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SOME RESULTS OF MONOMIAL IDEALS ON REGULAR SEQUENCES

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Abstract. Let R denote a commutative noetherian ring, and let $\mathbf{x} := x_1, \dots, x_d$ be an R-regular sequence. Suppose that $\mathfrak{a}$ denotes a monomial ideal with respect to $\mathbf{x}$. The first purpose of this article is to show that $\mathfrak{a}$ is irreducible if and only if $\mathfrak{a}$ is a generalized-parametric ideal. Next, it is shown that, for any integer $n \geq 1$, $(x_1, \dots, x_d)^n = \bigcap \mathbf{P}(f)$, where the intersection (irredundant) is taken over all monomials $f = x_1^{e_1} \cdots x_d^{e_d}$ such that $\deg(f) = n - 1$ and $\mathbf{P}(f) := (x_1^{e_1+1}, \dots, x_d^{e_d+1})$. The second main result of this paper shows that if $\mathfrak{q} := (\mathbf{x})$ is a prime ideal of R which is contained in the Jacobson radical of R and R is $\mathfrak{q}$-adically complete, then $\mathfrak{a}$ is a parameter ideal if and only if $\mathfrak{a}$ is a monomial irreducible ideal and $\mathrm{Rad}(\mathfrak{a}) = \mathfrak{q}$. In addition, if $\mathfrak{a}$ is generated by monomials $m_1, \dots, m_r$, then $\mathrm{Rad}(\mathfrak{a})$, the radical of $\mathfrak{a}$, is also monomial and $\mathrm{Rad}(\mathfrak{a}) = (\omega_1, \dots, \omega_r)$, where $\omega_i = \mathrm{rad}(m_i)$ for all $i = 1, \dots, r$.

1. Introduction

Let R be a commutative noetherian ring with the identity element 1_R, and let $\mathbf{x} := x_1, \dots, x_d$ be an R-regular sequence. A *monomial* with respect to $\mathbf{x}$ is a power product $x_1^{e_1} \cdots x_d^{e_d}$, where $e_1, \dots, e_d$ are non-negative integers (so a monomial is either a non-unit or the element 1_R), and a *monomial ideal* is a proper ideal generated by monomials. Monomial ideals are important in several areas of current research in commutative algebra and algebraic geometry, and they have been studied in their own right in several papers (for example see [2, 3, 6, 8, 9]), so many interesting results are proved about such ideals. A monomial ideal $\mathfrak{a}$ is called a *monomial irreducible ideal* if it cannot be written as proper intersection of two other monomial ideals. Suppose that s is an integer with $1 \leq s \leq d$, and let σ be a permutation of $\{1, \dots, d\}$ and let $e_1, \dots, e_s$ be non-negative integers. If $f = x_{\sigma(1)}^{e_1} \dots x_{\sigma(s)}^{e_s}$ is a monomial with

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respect to $\mathbf{x}$, then the monomial ideal $\mathbf{P}(f) := (x_{\sigma(1)}^{e_1+1}, \ldots, x_{\sigma(s)}^{e_s+1})$ is called a *generalized-parametric ideal* with respect to $\mathbf{x}$. In particular, an ideal of the form $(x_1^{a_1}, \ldots, x_d^{a_d})$ is called a *parameter ideal*, where $a_1, \ldots, a_d$ are positive integers.

The first observation of this paper is concerned with what might be considered a natural generalization of the Herzog-Hibi's result (see [5, Corollary 1.3.2]) for monomial ideals $\mathfrak{a}$ with respect to an R-regular sequence. More precisely, we shall show that:

Theorem 1.1. *Let R denote a noetherian ring, let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence and let $\mathfrak{a}$ be a non-zero monomial ideal of R with respect to $\mathbf{x}$. Then $\mathfrak{a}$ is a monomial irreducible ideal if and only if $\mathfrak{a}$ is a generalized-parametric ideal.*

The result of Theorem 1.1 is proved in Theorem 2.3. Pursuing this point of view further we derive the following consequence of Theorem 1.1.

Corollary 1.2. *Let R denote a noetherian ring and let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence contained in the Jacobson radical of R such that the ideal $\mathfrak{q} := (\mathbf{x})$ is prime and R is $\mathfrak{q}$-adically complete. Then, for any monomial ideal $\mathfrak{a}$ of R with respect to $\mathbf{x}$, the following conditions are equivalent*:

(i) $\mathfrak{a}$ *is a parameter ideal.*
(ii) $\mathfrak{a}$ *is monomial irreducible and* $\mathrm{Rad}(\mathfrak{a}) = \mathfrak{q}$.
(iii) $\mathfrak{a}$ *has a decomposition of parameter ideals.*

One of our tools for proving Corollary 1.2 is the following.

Proposition 1.3. *Let R denote a noetherian ring and let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence contained in the Jacobson radical of R such that the ideal $\mathfrak{q} := (\mathbf{x})$ is prime and R is $\mathfrak{q}$-adically complete. Suppose that $\mathfrak{a}$ is a monomial ideal of R generated by the monomials $m_1, \ldots, m_r$. Then* $\mathrm{Rad}(\mathfrak{a})$ *is also monomial and that* $\mathrm{Rad}(\mathfrak{a}) = (\omega_1, \ldots, \omega_r)$, *where* $\omega_i = \mathrm{rad}(m_i)$, *for all* $i = 1, \ldots, r$.

