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FERMAT'S EQUATION OVER 2-BY-2 MATRICES

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Abstract. We study the solvability of the Fermat's matrix equation in some classes of 2-by-2 matrices. We prove the Fermat's matrix equation has infinitely many solutions in a set of 2-by-2 positive semidefinite integral matrices, and has no nontrivial solutions in some classes including 2-by-2 symmetric rational matrices and stochastic quadratic field matrices.

1. Introduction

Pierre de Fermat mentioned in 1637 that for any integer n greater than 2, no positive integers a, b, c satisfy the equation

$$a^n + b^n = c^n. \tag{1}$$

The Fermat's last theorem had become a conjecture since then. Andrew Wiles [15] confirmed the conjecture is true. Subsequent research has extended the problem of Fermat's last theorem over some number fields (cf. [5,9]). In contrast to the classical Fermat's last theorem in integers, there have been a number of papers on the Fermat's equation in matrices (cf. [6, 11–14]). In particular, the Fermat's equation has been investigated in 2-by-2 integer matrices [3], rational matrices [7], general linear group $GL_2(\mathbb{Z})$ [3] and special linear group $SL_2(\mathbb{Z})$ of 2-by-2 matrices with det = 1 [11].

In this paper, we study the solvability of Fermat's matrix equation

$$A^n + B^n = C^n \tag{2}$$

in some classes of 2-by-2 matrices for $n \geq 4$. We prove the Fermat's matrix equation (2) has infinitely many solutions in a commuting family of 2-by-2 symmetric positive semidefinite integral matrices, and the equation (2) has no nontrivial solutions in some 2-by-2 symmetric rational matrices and m-by-m complex row stochastic matrices with row sums belonging to the quadratic field $\mathbb{Q}(\sqrt{2})$.

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2. Fermat's matrix equation

It is no surprising the Fermat's matrix equation (2) has solutions in positive integral matrices, such as

$$\begin{pmatrix} 2 & 6 \\ 6 & 3 \end{pmatrix}^3 + \begin{pmatrix} 7 & 3 \\ 3 & 3 \end{pmatrix}^3 = \begin{pmatrix} 3 & 6 \\ 6 & 6 \end{pmatrix}^3 .$$

The three matrices are symmetric, but not positive semidefinite. Observe that the mutual commutativity is invalid for the three matrices. In the following, we determine a class of commuting family of 2-by-2 matrices.

Lemma 1. *Let $\mathbb{K}$ be a subset of complex numbers and q be a complex number. Then $H(q, \mathbb{K}) = \left\{ \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in M_2(\mathbb{K}), a - c = bq \right\}$ is a commuting family.*

Proof. Suppose $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right)$ are matrices in $H(q, \mathbb{K})$. Then $(a - c)y = b(x - z)$, which implies that the two matrices commute since

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x & y \\ y & z \end{pmatrix} = \begin{pmatrix} ax + by & ay + bz \\ bx + cy & by + cz \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} x & y \\ y & z \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} ax + by & bx + cy \\ ay + bz & by + cz \end{pmatrix} .$$ □

Denote $H^+(q, \mathbb{K})$ the positive definite matrices in $H(q, \mathbb{K})$. We give a subclass of positive definite matrices for which the Fermat's matrix equation (2) has infinitely many solutions when $n = 3$.

Theorem 2. *The Fermat's matrix equation* (2) *has infinitely many solutions in $H^+(\pm 1, \mathbb{N})$ for $n = 3$.*

Proof. Firstly, we find three particular matrices in $H^+(1, \mathbb{N})$ satisfying the Fermat's matrix equation (2):

$$\begin{pmatrix} 7 & 3 \\ 3 & 4 \end{pmatrix}^3 + \begin{pmatrix} 11 & 6 \\ 6 & 5 \end{pmatrix}^3 = \begin{pmatrix} 12 & 6 \\ 6 & 6 \end{pmatrix}^3 . \tag{3}$$

