega)} \|\phi\|_{L^\infty(\mathbb{R}^n)} \int_{\Omega\cap Q} |a_j(y)|\,dy \\ &\le \sup_{\phi\in F_x(\Omega)} \frac{1}{|Q|} \int_{\Omega\cap Q} |a_j(y)|\,dy \le \|a_j\|_{L^\infty(\mathbb{R}^n)} \le \left\|\mathbf{1}_{Q_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}, \end{aligned}$$

which, together with Remark 2.4, implies that

$$\begin{aligned} \mathrm{I} &\lesssim \left\|\left\{\sum_{j=1}^{\infty} \left[\frac{|\lambda_j| \mathbf{1}_{4Q_j}}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}\right) \sim \|f\|_{H_z^{p(\cdot)}(\Omega)}. \end{aligned} \tag{2.6}$$

For II, from $x \notin 4Q_j$, $\phi \in F_x(\Omega)$ with $\operatorname{supp}(\phi) \subset Q$, and the fact that $\int_{\mathbb{R}^n} a_j(y)dy = 0$, we deduce that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a_j(y)\phi(y)dy\right| &= \left|\int_{Q_j} a_j(y)\left[\phi(y)-\phi\left(c_{Q_j}\right)\right]dy\right| \\ &\lesssim \|a_j\|_{L^\infty(\mathbb{R}^n)}\left[\ell(Q_j)\right]^{n+1}\|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \|\nabla\phi\|_{L^\infty(\mathbb{R}^n)}\left[\ell(Q_j)\right]^{n+1}\left\|\mathbf{1}_{Q_j}\right\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}. \qquad (2.7)\end{aligned}$$

If $\operatorname{supp}(a_j)\cap\operatorname{supp}(\phi)=\emptyset$, then $\int_{\mathbb{R}^n} a_j(y)\phi(y)dy = 0$. If $\operatorname{supp}(a_j)\cap\operatorname{supp}(\phi)\neq\emptyset$, then $Q_j\cap Q\neq\emptyset$. Notice that $x\notin 4Q_j$ and $x\in Q$, and hence $\frac{3}{2}\ell(Q_j)\le |x-y|<\ell(Q)$ for each $y\in Q_j\cap Q$, which implies that

$$\left|x-c_{Q_j}\right| \le |x-y| + \left|y-c_{Q_j}\right| \lesssim \ell(Q).$$

By this and (2.7), we find that

$$\left|\int_{\mathbb{R}^n} a_j(y)\phi(y)dy\right| \lesssim \frac{[\ell(Q_j)]^{n+1}}{[\ell(Q)]^{n+1}}\left\|\mathbf{1}_{Q_j}\right\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \frac{[\ell(Q_j)]^{n+1}}{|x-c_{Q_j}|^{n+1}}\left\|\mathbf{1}_{Q_j}\right\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)},$$

which implies that, for any $x\notin 4Q_j$,

$$\begin{aligned}M_z(a_j)(x) &\lesssim \frac{[\ell(Q_j)]^{n+1}}{|x-c_{Q_j}|^{n+1}}\left\|\mathbf{1}_{Q_j}\right\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \left[M\left(\mathbf{1}_{Q_j}\right)(x)\right]^{(n+1)/n}\left\|\mathbf{1}_{Q_j}\right\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

From this, $\theta := (n+1)/n$, Lemmas 2.2 and 2.3, and Remark 2.4, we deduce that

$$\begin{aligned}\mathrm{II} &\lesssim \left\|\sum_{j=1}^\infty \frac{|\lambda_j|}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\left[M\left(\mathbf{1}_{Q_j}\right)\right]^\theta\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\sim \left\|\left\{\sum_{j=1}^\infty \frac{|\lambda_j|}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\left[M\left(\mathbf{1}_{Q_j}\right)\right]^\theta\right\}^{1/\theta}\right\|^\theta_{L^{\theta p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \left\|\left[\sum_{j=1}^\infty \frac{|\lambda_j|}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\left(\mathbf{1}_{Q_j}\right)^\theta\right]^{1/\theta}\right\|^\theta_{L^{\theta p(\cdot)}(\mathbb{R}^n)} \\ &\sim \left\|\sum_{j=1}^\infty \frac{|\lambda_j|\mathbf{1}_{Q_j}}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}_z(\Omega)}. \qquad (2.8)\end{aligned}$$

Combining (2.5) (2.6), and (2.8), we obtain the desired result. This finishes the proof of Lemma 2.7. □

In consideration of Lemma 2.7, in order to prove (i) and (ii) of Theorem 1.3, it remains to prove that $\|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}$. To do this, we need the following lemma, which is a slight modification of [8, Lemma 2.7], with $L^p(\Omega)$ norm therein replaced by $L^{p(\cdot)}(\Omega)$ norm here, the details being omitted.