Another main result of this paper is to construct an irredundant generalized-parametric decomposition for ideal $(x_1, \ldots, x_d)^n$ for all integers $n \geq 1$. In fact, we shall show the following result which is identical with [4, Theorem 2.4] by a different proof.

Theorem 1.4. *Let R denote a noetherian ring and let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence. Put $\mathfrak{b} := (x_1, \ldots, x_d)$. Then, for any integer $n \geq 1$, we have*

$$\mathfrak{b}^n = \bigcap_{\deg(f)=n-1} \mathbf{P}(f),$$

where the intersection is taken over all monomials f with respect to $\mathbf{x}$ such that $\deg(f) = n - 1$. *Moreover, this intersection is irredundant.*

Throughout this paper all rings are commutative and noetherian, with identity, unless otherwise specified. We shall use R to denote such a ring and $\mathfrak{a}$ an ideal of R. The *radical* of $\mathfrak{a}$, denoted by $\mathrm{Rad}(\mathfrak{a})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}\}$. We say that $x_1, \ldots, x_d$ form an *R-regular sequence* (of elements of R) precisely when $(x_1, \ldots, x_d) \neq R$ and for each $i = 1, \ldots, d$, the element x_i is a non-zero divisor on the R-module $R/(x_1, \ldots, x_{i-1})$. For any unexplained notation and terminology we refer the reader to [1] or [7].

2. The results

The following proposition will be one of main tools in this paper. Before we state it, let us firstly recall some important notions on monomials. To this end, assume that $\mathfrak{q} := (\mathbf{x})$ and let $\mathrm{gr}_{\mathfrak{q}}(R) := \oplus_{n\geq 0}\mathfrak{q}^n/\mathfrak{q}^{n+1}$ denote the associated graded ring with respect to $\mathfrak{q}$. For every non-zero element ω of R with $\omega \notin \bigcap_{n\geq 0} \mathfrak{q}^n$, we define the order $\mathrm{ord}(\omega)$ of ω to be the largest integer t such that $\omega \in \mathfrak{q}^t$. Also, we define the initial form of ω as $\mathrm{In}(\omega) := \omega + \mathfrak{q}^{t+1} \in \oplus_{n\geq 0}\mathfrak{q}^n/\mathfrak{q}^{n+1}$. Then $\mathrm{In}(\omega)$ is a homogeneous non-zero polynomial of degree $t = \mathrm{ord}(\omega)$; and so there exist uniquely determined and pairwise distinct monomials $m_1, \ldots, m_r$ having degree $\mathrm{ord}(\omega)$ and elements $c_1, \ldots, c_r \in R \setminus \mathfrak{q}$ such that

$$\mathrm{In}(\omega) = \mathrm{In}(c_1 m_1 + \cdots + c_r m_r),$$

so we define the *set of terms* of ω by $\mathrm{Tm}(\omega) := \{m_1, \ldots, m_r\}$.

Definition. Let R be a ring and let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence. Then

(i) A monomial with respect to $\mathbf{x}$ is a power product $x_1^{e_1} \ldots x_d^{e_d}$, where $e_1, \ldots, e_d$ are non-negative integers, and a monomial ideal is a proper ideal generated by monomials.

(ii) A parameter ideal with respect to $\mathbf{x}$ is an ideal of the form $(x_1^{a_1}, \ldots, x_d^{a_d})$, where $a_1, \ldots, a_d$ are positive integers.

(iii) If $m = x_1^{e_1} \cdots x_d^{e_d}$ is a monomial with respect to $\mathbf{x}$, then we let $\mathbf{P}(m)$ denote the parameter ideal $(x_1^{e_1+1}, \ldots, x_d^{e_d+1})$. Note that, if $m = 1$, then $\mathbf{P}(m) = (\mathbf{x})$.

(iv) For every d-tuple $\mathbf{i} := (i_1, \ldots, i_d) \in \mathbb{N}_0^d$, we define $\deg(\mathbf{i}) := i_1 + \cdots + i_d$, the degree of $\mathbf{i}$, and we write $\mathbf{x}^{\mathbf{i}} := x_1^{i_1} \cdots x_d^{i_d}$. Since $\mathbf{x}$ is an R-regular sequence, it is easy to see that, for $\mathbf{i}, \mathbf{j} \in \mathbb{N}_0^d$, $\mathbf{x}^{\mathbf{i}} = \mathbf{x}^{\mathbf{j}}$ if and only if $\mathbf{i} = \mathbf{j}$.

(v) If $m = \mathbf{x}^{\mathbf{i}}$ is a monomial with respect to $\mathbf{x}$, then $\mathbf{i}$ is determined uniquely by m. We call $\deg(m) := \deg(\mathbf{i})$ the degree of m.