Let $\left(\begin{smallmatrix} a & b \\ b & a-b \end{smallmatrix}\right)$ be an arbitrary matrix in $H^+(1, \mathbb{N})$, which has determinant $a^2 - ab - b^2 > 0$. Direct computations show that

$$\begin{pmatrix} a & b \\ b & a - b \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7a + 3b & 3a + 4b \\ 3a + 4b & 4a - b \end{pmatrix}$$

which is symmetric and has positive determinant $19(a^2 - ab - b^2)$, and thus belongs to $H^+(1, \mathbb{N})$. Since $H(1, \mathbb{N})$ is a commuting family, it follows that

$$\begin{pmatrix} a & b \\ b & a - b \end{pmatrix}^3 \begin{pmatrix} 7 & 3 \\ 3 & 4 \end{pmatrix}^3 = \left(\begin{pmatrix} a & b \\ b & a - b \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 3 & 4 \end{pmatrix} \right)^3 .$$

Similarly,

$$\begin{pmatrix} a & b \\ b & a - b \end{pmatrix} \begin{pmatrix} 11 & 6 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 11a + 6b & 6a + 5b \\ 6a + 5b & 5a + b \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ b & a-b \end{pmatrix} \begin{pmatrix} 12 & 6 \\ 6 & 6 \end{pmatrix} = \begin{pmatrix} 12a+6b & 6a+6b \\ 6a+6b & 6a \end{pmatrix}$$

which are in $H^+(1,\mathbb{N})$. Multiplying $\left(\begin{smallmatrix} a & b \\ b & a-b \end{smallmatrix}\right)^3$ to both sides of equation (3), we obtain that

$$\begin{pmatrix} 7a+3b & 3a+4b \\ 3a+4b & 4a-b \end{pmatrix}^3 + \begin{pmatrix} 11a+6b & 6a+5b \\ 6a+5b & 5a+b \end{pmatrix}^3 = \begin{pmatrix} 12a+6b & 6a+6b \\ 6a+6b & 6a \end{pmatrix}^3.$$

Note that $H^+(-1,\mathbb{N}) = \{A = P^TBP : B \in H^+(1,\mathbb{N})\}$, where $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Suppose $A, B, C \in H^+(1,\mathbb{N})$ satisfy the Fermat's matrix equation $A^3 + B^3 = C^3$. Then,

$$P^TA^3P + P^TB^3P = P^TC^3P,$$

which yields

$$(P^TAP)^3 + (P^TBP)^3 = (P^TCP)^3.$$

Set $X = P^TAP$, $Y = P^TBP$, $Z = P^TCP$. Then $X, Y, Z \in H^+(-1,\mathbb{N})$ and $X^3 + Y^3 = Z^3$. ☐

Jarvis and Meekin [9] proved that the equation $x^n + y^n = z^n$ with $x, y, z \in \mathbb{Q}(\sqrt{2})$ has no nontrivial solutions, $xyz \neq 0$, when $n \geq 4$, where $\mathbb{Q}(\sqrt{2})$ is the real quadratic field consisting of $a + b\sqrt{2}$, $a, b \in \mathbb{Q}$. The result is helpful for studying of the Fermat's matrix equation (2) which has no nontrivial solutions in some matrix classes.

Theorem 3. *The Fermat's matrix equation $A^n + B^n = C^n$ has no nontrivial solutions in $H(q,\mathbb{Q})$ for $q = \pm 2, \pm 3, \pm 6$ and $n \geq 4$.*

Proof. Assume $q = 2$. Suppose $A, B, C \in H(2,\mathbb{Q})$ are nontrivial solutions satisfying $A^n + B^n = C^n$ with $n \geq 4$. Let $T = \left(\begin{smallmatrix} t_1 & t_2 \\ t_2 & t_3 \end{smallmatrix}\right) \in H(2,\mathbb{Q})$, then $t_1 - t_3 = 2t_2$. It is easy to see that the eigenvalues of T are

$$\lambda_{\pm}(T) = \frac{t_1 + t_3 \pm 2t_2\sqrt{2}}{2} \in \mathbb{Q}(\sqrt{2}). \tag{4}$$