Lemma 2.8. *Let Ω be a special Lipschitz domain, $p(\cdot) \in C^{\log}(\Omega)$ and $\frac{n}{n+1} < p_- \le p_+ \le 1$, where p_- and p_+ are as in (1.1). Then $f \in H_z^{p(\cdot)}(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp}(f) \subset \overline{\Omega}$ and $M_z(f) \in L^{p(\cdot)}(\Omega)$. Moreover, there exists a positive constant C independent of f, such that*

$$C^{-1} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \le \|f\|_{H_z^{p(\cdot)}(\Omega)} \le C \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

Now let us deal with the general case of Ω. In what follows, for each strongly Lipschitz domain Ω and $\epsilon \in (0, \infty)$, we assume $\Omega_\epsilon := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < \epsilon\}$. The following lemma is an analogue of [8, Lemma 2.8], since the proof is regular, its proof is omitted.

Lemma 2.9. *Let Ω be a strongly Lipschitz domain, $p(\cdot) \in C^{\log}(\Omega)$ and $\frac{n}{n+1} < p_- \le p_+ \le 1$, where p_- and p_+ are as in (1.1). If $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp}(f) \subset \overline{\Omega}$ and $M_z(f) \in L^{p(\cdot)}(\Omega)$. Then there exist positive constants C and ϵ independent of f, such that*

$$\|M_z(f)\|_{L^{p(\cdot)}(\Omega_\epsilon \setminus \overline{\Omega})} \le C \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

By [9, Corollary 2.27], we have the following embedding relationship between the variable and classical Lebesgue spaces.

Lemma 2.10. *Given Ω and $p(\cdot) : \Omega \to [1, \infty)$, let $|\Omega| < \infty$. Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \|f\|_{L^{p_-}(\Omega)} \le \|f\|_{L^{p(\cdot)}(\Omega)} \le c_2 \|f\|_{L^{p_+}(\Omega)},$$

where $1 \le p_- \le p_+ < \infty$. In particular, given any Ω, if $f \in L^{p(\cdot)}(\Omega)$, then f is locally integrable.

The following Lemma 2.11 extends [8, Lemma 2.9] from constant exponent case to the variable exponent case.

Lemma 2.11. *Let Ω be a bounded Lipschitz domain, $p(\cdot) \in C^{\log}(\Omega)$ and $\frac{n}{n+1} < p_- \le p_+ \le 1$, where p_- and p_+ are as in (1.1). Then $f \in H_z^{p(\cdot)}(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\operatorname{supp}(f) \subset \overline{\Omega}$, $M_z(f) \in L^{p(\cdot)}(\Omega)$ and $\langle f, \phi \rangle = 0$ for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω. Moreover, there exists a positive constant C independent of f, such that*

$$C^{-1} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \le \|f\|_{H_z^{p(\cdot)}(\Omega)} \le C \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.$$

Proof. Let $f \in H_z^{p(\cdot)}(\Omega)$. From the fact that $f \in H^{p(\cdot)}(\mathbb{R}^n)$ with $\operatorname{supp}(f) \subset \overline{\Omega}$, $\langle f, \phi \rangle = 0$ for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω and Lemma 2.7, it follows that $\|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \lesssim \|f\|_{H_z^{p(\cdot)}(\Omega)}$.

On the contrary, by Lemma 2.7, we know that $\|f\|_{H_z^{p(\cdot)}(\Omega)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, which implies that, to prove $\|f\|_{H_z^{p(\cdot)}(\Omega)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}$, it suffices to show $\|M_z(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}$. This, further implies that, it suffices to show $\|M_z(f)\|_{L^{p(\cdot)}(\Omega^\complement)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}$.