Note that if $\mathbf{x}^{\mathbf{i}} \in (\mathbf{x}^{\mathbf{j}})$, then it is easy to see that $i_1 \geq j_1, \ldots, i_d \geq j_d$ and $\mathbf{x}^{\mathbf{i}} = \mathbf{x}^{\mathbf{j}}.\mathbf{x}^{\mathbf{i}-\mathbf{j}}$.

(vi) If $f = x_1^{e_1} \cdots x_d^{e_d}$ is a monomial with respect to $\mathbf{x}$, then the *support* of f, denoted by $\mathrm{supp}(f)$, is defined to be the set $\{j \,|\, j \in \{1, \ldots, d\}$ and $e_j \neq 0\}$. Also, the *radical* of f, denoted by $\mathrm{rad}(f)$, is defined by $\mathrm{rad}(f) := \Pi_{j\in \mathrm{supp}(f)} x_j$.

The following proposition which plays a key role in the proof of the main theorems, shows that if $\mathbf{x} := x_1, \ldots, x_d$ is an R-regular sequence contained in the Jacobson radical of R such that the ideal $(\mathbf{x})$ is prime, then the radical of a monomial ideal is again a monomial ideal.

Proposition 2.1. *Let R be a noetherian ring and let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence contained in the Jacobson radical of R such that the ideal $\mathfrak{q} := (\mathbf{x})$ is prime. Suppose that R is complete with respect to the $\mathfrak{q}$-adic topology, and let $\mathfrak{a}$ be a monomial ideal with respect to $\mathbf{x}$. Then $\operatorname{Rad}(\mathfrak{a})$ is also a monomial ideal with respect to $\mathbf{x}$.*

Proof. In view of [6, Proposition 3], it is enough to show that, for every $w \in \operatorname{Rad}(\mathfrak{a})$ with $w \neq 0$, we have $\operatorname{Tm}(w) \subseteq \operatorname{Rad}(\mathfrak{a})$. To do this, since $\mathfrak{q}$ is a prime ideal contained in the Jacobson radical of R, it follows from [6, Proposition 2] that w admits a monomial representation (with respect to $\mathbf{x}$), i.e., there exist elements $e_1, \ldots, e_s \in R\backslash\mathfrak{q}$ such that $w = e_1m_1 + \cdots + e_sm_s$, where $m_1, \ldots, m_s$ are distinct non-zero monomials having the same degree $\operatorname{ord}(w)$; so that $\operatorname{Tm}(w) = \{m_1, \ldots, m_s\}$. Now, we show that $m_1, \ldots, m_s \in \operatorname{Rad}(\mathfrak{a})$.

We use induction on s. Consider the case in which $s = 1$. Then as $w \in \operatorname{Rad}(\mathfrak{a})$, there exists an integer $k \geq 1$ such that $w^k \in \mathfrak{a}$. Hence, in view of [6, Proposition 3], $\operatorname{Tm}(w^k) \subseteq \mathfrak{a}$. That is $\{m_1^k\} \subseteq \mathfrak{a}$, and so $m_1^k \in \mathfrak{a}$. Thus $m_1 \in \operatorname{Rad}(\mathfrak{a})$, as required. Suppose now that $s > 1$ and that the result has been proved for all non-zero monomials w' of R with $|(\operatorname{Tm}(w'))| \leq s - 1$. Set

$$\mathcal{M} := \{m_1^{\alpha_1}m_2^{\alpha_2}\cdots m_s^{\alpha_s} \mid 0 \leq \alpha_i \in \mathbb{Z}, \text{ and } \Sigma_{i=1}^s \alpha_i = k\}.$$

Then it is clear that $\operatorname{Tm}(w^k) \subseteq \mathcal{M}$. Next, we claim that there exists $j \in \{1, \ldots, s\}$ such that the monomial m_j^k cannot cancel against other elements of $\mathcal{M}$. To do this end, for all $i \in \{1, \ldots, s\}$, let us consider

$$m_i = x_1^{\beta_{i1}} \cdots x_d^{\beta_{id}} := \mathbf{x}^{\beta_\mathbf{i}},$$

where $\beta_\mathbf{i} = (\beta_{i1}, \ldots, \beta_{id}) \in \mathbb{N}_0^d$ and $\sum_{j=1}^d \beta_{ij} = \operatorname{ord}(w)$. Since the convex hull of the finite set $\{\beta_1, \ldots, \beta_s\}$ of $\mathbb{R}^d$ is a convex polytope, without loss of generality we may assume that β_1 is not in the convex hull of the set $\{\beta_2, \ldots, \beta_s\}$. Now, in order to establish the claim, let us suppose, on the contrary, that

$$m_1^k = (\mathbf{x}^{\beta_1})^k = m_1^{k_1}m_2^{k_2}\cdots m_s^{k_s} = (\mathbf{x}^{\beta_1})^{k_1}(\mathbf{x}^{\beta_2})^{k_2}\cdots(\mathbf{x}^{\beta_s})^{k_s},$$