By Lemma 1, the family $H(2,\mathbb{Q})$ is commuting. Hence, by [8, Theorem 2.2.3], there exists a unitary matrix U which simultaneously upper triangularizes the matrices A, B, C. The assumption $A^n + B^n = C^n$ implies that

$$(U^*AU)^n + (U^*BU)^n = (U^*CU)^n.$$

Comparing the (1,1) entries on both sides, we have that

$$\lambda_\epsilon(A)^n + \lambda_\xi(B)^n = \lambda_\eta(C)^n, \tag{5}$$

where $\epsilon, \xi, \eta \in \{+,-\}$ according to the choice of the eigenvalues of (4). The eigenvalues, according to (4), are elements of $\mathbb{Q}(\sqrt{2})$, and thus by a result of Jarvis and Meekin [9], one of the eigenvalues should be 0, say $\lambda_\epsilon(A)$. From (4), if $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_2 & a_3 \end{smallmatrix}\right)$, $a_1 - a_3 = 2a_2$, then $\lambda_\epsilon(A) = \frac{a_1 + a_3 \pm 2a_2\sqrt{2}}{2}$. The condition $\lambda_\epsilon(A) = 0$ implies that $a_2 = 0$, and thus $a_1 = a_3$ which is nonzero for $A \neq 0$, gives a contradiction. Therefore, the matrix equation has no nontrivial solutions in

$H(2,\mathbb{Q})$ for $n \geq 4$. Apply the same reasoning to prove Theorem 2, we obtain that the Fermat's matrix equation $A^n + B^n = C^n$ has no solution in $H(-2,\mathbb{Q})$ for $n \geq 4$.

Assume $q = \pm 3, \pm 6$. We quote a result due to Freitas and Siksek [5]: The Fermat's equation $x^n + y^n = z^n$ has no nontrivial solutions in $\mathbb{Q}(\sqrt{d})$ for $n \geq 4$ when $3 \leq d \neq 5, 17 \leq 23$ is a square free integer. Let $T = \left(\begin{smallmatrix} t_1 & t_2 \\ t_2 & t_3 \end{smallmatrix}\right) \in H(q,\mathbb{Q})$. Then $t_1 - t_3 = qt_2$. It is easy to see that the eigenvalues of T are

$$\lambda_{\pm}(T) = \frac{t_1 + t_3 \pm t_2\sqrt{q^2+4}}{2} \in \mathbb{Q}(\sqrt{q^2+4}).$$

If $q = \pm 3$, $q^2 + 4 = 13$. Suppose $A^n + B^n = C^n$ has no nontrivial solutions in $H(q,\mathbb{Q})$. Repeating the argument used for $q = \pm 2$, we obtain the equation (5) has a solution in $\mathbb{Q}(\sqrt{13})$, a contradiction. For $q = \pm 6$, in this case, $q^2 + 4 = 40$, and $\mathbb{Q}(\sqrt{40}) = \mathbb{Q}(\sqrt{10})$. The conclusion follows a similar way. □

For $q = 0$, we have the following result.

Theorem 4. *The Fermat's matrix equation $A^n + B^n = C^n$ has infinitely many solutions in $H(0,\mathbb{Z})$ for all positive integer n.*

Proof. We claim that for arbitrary $a, b \in \mathbb{C}$ and positive integer n,

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix}^n + \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}^n = \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}^n. \tag{6}$$

Direct computations show that

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix}^n = a^n \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^n = a^n \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$$

and

$$\begin{pmatrix} b & -b \\ -b & b \end{pmatrix}^n = a^n \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^n = b^n \begin{pmatrix} 2^{n-1} & -2^{n-1} \\ -2^{n-1} & 2^{n-1} \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} \begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix}^n &= \left(\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \right)^n \\ &= \begin{pmatrix} \frac{1}{2}((2a)^n + (2b)^n) & \frac{1}{2}((2a)^n - (2b)^n) \\ \frac{1}{2}((2a)^n - (2b)^n) & \frac{1}{2}((2a)^n + (2b)^n) \end{pmatrix}. \end{aligned}$$

This proves the identity (6). □

Remark. It is explicitly proved in [4] that there exist solutions of the Fermat's matrix equation $A^4 + B^4 = C^4$ in 2-by-2 integral matrices. In addition, it is shown [10] that the Fermat's matrix equation $A^n + B^n = C^n$ has infinitely many solutions in weighted shift integral matrices. Theorem 4 provides a class of positive semidefinite integral matrices which is a subset of $H(0,\mathbb{Z})$ and assures the solvability of Fermat's matrix equation (6).