For $x \in \Omega_\epsilon \backslash \overline{\Omega}$, from Lemma 2.9, we deduce that there exists an $\epsilon \in (0, \infty)$ such that

$$\|M_z(f)\|_{L^{p(\cdot)}(\Omega_\epsilon \backslash \overline{\Omega})} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}. \tag{2.9}$$

For $x \in 2B \backslash \overline{\Omega}_\epsilon$, we choose a ball $B := B(\vec{0}_n, r_B)$ with r_B large enough such that $\Omega_\epsilon \subset B$. For any $x \in 2B \backslash \overline{\Omega}_\epsilon$ and any $\widetilde{x} \in \Omega$, by the proof of [8, Lemma 2.9], we know that, $M_z(f)(x) \lesssim M_z(f)(\widetilde{x})$ for each $x \in 2B \backslash \overline{\Omega}_\epsilon$, and hence

$$\begin{aligned}
&\frac{1}{|2B \backslash \overline{\Omega}_\epsilon|} \int_{2B \backslash \overline{\Omega}_\epsilon} \left[\frac{|M_z(f)(x)|}{\lambda}\right]^{p(x)} dx \\
&\leq \sup_{x \in 2B \backslash \overline{\Omega}_\epsilon} \left[\frac{|M_z(f)(x)|}{\lambda}\right]^{p(x)} \lesssim \inf_{x \in \Omega} \left[\frac{|M_z(f)(x)|}{\lambda}\right]^{p(x)} \\
&\lesssim \frac{1}{|\Omega|} \int_\Omega \left[\frac{|M_z(f)(x)|}{\lambda}\right]^{p(x)} dx,
\end{aligned}$$

which further implies that, for any $x \in 2B \backslash \overline{\Omega}_\epsilon$,

$$\begin{aligned}
\|M_z(f)\|_{L^{p(\cdot)}(2B \backslash \overline{\Omega}_\epsilon)} &= \inf \left\{\lambda \in (0, \infty) : \int_{2B \backslash \overline{\Omega}_\epsilon} \left[\frac{|M_z(f)(x)|}{\lambda}\right]^{p(x)} dx \leq 1\right\} \\
&\lesssim \inf \left\{\lambda \in (0, \infty) : \int_\Omega \left[\frac{|M_z(f)(x)|}{\lambda}\right]^{p(x)} dx \leq 1\right\} \\
&\sim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.
\end{aligned} \tag{2.10}$$

For $x \in (2B)^\complement$, let $\phi \in F_x(\Omega)$ with $\operatorname{supp}(\phi) \subset Q$ and $x \in Q$. If $Q \cap \Omega = \emptyset$, then $\langle f, \phi \rangle = 0$. Otherwise, we have $|x| \lesssim \ell(Q)$ and hence $[\ell(Q)]^{-n-1} \lesssim |x|^{-n-1}$. Thus, by choosing I to be a cube centered at origin such that $2B \subset I$ and $\ell(I) \sim \operatorname{diam}(\Omega)$ (see, [8, Lemma 2.9]), where and in what follows, $\operatorname{diam}(\Omega)$ denotes the *diameter* of Ω, namely, $\operatorname{diam}(\Omega) := \sup\{|x - y| : x, y \in \Omega\}$ (see, for example, [1]), and similarly to the proof of [8, Lemma 2.9], we see that

$$\begin{aligned}
M_z(f)(x) &\lesssim \frac{|I| \operatorname{diam}(\Omega)}{|x|^{n+1}} \inf_{y \in \Omega} M_z(f)(y) \\
&\lesssim \frac{|I| \operatorname{diam}(\Omega)}{|x|^{n+1}} \frac{1}{\|\mathbf{1}_\Omega\|_{L^{p(\cdot)}(\Omega)}} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)},
\end{aligned}$$

which, together with Lemmas 2.2 and 2.10, $\frac{n}{n+1} < p_+ \leq 1$, further implies that

$$\|M_z(f)\|_{L^{p(\cdot)}((2B)^\complement)} \lesssim \frac{|I| \operatorname{diam}(\Omega)}{\|\mathbf{1}_\Omega\|_{L^{p(\cdot)}(\Omega)}} \left\|\frac{1}{|x|^{n+1}}\right\|_{L^{p(\cdot)}((2B)^\complement)} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}$$