where $k_1 + k_2 + \cdots + k_s = k$ and $k_1 < k$. Then we have

$$(k - k_1)\beta_1 = k_2\beta_2 + \cdots + k_s\beta_s,$$

and so $\beta_1 = \sum_{j=2}^s (k_j/k - k_1)\beta_j$. As $\sum_{j=2}^s (k_j/k - k_1) = 1$, we obtain a contradiction with the choice of β_1. Therefore we have $m_1^k \in \operatorname{Tm}(w^k)$, and hence it follows from $\operatorname{Tm}(w^k) \subseteq \mathfrak{a}$ that $m_1^k \in \mathfrak{a}$; so that $m_1 \in \operatorname{Rad}(\mathfrak{a})$. Consequently we have $w' := w - e_1m_1 \in \operatorname{Rad}(\mathfrak{a})$. Since $|(\operatorname{Tm}(w'))| \leq s - 1$, we can now use the inductive hypothesis in order to see $\operatorname{Tm}(w') \subseteq \operatorname{Rad}(\mathfrak{a})$; so that $m_2, \ldots, m_s \in \operatorname{Rad}(\mathfrak{a})$. This completes the inductive step, and the proof. $\square$

We are now ready to state and prove the first main result of this paper which shows that the radical of a monomial ideal with respect to an R-regular sequence $\mathbf{x} := x_1, \ldots, x_d$ can be computed explicitly. Recall that for a monomial $f = x_1^{e_1} \cdots x_d^{e_d}$ with respect to $\mathbf{x}$, the radical of f is $\mathrm{rad}(f) := \Pi_{j \in \mathrm{supp}(f)} x_j$, where $\mathrm{supp}(f) = \{j \mid j \in \{1, \ldots, d\} \text{ and } e_j \neq 0\}$.

Theorem 2.2. *Let R be a noetherian ring and let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence contained in the Jacobson radical of R such that the ideal $\mathfrak{q} := (\mathbf{x})$ is prime. Suppose that R is complete with respect to the $\mathfrak{q}$-adic topology, and let $\mathfrak{a} = (m_1, \ldots, m_r)$ be a monomial ideal with respect to $\mathbf{x}$. Then*

$$\mathrm{Rad}(\mathfrak{a}) = (\omega_1, \ldots, \omega_r),$$

where $\omega_i = \mathrm{rad}(m_i)$ for all $i = 1, \ldots, r$.

Proof. It is easy to see that $\omega_i = \mathrm{rad}(m_i) \in \mathrm{Rad}(\mathfrak{a})$ for all $i = 1, \ldots, r$. Hence

$$(\omega_1, \ldots, \omega_r) \subseteq \mathrm{Rad}(\mathfrak{a}).$$

Now, in order to show the opposite inclusion let us put

$$\mathfrak{b} := (\omega_1, \ldots, \omega_r).$$

Then, since in view of Proposition 2.1, $\mathrm{Rad}(\mathfrak{a})$ is a monomial ideal, it is enough for us to show that for each monomial $u \in \mathrm{Rad}(\mathfrak{a})$ we have $u \in \mathfrak{b}$. To this end, there exists an integer $k \geq 1$ such that $u^k \in \mathfrak{a}$. Hence, it follows from [6, Corollary 3] that $m_i \mid u^k$ for some $i = 1, \ldots, r$. Now, let $u = x_{s_1}^{c_1} \cdots x_{s_t}^{c_t}$ and $m_j = x_{k_1}^{a_1} \cdots x_{k_n}^{a_n}$. Then $u^k = x_{s_1}^{kc_1} \cdots x_t^{kc_t}$. As $m_j \mid u^k$, it follows from [6, Remark 1] that there is a monomial w such that $u^k = w m_j$. Now, it is easy to see that $\mathrm{rad}(m_j) \mid u$, and so $u \in \mathfrak{b}$, as required. □

The next main result of this paper is a generalization of [5, Corollary 1.3.2]. For a monomial ideal $\mathfrak{a}$ with respect to $\mathbf{x}$, of a noetherian ring R, we say that the monomials $f_1, \ldots, f_k$ are an *irredundant monomial generating sequence* for $\mathfrak{a}$ if f_i is not a monomial multiple of f_j, whenever $i \neq j$, for all $i, j \in \{1, \ldots, k\}$. Recall that if $u = x_1^{i_1} \ldots x_d^{i_d}$ and $v = x_1^{j_1} \ldots x_d^{j_d}$ be two monomials with respect to $\mathbf{x}$, then the least common multiple of u and v is defined by $\mathrm{lcm}(u, v) := x_1^{k_1} \ldots x_d^{k_d}$, where $k_r := \max\{i_r, j_r\}$ for $1 \leq r \leq d$.