Vaserstein [14] proved that the Fermat's matrix equation $A^n + B^n = C^n$ in $GL_2(\mathbb{Z})$ with det $= \pm 1$ has a nontrivial solution if and only if n is not a multiple of 4 or 6. In $SL_2(\mathbb{Z})$, Khazanov [11] proved that the Fermat's matrix equation has a nontrivial solution if and only if n is not a multiple of 3 or 4. When $n = 4$, there is a solution in 2-by-2 invertible matrices, for instance,

$$\begin{pmatrix} 1 & 6 \\ 9 & 4 \end{pmatrix}^4 + \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix}^4 = \begin{pmatrix} 5 & 10 \\ 5 & 5 \end{pmatrix}^4.$$

For $n = 4, 6$, we have the following result.

Theorem 5. *The Fermat's matrix equation $A^n + B^n = C^n$ has no nontrivial solutions in $H(q, \mathbb{Q})$ for any integer $q \neq 0$ when $n = 4$ and 6.*

Proof. As indicated in the proof of Theorem 3, the eigenvalues of a matrix in $H(q, \mathbb{Q})$ are elements in $\mathbb{Q}(\sqrt{q^2+4})$. Suppose there are matrices $A, B, C \in H(q, \mathbb{Q})$ satisfying $A^4 + B^4 = C^4$. Apply the same technique used in the proof of Theorem 3, it yields

$$\lambda_\epsilon(A)^4 + \lambda_\xi(B)^4 = \lambda_\eta(C)^4, \tag{7}$$

where $\epsilon, \xi, \eta \in \{1, -1\}$ according to the choice of the respective eigenvalues. By a theorem of [1]: The equation $x^4 + y^4 = z^4$ has nontrivial solutions in the field $\mathbb{Q}(\sqrt{d})$, where d is a square-free integer, $d \neq 0, 1$, if and only if $d = -7$. Since $q^2 + 4 \neq -7$ and $q^2 + 4$ is square free for $q \neq 0$, it follows that the equation $x^4 + y^4 = z^4$ has no nontrivial solutions in $\mathbb{Q}(\sqrt{q^2+4})$, a contradiction to the fact of (7).

The assertion for $n = 6$ can be followed by using the similar argument to the Fermat's matrix equation $n = 4$, and applying a known result by Aigner [2] that the equation $x^6 + y^6 = z^6$ has no nontrivial solutions in the field $\mathbb{Q}(\sqrt{d})$ if d is a square-free integer, $d \neq 0, 1$. □

We summarize in Table 1 the solvability of the Fermat's matrix equation $A^n + B^n = C^n, A, B, C \in H(q, \mathbb{Q}),\ q \in \mathbb{Z}$.

TABLE 1. Solvability of $A^n + B^n = C^n$ in $H(q, \mathbb{Q})$

q	n	nontrivial solutions
0	≥ 3	∞ (Theorem 4)
± 1	3	∞ (Theorem 2)
$\pm 2, \pm 3, \pm 6$	≥ 4	$\emptyset$ (Theorem 3)
$\neq 0$	$4, 6$	$\emptyset$ (Theorem 5)

Although our discussion, so far, is restricted to 2-by-2 matrices, we may relax our results to higher dimensions of matrices. One typical generalization is given as follows.

For any positive integer m, denote

$$H_{2m}(q,\mathbb{K}) = \left\{ \begin{pmatrix} T_{11} & \cdots & T_{1m} \\ \vdots & \ddots & \vdots \\ T_{m1} & \cdots & T_{mm} \end{pmatrix} \in M_{2m}(\mathbb{K}), T_{ij} \in H(q,\mathbb{K}), i,j = 1,\ldots,m \right\}.$$

Theorem 6. *The Fermat's matrix equation* (2) *has infinitely many solutions in* $H_{2m}(\pm 1,\mathbb{N})$ *for* $n = 3$.