$$
\begin{aligned}
&\lesssim [\mathrm{diam}(\Omega)]^{n+1} \|(\mathbf{1}_\Omega)^{p_-}\|_{L^{p(\cdot)/p_-}(\Omega)}^{-1/p_-} \\
&\quad \times \left\| |x|^{(-n-1)p_-} \right\|_{L^{p(\cdot)/p_-}((2B)^{\complement})}^{1/p_-} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \\
&\lesssim \|(\mathbf{1}_\Omega)^{p_-}\|_{L^{p_+/p_-}(\Omega)}^{-1/p_-} \left\| |x|^{(-n-1)p_-} \right\|_{L^{p_+/p_-}((2B)^{\complement})}^{1/p_-} \\
&\quad \times \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \\
&\lesssim \left[\int_{2r_B}^{\infty} \int_{|x|=\lambda} \left[|x|^{(-n-1)} \right]^{p_+} \lambda^{n-1}\, dx d\lambda \right]^{1/p_+} \\
&\quad \times \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \\
&\lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}. \qquad (2.11)
\end{aligned}
$$

Combining the estimates of (2.9), (2.10) and (2.11), we obtain the desired inequality. This finishes the proof of Lemma 2.11. □

To prove Theorem 1.3(ii), we need the following Lemma 2.12, its proof is a repetition of the argument in [8, Lemma 2.10] except that [8, Lemma 2.7] is modified to be Lemma 2.8 and we omit the details here.

Lemma 2.12. *Let Ω be an unbounded strongly Lipschitz domain, $p(\cdot)\in C^{\log}(\Omega)$ and $\frac{n}{n+1} < p_- \le p_+ \le 1$, where p_- and p_+ are as in* (1.1)*. Then $f \in H_z^{p(\cdot)}(\Omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$, $\mathrm{supp}\,(f) \subset \overline{\Omega}$ and $M_z(f) \in L^{p(\cdot)}(\Omega)$. Moreover, there exists a positive constant C independent of f, such that*

$$
C^{-1} \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \le \|f\|_{H_z^{p(\cdot)}(\Omega)} \le C \|M_z(f)\|_{L^{p(\cdot)}(\Omega)}.
$$

Proofs of (i) *and* (ii) *of Theorem 1.3.* By Lemmas 2.11 and 2.12, we obtain that the desired of (i) and (ii) of Theorem 1.3, respectively. This finishes the proofs of (i) and (ii) of Theorem 1.3. □

Proofs of (iii) *and* (iv) *of Theorem 1.3.* We only prove for Theorem 1.3(iii), since the proof of Theorem 1.3(iv) is analogous to that of Theorem 1.3(iii) and we omit the details here. From the fact that $F_x(\Omega) \subset F_x^q(\Omega)$, it follows that $M_z(f)(x) \le M_z^{(q)}(f)(x)$ for all $x \in \mathbb{R}^n$, which, combined with Theorem 1.3(i), further implies that

$$
\|f\|_{H_z^{p(\cdot)}(\Omega)} \lesssim \|M_z(f)\|_{L^{p(\cdot)}(\Omega)} \lesssim \left\| M_z^{(q)}(f) \right\|_{L^{p(\cdot)}(\Omega)}.
$$

On the contrary, let Ω be bounded and $f \in H_z^{p(\cdot)}(\Omega)$. Then $f \in H^{p(\cdot)}(\mathbb{R}^n)$ with $\mathrm{supp}\,(f) \subset \overline{\Omega}$, for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \equiv 1$ on Ω, it follows that $\langle f, \phi\rangle = 0$. Suppose that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ is an atomic decomposition in $H^{p(\cdot)}(\mathbb{R}^n)$, where $\{a_j\}_{j=1}^{\infty}$ are $(p(\cdot),\infty)$-atoms and $\{\lambda_j\}_{j=1}^{\infty} \subset \mathbb{C}$ satisfy

$$
\|f\|_{H_z^{p(\cdot)}(\Omega)} \sim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \mathcal{A}\left(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}\right).
$$

Now let us to prove that $M_z^{(q)}(f) \in L^{p(\cdot)}(\Omega)$. For $x \in 4Q_j$ and a $(p(\cdot),\infty)$-atom a_j with $\operatorname{supp}(a_j) \subset Q_j$, by the proof of [8, (2.18)] and $\|a\|_{L^\infty(\mathbb{R}^n)} \leq \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}$, we find that, for any $x \in 4Q_j$,

$$M_z^{(q)}(a_j)(x) \leq \|a_j\|_{L^\infty(\mathbb{R}^n)} \leq \left\|\mathbf{1}_{Q_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}. \tag{2.12}$$