Theorem 2.3. *Let R be a noetherian ring, let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence and let $\mathfrak{a}$ be a non-zero monomial ideal of R with respect to $\mathbf{x}$. Then $\mathfrak{a}$ is a monomial irreducible ideal if and only if $\mathfrak{a}$ is a generalized-parametric ideal.*

Proof. ($\Rightarrow$) Let $\mathfrak{a}$ be a non-zero monomial irreducible ideal with respect to $\mathbf{x}$. Then in view of [6, Remark 3] $\mathfrak{a}$ admits an irredundant monomial generating sequence $f_1, \ldots, f_k$. It is sufficient for us to show that every f_i is of the form $x_{t_i}^{e_i}$. Suppose by way of contradiction that one of the f_i is not of this form. After an appropriate reordering of the f_j if necessary we may assume that f_1 is not of the form $x_{t_i}^{e_i}$. This means that we can write $f_1 = x_{t_i}^{e_i} g$, where $e_i \geq 1$

and g is not divisible by x_{t_i}, and that $g \neq 1$ is a monomial with respect to $\mathbf{x}$. Now, we set

$$\mathfrak{b} := (x_{t_i}^{e_i}, f_2, \ldots, f_k) \text{ and } \mathfrak{c} := (g, f_2, \ldots, f_k).$$

Then, in view of [6, Proposition 1], we have

$$\begin{aligned} \mathfrak{b} \cap \mathfrak{c} = &(\mathrm{lcm}(x_{t_i}^{e_i}, g)) + (\mathrm{lcm}(x_{t_i}^{e_i}, f_2)) + \cdots + (\mathrm{lcm}(x_{t_i}^{e_i}, f_k)) + \cdots \\ &+ (\mathrm{lcm}(f_k, g)) + (\mathrm{lcm}(f_k, f_2)) + \cdots + (\mathrm{lcm}(f_k, f_k)). \end{aligned}$$

Hence

$$\mathfrak{b} \cap \mathfrak{c} \subseteq (\mathrm{lcm}(x_{t_i}^{e_i}, g)) + (f_2) + \cdots + (f_k) = \mathfrak{a}.$$

As $\mathfrak{a} \subseteq \mathfrak{b} \cap \mathfrak{c}$, therefore it follows that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$. Now, in order to complete the proof, we show that $\mathfrak{a} \neq \mathfrak{b}$ and $\mathfrak{a} \neq \mathfrak{c}$. In order to show $\mathfrak{a} \neq \mathfrak{b}$, it suffices to show that $x_{t_i}^{e_i} \notin \mathfrak{a}$. Suppose by way of contradiction that $x_{t_i}^{e_i} \in \mathfrak{a}$. Then, in view of [6, Corollary 3], $f_j \mid x_{t_i}^{e_i}$ for some index j. Since $x_{t_i}^{e_i} \mid f_1$, it follows that $f_j \mid f_1$. As the sequence $f_1, \ldots, f_k$ is irredundant, so we have $f_j = f_1$. Thus $f_1 = x_{t_i}^{e_i} g \mid x_{t_i}^{e_i}$. By comparing exponent vectors, we conclude that $g = 1$, which is a contradiction. Similarly, we have $\mathfrak{a} \neq \mathfrak{c}$. Consequently, we have $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$, where $\mathfrak{a} \subset \mathfrak{b}$ and $\mathfrak{a} \subset \mathfrak{c}$. This contradicts the assumption that $\mathfrak{a}$ is a monomial irreducible ideal.

($\Leftarrow$) For the converse, assume that $\mathfrak{a}$ is a generalized-parametric ideal. That is there are positive integers $k, t_1, \ldots, t_k, e_1, \ldots, e_k$ such that $t_1 < \cdots < t_k \leq d$ and $\mathfrak{a} = (x_{t_1}^{e_1}, \ldots, x_{t_k}^{e_k})$. We show that $\mathfrak{a}$ is a monomial irreducible ideal. Suppose on the contrary that there exist two monomial ideals $\mathfrak{b}$ and $\mathfrak{c}$ such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{a} \neq \mathfrak{b}$ and $\mathfrak{a} \neq \mathfrak{c}$. Then there are monomials f_1, f_2 such that $f_1 \in \mathfrak{b} \backslash \mathfrak{a}$ and $f_2 \in \mathfrak{c} \backslash \mathfrak{a}$. Now, let $f_1 = x_1^{m_1} \cdots x_d^{m_d}$ and $f_2 = x_1^{n_1} \cdots x_d^{n_d}$. Write $p_i = \max\{m_i, n_i\}$ for $i = 1, \ldots, d$. Then, for all $j = 1, \ldots, k$, we have $m_{t_j} < e_j$; because if $m_{t_j} \geq e_j$ for some j, then a comparison of exponent vectors shows that $f_1 \in (x_{t_i}^{e_i}) \subseteq \mathfrak{a}$, which is a contradiction. Similarly, for $i = 1, \ldots, k$, we have $n_{t_i} < e_i$, and hence $p_{t_i} < e_i$. Consequently, in view of [6, Corollary 3], $\mathrm{lcm}(f_1, f_2) = x_1^{p_1} \cdots x_d^{p_d} \notin \mathfrak{a}$. On the other hand, we have

$$\mathrm{lcm}(f_1, f_2) \in \mathfrak{b} \cap \mathfrak{c} = \mathfrak{a},$$

which is a contradiction. $\square$

As the first application of Theorems 2.2 and 2.3 we derive the following result which shows that a monomial ideal $\mathfrak{a}$ with respect to $\mathbf{x}$ is a parameter ideal if and only if it is monomial irreducible ideal and $\mathrm{Rad}(\mathfrak{a}) = (\mathbf{x})$.