Proof. We prove the case for $H_{2m}(1,\mathbb{N})$, and it can be proved analogously for the case $H_{2m}(-1,\mathbb{N})$. Clearly, $H_{2m}(q,\mathbb{K})$ is a commuting family. By Theorem 2, there are matrices $A, B, C \in H(1,\mathbb{N})$ satisfying $A^3+B^3 = C^3$. Let $\hat{A}, \hat{B}, \hat{C} \in M_{2m}(\mathbb{N})$ be three block diagonal matrices with block diagonals being A, B and C, respectively. Then, we have $\hat{A}^3 + \hat{B}^3 = \hat{C}^3$. Let $T = (T_{ij}) \in H_{2m}(1,\mathbb{N})$. Since T_{ij} commutes with A, B and C for $i,j = 1,\ldots,m$, it follows that T commutes with $\hat{A}, \hat{B}$ and $\hat{C}$, and thus

$$(T\hat{A})^3 + (T\hat{B})^3 = (T\hat{C})^3,$$

where $T\hat{A}, T\hat{B}, T\hat{C} \in H_{2m}(1,\mathbb{N})$. □

Finally, we give another matrix class for which the Fermat's matrix equation has no solutions.

Theorem 7. *The Fermat's matrix equation* $A^n + B^n = C^n$, $n \geq 4$, *has no nontrivial solutions in the class of* $m \times m$ *complex row stochastic matrices with nonzero row sums belonging to the quadratic field* $\mathbb{Q}(\sqrt{2})$.

Proof. Let $\mathcal{C} = \{A = (a_{ij}) \in M_m(\mathbb{C}) : \sum_{j=1}^m a_{ij} = r(A) \in \mathbb{Q}(\sqrt{2}),\ i = 1,2,\ldots,m\}$. Then the class $\mathcal{C}$ has a common eigenvector $u_1 = \frac{1}{\sqrt{m}}(1,1,\ldots,1)^T \in \mathbb{C}^m$ corresponding to the eigenvalue r_A for every $A \in \mathcal{C}$. Extend the vector u_1 to an orthonormal basis $u_1, u_2, \ldots, u_m$ for $\mathbb{C}^m$, and denote the unitary matrix $U = [u_1\, u_2 \cdots u_m]$, we obtain that, for $A \in \mathcal{C}$,

$$U^*AU = \left[\begin{array}{c|cccc} r(A) & t_{12} & t_{13} & \cdots & t_{1k} \\ \hline 0 & & & & \\ 0 & & A_1 & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right].$$

As a consequence, we have

$$A^n = U \left[\begin{array}{c|cccc} r(A)^n & \tilde{t}_{12} & \tilde{t}_{13} & \cdots & \tilde{t}_{1k} \\ \hline 0 & & & & \\ 0 & & \tilde{A}_1 & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right] U^*.$$

Hence, if $A, B, C \in \mathcal{C}$ satisfy the matrix equation $A^n + B^n = C^n$, $n \geq 4$, then

$$r(A)^n + r(B)^n = r(C)^n, \ n \geq 4$$

which, by [9], should not have nontrivial solutions $\mathbb{Q}(\sqrt{2})$, and leads to a contradiction. □

We have the following immediate consequence.

Theorem 8. *The Fermat's matrix equation* $A^n + B^n = C^n$ *has no nontrivial solutions in the circulant matrices with entries from* $\mathbb{Q}(\sqrt{2})$ *and nonzero row sum for* $n \geq 4$.

Remark. It is obvious that matrices in $H(0, \mathbb{N})$ are circulant with nonzero row sum. In contrast with Theorem 4, the Fermat's matrix equation $A^n + B^n = C^n$, according to Theorem 8, has no nontrivial solutions in $H(0, \mathbb{N})$ for $n \geq 4$.

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