For $x \notin 4Q_j$ and $\phi \in F_x(\Omega)$ with $\operatorname{supp}(\phi) \subset Q$, from the fact that $\int_{\mathbb{R}^n} a_j(y)dy = 0$, $\|a\|_{L^\infty(\mathbb{R}^n)} \leq \|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}$, and similar to that of [8, (2.20)], it follows that, for any $x \notin 4Q_j$,

$$\begin{aligned} M_z^{(q)}(a_j)(x) &\lesssim \left[\frac{\ell(Q_j)}{|x - c_{Q_j}|}\right]^{1+n/q'} \left\|\mathbf{1}_{Q_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \\ &\lesssim \left[M\left(\mathbf{1}_{Q_j}\right)(x)\right]^{(1/n+1/q')} \left\|\mathbf{1}_{Q_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}. \end{aligned} \tag{2.13}$$

Moreover, by the fact that $p^* = \frac{np_-}{n-p_-}$ and $\frac{p^*}{p^*-1} < q < \infty$, we know that

$$\theta := 1/n + 1/q' = 1/n + 1 - 1/q > 1/n + 1/p^* = 1/p_- \geq 1.$$

Thus, by (2.12), (2.13) with $\theta > 1$, and using the same estimates of (2.6) and (2.8), we find that $\|M_z^{(q)}(f)\|_{L^{p(\cdot)}(\Omega)} \lesssim \|f\|_{H_z^{p(\cdot)}(\Omega)} < \infty$. This finishes the proof of Theorem 1.3(iii). □

Proof of Theorem 1.4(i). Let $f \in H_r^{p(\cdot)}(\Omega)$. From the definition of $H_r^{p(\cdot)}(\Omega)$, we choose an extension F of f on $\mathbb{R}^n$ such that $\|F\|_{H^{p(\cdot)}(\mathbb{R}^n)} \leq 2\|f\|_{H_r^{p(\cdot)}(\Omega)}$. Moreover, by Lemma 2.6 and the proof of [8, Theorem 1.2], we conclude that, for all $x \in \Omega$, $M_r(f)(x) \lesssim M_z(F)(x)$, and hence $\|M_r(f)\|_{L^{p(\cdot)}(\Omega)} \lesssim \|M_z(F)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, which, combined with Lemma 2.7, implies that

$$\|M_r(f)\|_{L^{p(\cdot)}(\Omega)} \lesssim \|M_z(F)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|F\|_{H^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_r^{p(\cdot)}(\Omega)}.$$

On the other hand, proceeding as in the proof of [12, Theorem 1.12], we know that $\|f\|_{H_r^{p(\cdot)}(\Omega)} \lesssim \|f\|_{H^{p(\cdot)}(\Omega)}$ when Ω is a bounded Lipschitz domain. Furthermore, by the proof of [12, Theorem 1.5], we find that $\|f\|_{H^{p(\cdot)}(\Omega)} \lesssim \|M_r(f)\|_{L^{p(\cdot)}(\Omega)}$ when Ω is a proper open subset. Thus, we have $\|f\|_{H_r^{p(\cdot)}(\Omega)} \lesssim \|M_r(f)\|_{L^{p(\cdot)}(\Omega)}$. This finishes the proof of Theorem 1.4(i). □

Proof of Theorem 1.4(ii). From the fact that $G_x(\Omega) \subset G_x^q(\Omega)$, it follows that $M_r(f)(x) \leq M_r^{(q)}(f)(x)$ for all $x \in \Omega$, which together with Theorem 1.4(i), implies that $\|f\|_{H_r^{p(\cdot)}(\Omega)} \lesssim \|M_r^{(q)}(f)\|_{L^{p(\cdot)}(\Omega)}$. Conversely, by the proofs of [8, Theorem 2.12] and Theorem 1.3(iii), we can get the desired results and leave the details to the interested readers. This completes the proof of Theorem 1.4(ii). □

3. Proof of Theorem 1.9

In this section, we give the proof of Theorem 1.9. Let us recall the following notion of local $(p(\cdot), q)$-atoms (see [17, Definitions 1.3 and 1.4]).