Proposition 2.4. *Let R be a noetherian ring and let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence contained in the Jacobson radical of R such that the ideal $\mathfrak{q} := (\mathbf{x})$ is prime. Suppose that R is complete with respect to the $\mathfrak{q}$-adic topology, and let $\mathfrak{a}$ be a non-zero monomial ideal of R with respect to $\mathbf{x}$. Then $\mathfrak{a}$ is a parameter ideal if and only if $\mathfrak{a}$ is a monomial irreducible ideal and* $\mathrm{Rad}(\mathfrak{a}) = \mathfrak{q}$.

Proof. ($\Rightarrow$) Let $\mathfrak{a}$ be a parameter ideal. Then $\mathfrak{a} = (x_1^{e_1}, \dots, x_d^{e_d})$, where $e_1, \dots, e_d$ are positive integers. Therefore, in view of Theorem 2.2,

$$\mathrm{Rad}(\mathfrak{a}) = (\mathrm{rad}(x_1^{e_1}), \dots, \mathrm{rad}(x_d^{e_d})) = (x_1, \dots, x_d) = \mathfrak{q}.$$

Now, we show that $\mathfrak{a}$ is a monomial irreducible ideal. To do this, assume the contrary. Then there exist monomial ideals $\mathfrak{b}$ and $\mathfrak{c}$ such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{a} \subset \mathfrak{b}$ and $\mathfrak{a} \subset \mathfrak{c}$. Hence there exist monomials f_1, f_2 such that $f_1 \in \mathfrak{b}$, $f_2 \in \mathfrak{c}$ and that $f_1, f_2 \notin \mathfrak{a}$. Let us consider

$$f_1 = x_1^{m_1} \cdots x_d^{m_d} \text{ and } f_2 = x_1^{n_1} \cdots x_d^{n_d}.$$

Also, we set $p_i = \max\{m_i, n_i\}$ for every $i = 1, 2, \dots, d$. Then, as $f_1, f_2 \notin \mathfrak{a}$, it follows that $p_i < e_i$ for all $i = 1, 2, \dots, d$. Consequently,

$$\mathrm{lcm}(f_1, f_2) = x_1^{p_1} \cdots x_d^{p_d} \notin \mathfrak{a}.$$

On the other hand, $\mathrm{lcm}(f_1, f_2) \subseteq \mathfrak{b} \cap \mathfrak{c}$, and so $\mathrm{lcm}(f_1, f_2) \in \mathfrak{a}$, which is a contradiction.

($\Leftarrow$) Let $\mathfrak{a}$ be a monomial irreducible ideal and $\mathrm{Rad}(\mathfrak{a}) = \mathfrak{q}$. Then, the condition $\mathrm{Rad}(\mathfrak{a}) = \mathfrak{q}$ implies that $\mathfrak{a} \neq 0$, and in view of Theorem 2.3, $\mathfrak{a}$ is a generalized-parametric ideal. Hence there exist positive integers

$$k, t_1, \dots, t_k, e_1, \dots, e_k$$

such that

$$1 \leq t_1 < \cdots < t_k \leq d \text{ and } \mathfrak{a} = (x_{t_1}^{e_1}, \dots, x_{t_k}^{e_k}).$$

Now, as $\mathrm{Rad}(\mathfrak{a}) = \mathfrak{q}$, it follows from Theorem 2.2 that

$$\mathrm{Rad}(\mathfrak{a}) = (x_{t_1}, \dots, x_{t_k}) = (x_1, \dots, x_d),$$

and so the irredundant monomial ideal $\mathfrak{a}$ generated by the sequence $x_{t_1}^{e_1}, \dots, x_{t_k}^{e_k}$ contains a power of each element x_i. That is, we obtain that $\mathfrak{a} = (x_1^{e_1}, \dots, x_d^{e_d})$, and so $\mathfrak{a}$ is a parameter ideal. □

Let $\mathfrak{a}$ be a monomial ideal of R with respect to $\mathbf{x}$. Recall that a monomial f with respect to an R-regular sequence $\mathbf{x} := x_1, \dots, x_d$ is called an $\mathfrak{a}$-*corner-element* if $f \notin \mathfrak{a}$ and $x_1 f, \dots, x_d f \in \mathfrak{a}$. The notion of the corner-element was introduced by Heinzer et al. in [4].