Definition 3.1. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $q \in (1, \infty]$. Assume that p_- and $\underline{p}$ are as in (1.1). Fix an integer $d \geq d_{p(\cdot)} := \min\{d \in \mathbb{Z}_+ : p_-(n + d + 1) > n\}$. A measurable function a on $\mathbb{R}^n$ is called a *type* (a) *local* $(p(\cdot), q)$*-atom* if there exists a cube Q such that

(i) $\operatorname{supp}(a) \subset Q$ with $0 < \ell(Q) < 1$;
(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq |Q|^{1/q} \|\mathbf{1}_Q\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}$;
(iii) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$ for $|\alpha| \leq d$.

Moreover, a measurable function b on $\mathbb{R}^n$ is called a *type* (b) *local* $(p(\cdot), q)$*-atom* if $\operatorname{supp}(b) \subset \widetilde{Q}$ with $\ell(\widetilde{Q}) \geq 1$ and $\|b\|_{L^q(\mathbb{R}^n)} \leq |\widetilde{Q}|^{1/q} \|\mathbf{1}_{\widetilde{Q}}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}$.

For sequences of $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and cubes $\{Q_j\}_{j=1}^\infty$, define that

$$\mathcal{A}'\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\right) := \left\| \left[\sum_{j=1}^\infty \left(\frac{|\lambda_j| \mathbf{1}_{Q_j}}{\|\mathbf{1}_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right)^{\underline{p}} \right]^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Proof of Theorem 1.9. Let $f \in h_r^{p(\cdot)}(\Omega)$. Then by the proof of [12, Theorem 1.12(i)], we have the atomic decomposition of $h_r^{p(\cdot)}(\Omega)$. Conversely, this part of the proof largely follows [7, Theorem 2.7] and we include it here primarily for the reader's convenience. Then, we know that there exist two sequences $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ and $\{\kappa_j\}_{j=1}^\infty \subset \mathbb{C}$, a sequence $\{a_j\}_{j=1}^\infty$ of type (a) $(p(\cdot), q)_\Omega$-atoms and a sequence $\{b_j\}_{j=1}^\infty$ of type (b) $(p(\cdot), q)_\Omega$-atoms such that

$$f = \sum_{j=1}^\infty \lambda_j a_j + \sum_{j=1}^\infty \kappa_j b_j \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n), \tag{3.1}$$

and

$$\mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\right) + \mathcal{A}\left(\{\kappa_j\}_{j=1}^\infty, \{\widetilde{Q}_j\}_{j=1}^\infty\right) < \infty, \tag{3.2}$$

where $\{Q_j\}_{j=1}^\infty$ and $\{\widetilde{Q}_j\}_{j=1}^\infty$, respectively, denote the supports of $\{a_j\}_{j=1}^\infty$ and $\{b_j\}_{j=1}^\infty$. The type (a) $(p(\cdot), q)_\Omega$-atoms a_j in (3.1) are already type (a) local $(p(\cdot), q)$-atoms and

$$\mathcal{A}'\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\right) = \mathcal{A}\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\right) < \infty, \tag{3.3}$$

hence we let those atoms stand unchanged. For type (b) $(p(\cdot), q)_\Omega$-atoms b_j in (3.1), there are two different cases.

Case 1: If b_j is a type (b) $(p(\cdot), q)_\Omega$-atom as in (3.1) and is supported on a type (b) cube Q with $\ell(Q) < 1$, then we find a cube $\widetilde{Q} \subset (\overline{\Omega})^\complement$ which has the same size as Q. We further consider the extension $(b_j)_1$ of the function b_j as follows:

$$(b_j)_1 := \begin{cases} b_j(x), & \text{for } x \in Q, \\ -\dfrac{1}{|Q|}\displaystyle\int_Q b_j(y)\,dy, & \text{for } x \in \widetilde{Q}. \end{cases}$$

From this, we deduce that the function $(b_j)_1$ is supported on $Q \cup \widetilde{Q}$. Since the distance of Q and $\widetilde{Q}$ to $\partial\Omega$ are comparable to $\ell(Q)$, we may find another cube $\widehat{Q}$ such that $(Q \cup \widetilde{Q}) \subset \widehat{Q}$ and $|Q| \le |\widehat{Q}| \lesssim |Q|$. By this and the Hölder inequality, we know that

$$\begin{aligned}\left\|(b_j)_1\right\|_{L^q(\mathbb{R}^n)} &\le \|b_j\|_{L^q(\mathbb{R}^n)} + \left\|\left(-\frac{1}{|Q|}\int_Q b_j(y)\,dy\right)\mathbf{1}_{\widetilde{Q}}\right\|_{L^q(\mathbb{R}^n)} \\ &\le \|b_j\|_{L^q(\mathbb{R}^n)} + \left|\widetilde{Q}\right|^{1/q}|Q|^{-1}\left|\int_Q b_j(y)\,dy\right| \\ &\le \|b_j\|_{L^q(\mathbb{R}^n)} + \left|\widetilde{Q}\right|^{1/q}|Q|^{-1+1/q'}\|b_j\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|\widehat{Q}|^{1/q}}{\|\mathbf{1}_{\widehat{Q}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}(b_j)_1(x)\,dx &= \int_Q b_j(x)\,dx - \int_{\widetilde{Q}}\left(\frac{1}{|Q|}\int_Q b_j(x)\,dx\right)\mathbf{1}_{\widetilde{Q}}(y)\,dy \\ &= \int_Q b_j(x)\,dx - \left|\widetilde{Q}\right||Q|^{-1}\int_Q b_j(x)\,dx = 0\,,\end{aligned}$$

which further implies that $(b_j)_1$ is an acceptable type (a) local $(p(\cdot), q)$-atom in $\mathbb{R}^n$ and

$$\mathcal{A}'\left(\{\kappa_Q\}_{Q\in Q_1}, \{Q\}_{Q\in Q_1}\right) = \mathcal{A}\left(\{\kappa_Q\}_{Q\in Q_1}, \{Q\}_{Q\in Q_1}\right) < \infty, \tag{3.4}$$

where $Q_1 := \{Q \subset \Omega : Q \text{ is a type } (b) \text{ cube with } \ell(Q) < 1\}$.

Case 2: If b_j is a type (b) $(p(\cdot), q)_\Omega$-atom as in (3.1) and is supported on a type (b) cube Q with $\ell(Q) \ge 1$. Now we just define

$$(b_j)_2 := \begin{cases} b_j(x), & \text{for } x \in Q, \\ 0, & \text{elsewhere,} \end{cases}$$

which implies that $(b_j)_2$ is an acceptable type (b) local $(p(\cdot), q)$-atom in $\mathbb{R}^n$ and

$$\mathcal{A}'\left(\{\kappa_Q\}_{Q\in Q_2}, \{Q\}_{Q\in Q_2}\right) = \mathcal{A}\left(\{\kappa_Q\}_{Q\in Q_2}, \{Q\}_{Q\in Q_2}\right) < \infty, \tag{3.5}$$

where $Q_2 := \{Q \subset \Omega : Q \text{ is a type } (b) \text{ cube with } \ell(Q) \ge 1\}$. Moreover, from Definition 1.7, [17, Theorem 1.5], (3.3), (3.4), (3.5) and (3.2), we deduce that

$$\begin{aligned}\|f\|_{h_r^{p(\cdot)}(\Omega)} \le \|F\|_{h^{p(\cdot)}(\mathbb{R}^n)} &\lesssim \mathcal{A}'\left(\{\lambda_j\}_{j=1}^\infty, \{Q_j\}_{j=1}^\infty\right) + \mathcal{A}'\left(\{\kappa_Q\}_{Q\in Q_1}, \{Q\}_{Q\in Q_1}\right) \\ &\quad + \mathcal{A}'\left(\{\kappa_Q\}_{Q\in Q_2}, \{Q\}_{Q\in Q_2}\right) < \infty,\end{aligned}$$

which implies that $f \in h_r^{p(\cdot)}(\Omega)$. This finishes the proof of Theorem 1.9. □

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References

[1] P. Auscher and E. Russ, *Hardy spaces and divergence operators on strongly Lipschitz domains of* $\mathbb{R}^n$, J. Funct. Anal. **201** (2003), no. 1, 148–184. https://doi.org/10.1016/S0022-1236(03)00059-4

[2] P. Auscher and Ph. Tchamitchian, *Gaussian estimates for second order elliptic divergence operators on Lipschitz and* C^1 *domains*, in Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), 15–32, Lecture Notes in Pure and Appl. Math., 215, Dekker, New York, 2001.

[3] T. A. Bui and X. T. Duong, *Regularity estimates for Green operators of Dirichlet and Neumann problems on weighted Hardy spaces*, arXiv:1808.09639.