Corollary 2.5. *Let R be a noetherian ring and let $\mathbf{x} := x_1, \dots, x_d$ be an R-regular sequence contained in the Jacobson radical of R such that the ideal $\mathfrak{q} := (\mathbf{x})$ is prime. Suppose that R is complete with respect to the $\mathfrak{q}$-adic topology, and let $\mathfrak{a}$ be a non-zero monomial ideal of R with respect to $\mathbf{x}$. Then $\mathfrak{a}$ has a decomposition of parameter ideals if and only if $\mathrm{Rad}(\mathfrak{a}) = \mathfrak{q}$.*

Proof. ($\Rightarrow$) If $\mathfrak{a}$ has a decomposition $\mathfrak{a} = \cap_{i=1}^n \mathfrak{q}_i$, where $\mathfrak{q}_i$ is a parameter ideal for every $i = 1, \dots, n$, then in view of Proposition 2.4, $\mathrm{Rad}(\mathfrak{q}_i) = \mathfrak{q}$ for every $i = 1, \dots, n$. Hence

$$\mathrm{Rad}(\mathfrak{a}) = \mathrm{Rad}(\cap_{i=1}^n \mathfrak{q}_i) = \cap_{i=1}^n \mathrm{Rad}(\mathfrak{q}_i) = \mathfrak{q}.$$

($\Leftarrow$) In order to prove the converse, let $\mathrm{Rad}(\mathfrak{a}) = \mathfrak{q}$ and let $f_1, \ldots, f_s$ denote the $\mathfrak{a}$-corner-elements (note that in view of [4, Remark 3.15] the set of $\mathfrak{a}$-corner-elements is finite). Let $\mathfrak{b} := \bigcap_{i=1}^{s} \mathbf{P}(f_i)$, and we shall show that $\mathfrak{a} = \mathfrak{b}$. To this end, in view of [6, Proposition 1], $\mathfrak{b}$ is a monomial ideal of R with respect to $\mathbf{x}$; and [4, Corollary 3.3] shows that $\mathfrak{b}$ is the irredundant intersection of the s parameter ideals $\mathbf{P}(f_i)$ for every $i = 1, \ldots, s$. Now, let g be a non-zero monomial of R with respect to $\mathbf{x}$ such that $g \in \mathfrak{a}$ and $g \notin \mathfrak{b}$. Then, there exists i, $1 \leq i \leq s$ such that $g \notin \mathbf{P}(f_i)$. Hence, according to [4, Lemma 2.3], $f_i \in (g)$, and so $f_i \in \mathfrak{a}$, which is a contradiction. Thus $\mathfrak{a} \subseteq \mathfrak{b}$. In order to show the reverse inclusion, suppose on the contrary that $\mathfrak{b}$ is not a subset of $\mathfrak{a}$. Then there exists a monomial $f \in \mathfrak{b}$ such that $f \notin \mathfrak{a}$. Hence, in view of [4, Remark 3.15] there is a monomial $g \in R$ such that gf is an $\mathfrak{a}$-corner-element. Now, as $gf \in \mathfrak{b}$, it follows that $gf = f_i$ for some $i = 1, \ldots, s$. Therefore $f_i \in \mathbf{P}(f_i)$; and so by virtue of [4, Lemma 2.3] $f_i \notin (f_i)$, which is a contradiction. Therefore $\mathfrak{a} = \mathfrak{b}$, as required. □

Proposition 2.6. *Let R be a noetherian ring, let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence, and suppose that $\mathfrak{a}$ is a monomial ideal with respect to $\mathbf{x}$. Assume that f is a monomial with respect to $\mathbf{x}$. Then $\mathfrak{a} \subseteq \mathbf{P}(f)$ if and only if $f \notin \mathfrak{a}$.*

Proof. ($\Rightarrow$) Let $\mathfrak{a} \subseteq \mathbf{P}(f)$. We show that $f \notin \mathfrak{a}$. Suppose to the contrary that $f \in \mathfrak{a}$. Then $f \in \mathbf{P}(f)$, and so in view of [4, Lemma 2.3], $f \notin (f)$, which is a contradiction.

($\Leftarrow$) Let $f = x_1^{e_1} \cdots x_d^{e_d}$, and let $f \notin \mathfrak{a}$. Suppose that $\mathfrak{a} = (g_1, \ldots, g_s)$, where g_i is a monomial with respect to $\mathbf{x}$, for all $i = 1, \ldots, s$. Then $f \notin (g_1, \ldots, g_s)$, and so, in view of [4, Remark 2.2], $f \notin (g_i)$ for all $i = 1, \ldots, s$. Hence, it follows from [4, Lemma 2.3] that $g_i \in P(f)$ for all $i = 1, \ldots, s$, and so $\mathfrak{a} \subseteq \mathbf{P}(f)$, as required. □

Corollary 2.7. *Let R be a noetherian ring, let $\mathbf{x} := x_1, \ldots, x_d$ be an R-regular sequence, and let f, g be two monomials with respect to $\mathbf{x}$. Then the following conditions are equivalent*:

(i) $f \in (g)$,
(ii) $g \notin \mathbf{P}(f)$,
(iii) $\mathbf{P}(f) \subseteq \mathbf{P}(g)$,
(iv) $(\mathbf{P}(f) :_R g) \neq R$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from [4, Lemma 2.3], and (ii) $\Leftarrow$ (iii) follows from Proposition 2.6. In order to show the conclusion (iii) $\Rightarrow$ (iv), suppose on the contrary that $\mathbf{P}(f) :_R g = R$. Then $g \in \mathbf{P}(f)$, and so $g \in \mathbf{P}(g)$. Hence, in view of [4, Lemma 2.3] we have $g \notin (g)$, which is a contradiction.