[4] J. Cao, D. Chang, D. Yang, and S. Yang, *Weighted local Orlicz-Hardy spaces on domains and their applications in inhomogeneous Dirichlet and Neumann problems*, Trans. Amer. Math. Soc. **365** (2013), no. 9, 4729–4809. https://doi.org/10.1090/S0002-9947-2013-05832-1

[5] D.-C. Chang, *The dual of Hardy spaces on a bounded domain in* $\mathbf{R}^n$, Forum Math. **6** (1994), no. 1, 65–81. https://doi.org/10.1515/form.1994.6.65

[6] D.-C. Chang, G. Dafni, and E. M. Stein, *Hardy spaces, BMO, and boundary value problems for the Laplacian on a smooth domain in* $\mathbb{R}^n$, Trans. Amer. Math. Soc. **351** (1999), no. 4, 1605–1661. https://doi.org/10.1090/S0002-9947-99-02111-X

[7] D.-C. Chang, S. G. Krantz, and E. M. Stein, H^p *theory on a smooth domain in* $\mathbf{R}^N$ *and elliptic boundary value problems*, J. Funct. Anal. **114** (1993), no. 2, 286–347. https://doi.org/10.1006/jfan.1993.1069

[8] X. Chen, R. Jiang, and D. Yang, *Hardy and Hardy-Sobolev spaces on strongly Lipschitz domains and some applications*, Anal. Geom. Metr. Spaces **4** (2016), no. 1, 336–362. https://doi.org/10.1515/agms-2016-0017

[9] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Heidelberg, 2013. https://doi.org/10.1007/978-3-0348-0548-3

[10] D. Cruz-Uribe and L.-A. D. Wang, *Variable Hardy spaces*, Indiana Univ. Math. J. **63** (2014), no. 2, 447–493. https://doi.org/10.1512/iumj.2014.63.5232

[11] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[12] X. Liu, *Atomic characterizations of variable Hardy spaces on domains and their applications*, Banach J. Math. Anal. **15** (2021), no. 1, 26. https://doi.org/10.1007/s43037-020-00109-3

[13] A. Miyachi, *Maximal functions for distributions on open sets*, Hitotsubashi J. Arts Sci. **28** (1987), no. 1, 45–58.

[14] ———, H^p *spaces over open subsets of* $\mathbf{R}^n$, Studia Math. **95** (1990), no. 3, 205–228. https://doi.org/10.4064/sm-95-3-205-228

[15] E. Nakai and Y. Sawano, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal. **262** (2012), no. 9, 3665–3748. https://doi.org/10.1016/j.jfa.2012.01.004

[16] Y. Sawano, K. Ho, D. Yang, and S. Yang, *Hardy spaces for ball quasi-Banach function spaces*, Dissertationes Math. **525** (2017), 102 pp. https://doi.org/10.4064/dm750-9-2016

[17] J. Tan, *Atomic decompositions of localized Hardy spaces with variable exponents and applications*, J. Geom. Anal. **29** (2019), no. 1, 799–827. https://doi.org/10.1007/s12220-018-0019-1

[18] D. Yang, J. Zhang, and C. Zhuo, *Variable Hardy spaces associated with operators satisfying Davies-Gaffney estimates*, Proc. Edinb. Math. Soc. (2) **61** (2018), no. 3, 759–810. https://doi.org/10.1017/s0013091517000414

[19] D. Yang, C. Zhuo, and E. Nakai, *Characterizations of variable exponent Hardy spaces via Riesz transforms*, Rev. Mat. Complut. **29** (2016), no. 2, 245–270. https://doi.org/10.1007/s13163-016-0188-z

[20] C. Zhuo, Y. Sawano, and D. Yang, *Hardy spaces with variable exponents on RD-spaces and applications*, Dissertationes Math. **520** (2016), 74 pp. https://doi.org/10.4064/dm744-9-2015

[21] C. Zhuo and D. Yang, *Maximal function characterizations of variable Hardy spaces associated with non-negative self-adjoint operators satisfying Gaussian estimates*, Nonlinear Anal. **141** (2016), 16–42. https://doi.org/10.1016/j.na.2016.03.025

[22] C. Zhuo, D. Yang, and Y. Liang, *Intrinsic square function characterizations of Hardy spaces with variable exponents*, Bull. Malays. Math. Sci. Soc. **39** (2016), no. 4, 1541–1577. https://doi.org/10.1007/s40840-015-0266-2

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