Finally, in order to show (iv) $\Rightarrow$ (i), suppose that $f \notin (g)$. Then, according to [4, Lemma 2.3], $g \in \mathbf{P}(f)$, and so $(\mathbf{P}(f) :_R g) = R$, which is a contradiction. □

Lemma 2.8. *Let R be a noetherian ring, let $\mathbf{x} := x_1, \dots, x_d$ be an R-regular sequence and let f, g be two monomials with respect to $\mathbf{x}$. Then the following conditions hold*:

(i) *If $f \in (g)$, then* $\deg(f) \geq \deg(g)$.

(ii) *If* $\deg(f) = \deg(g)$ *and $g \in (f)$, then $g = f$.*

(iii) *If* $\deg(f) = \deg(g)$ *and $f \neq g$, then $f \in \mathbf{P}(g)$.*

Proof. (i) Let $f = x_1^{a_1} \cdots x_d^{a_d}$ and $g = x_1^{b_1} \cdots x_d^{b_d}$ and $f \in (g)$. Then in view of [4, Lemma 2.3], $g \notin \mathbf{P}(f)$. That is

$$x_1^{b_1} \cdots x_d^{b_d} \notin (x_1^{a_1+1}, \dots, x_d^{a_d+1}).$$

Hence $a_1 \geq b_1, \dots, a_d \geq b_d$, and so $\deg(f) \geq \deg(g)$, as required.

The part (ii) readily follows from the definition. Finally, in order to show (iii), suppose that $f \notin \mathbf{P}(g)$, then in view of [4, Lemma 2.3], we have $g \in (f)$. Hence, it follows from part (ii) that $f = g$, which is a contradiction. □

Lemma 2.9. *Let R be a noetherian ring and let $\mathbf{x} := x_1, \dots, x_d$ be an R-regular sequence. Suppose that f is a monomial with respect to $\mathbf{x}$ and let $n \geq 1$ be an integer. Then* $\deg(f) < n$ *if and only if there exists a monomial g with respect to $\mathbf{x}$ of degree $n-1$ such that $g \in (f)$.*

Proof. Let f be a monomial with respect to $\mathbf{x}$ and $n \geq 1$ an integer such that $\deg(f) < n$. Let

$$f = x_1^{e_1} \cdots x_d^{e_d} \text{ and } g = x_1^{n-(e_2+\cdots+e_d+1)} x_2^{e_2} \cdots x_d^{e_d}.$$

Then $\deg(g) = n-1$ and that $g \in (f)$. Note that $n - 1 \geq e_1 + \cdots + e_d$.

Conversely, let g be a monomial with respect to $\mathbf{x}$ such that $\deg(g) = n-1$ and $g \in (f)$. It follows from Lemma 2.8 that $\deg(g) \geq \deg(f)$. Therefore $\deg(f) \leq n-1$, as required. □

We end this section with the following final main result of the paper.

Theorem 2.10. *Let R be a noetherian ring, let $\mathbf{x} := x_1, \dots, x_d$ be an R-regular sequence, and suppose that $\mathfrak{q} := (x_1, \dots, x_d)$. Then, for any integer $n \geq 1$, we have*

$$\mathfrak{q}^n = \cap_{\deg(f)=n-1} \mathbf{P}(f),$$

where the intersection is taken over all monomials f with respect to $\mathbf{x}$ such that $\deg(f) = n-1$. *Moreover, this intersection is irredundant.*

Proof. Let $\mathfrak{a} = \bigcap_{\deg(f)=n-1} \mathbf{P}(f)$, where the intersection runs over all monomials f such that $\deg(f) = n-1$, and we show $\mathfrak{a} = \mathfrak{q}^n$. To this end, since each ideal $\mathbf{P}(f)$ is a monomial ideal with respect to $\mathbf{x}$, it follows from [6, Lemma 3] that $\mathfrak{a}$ is also a monomial ideal with respect to $\mathbf{x}$. Thus, in order to show $\mathfrak{a} = \mathfrak{q}^n$, it is enough for us to show that, if g is a monomial with respect to $\mathbf{x}$ in R, then $g \in \mathfrak{a}$ if and only if $g \in \mathfrak{q}^n$.

To do this, we have $g \notin \mathfrak{a}$ if and only if there exists a monomial f of degree $n-1$ such that $g \notin \mathbf{P}(f)$, by definition of $\mathfrak{a}$, that is, if and only if there exists a monomial f of degree $n-1$ such that $f \in (g)$. But Lemma 2.9 shows that this condition holds if and only if $\deg(g) \leq n-1$, and this is so if and only if $g \notin \mathfrak{q}^n$, by the definition of $\mathfrak{q}^n$.

To see that the intersection is irredundant, let g and f be distinct monomials with $\deg(g) = \deg(f) = n-1$. Now, Lemma 2.8 shows that $f \in \mathbf{P}(g)$ and so by Corollary 2.7, we have $\mathbf{P}(g)$ is not a subset of $\mathbf{P}(f)$. This completes the proof. $\square$

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