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INVARIANT MEAN VALUE PROPERTY AND $\mathcal{M}$-HARMONICITY ON THE HALF-SPACE

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Abstract. It is well known that every invariant harmonic function on the unit ball of the multi-dimensional complex space has the volume version of the invariant mean value property. In 1993 Ahern, Flores and Rudin first observed that the validity of the converse depends on the dimension of the underlying complex space. Later Lie and Shi obtained the analogues on the unit ball of multi-dimensional real space. In this paper we obtain the half-space analogues of the results of Liu and Shi.

1. Introduction

As is well known on the setting of the unit ball of the multi-dimensional complex space, the invariant harmonic (=harmonic with respect to the Bergman metric) functions satisfy the invariant mean value property; see [6, Section 3.3.6] for precise definition of the invariant mean value property. Conversely, provided that functions under consideration are continuous, the invariant mean value property implies the invariant harmonicity. Of course, the invariant mean value property yields its volume version, which we may refer to as the invariant volume mean value property. In 1993 Ahern, Flores and Rudin [1] investigated whether the invariant volume mean value property implies the invariant harmonicity. They answered in the positive for bounded functions. To our great surprise, when the boundedness is relaxed to integrability, they also discovered a cut-off phenomenon for the dimension d of the underlying complex space. More explicitly, when functions under consideration are just integrable, they showed that the invariant volume mean value property implies the invariant harmonicity if and only if $d \leq 11$.

In the setting of the unit ball of the multi-dimensional real space, Liu and Shi [5] obtained the real analogues of what are mentioned in the preceding

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paragraph; see Section 2.6 for precise statements. The purpose of the current paper is to obtain the analogues in the setting of the upper half-space.

To begin with, we set some notation and terminology. For a fixed positive integer $n > 1$, let $\mathbf{H} = \mathbf{R}^{n-1} \times \mathbf{R}_+$ be the upper half-space in the real n-space $\mathbf{R}^n$ where $\mathbf{R}_+$ denotes the set of all positive real numbers. We will often write a typical point $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$.

Given $z \in \mathbf{H}$, we denote by $\phi_z : \mathbf{H} \to \mathbf{H}$ the function defined by

$$\phi_z(w) := z_n w + (z', 0)$$

for $w \in \mathbf{H}$. Assume $n > 2$ for a moment. Given γ real, associated with the Riemannian metric $ds^2 = z_n^{2\gamma/(n-2)} \sum_{j=1}^n dz_j^2$ on $\mathbf{H}$ is the Laplace-Beltrami operator (see [3])

$$L_\gamma := z_n^{2\gamma/(2-n)} \left[\Delta + \frac{\gamma}{z_n} \frac{\partial}{\partial z_n} \right]. \tag{1.1}$$

Here, and elsewhere, Δ denotes the ordinary Laplacian on $\mathbf{R}^n$. One may check by straightforward calculation

$$L_\gamma(f \circ \phi_z) = z_n^{2(1-\frac{\gamma}{2-n})} L_\gamma f \circ \phi_z$$

and, in particular,

$$L_{2-n}(f \circ \phi_z) = (L_{2-n} f) \circ \phi_z \tag{1.2}$$

for $z \in \mathbf{H}$ and $f \in C^2(\mathbf{H})$.

Motivated by the observation in the preceding paragraph, we set

$$\widetilde{\Delta} = \widetilde{\Delta}_{\mathbf{H}} := z_n^2 \left[\Delta + \frac{2-n}{z_n} \frac{\partial}{\partial z_n} \right]$$

for $n \geq 2$. Note from (1.2)

$$\widetilde{\Delta}(f \circ \phi_z) = (\widetilde{\Delta} f) \circ \phi_z \tag{1.3}$$

for $n > 2$; this is also quite elementary for $n = 2$. Following [5] and [6], we now say that a function $f \in C^2(\mathbf{H})$ is $\mathcal{M}$-*harmonic* if f is annihilated by $\widetilde{\Delta}$. Note that the notions of the $\mathcal{M}$-harmonicity and the ordinary harmonicity coincide for $n = 2$, but not for $n > 2$. Also, note from (1.3) that the $\mathcal{M}$-harmonicity on $\mathbf{H}$ is invariant under composition with the maps ϕ_z. In this sense we refer to $\widetilde{\Delta}$ as the *invariant* Laplacian on $\mathbf{H}$.

In order to state our main results, we introduce more notation. Put

$$\mathbf{e} := (0, \ldots, 0, 1) \in \mathbf{H}$$

for a standard reference point in $\mathbf{H}$. Denote by $\mathbf{B}$ and $\mathbf{S} := \partial \mathbf{B}$ the unit ball and the unit sphere of $\mathbf{R}^n$, respectively. Given $z \in \mathbf{H}$ and $0 < r < 1$, let $E_r(z)$ be the pseudohyperbolic ball, which is to be defined in Section 2.3, with radius

r and center z. Let ω_r be the weighted surface area measure on $\partial E_r(\mathbf{e})$ given by

$$d\omega_r(\zeta) := \frac{1}{|r\mathbf{S}|}\left(\frac{2}{|\zeta+\mathbf{e}|^2}\right)^{n-1} d\nu_r(\zeta), \qquad \zeta \in \partial E_r(z), \tag{1.4}$$

where $|r\mathbf{S}|$ is the surface area of $r\mathbf{S}$, i.e., $|r\mathbf{S}| := r^{n-1}|\mathbf{S}|$ and ν_r denotes the surface area measure on $\partial E_r(\mathbf{e})$. Finally, we denote by μ the positive finite measure on $\mathbf{H}$ given by

$$d\mu(w) := \frac{1}{|\mathbf{B}|}\left(\frac{2}{|w+\mathbf{e}|^2}\right)^{n} dw, \qquad w \in \mathbf{H},$$

where $|\mathbf{B}|$ is the volume of $\mathbf{B}$.

We are now ready to introduce the notions of the invariant (volume) mean value property on $\mathbf{H}$. We say that a function $f \in C(\mathbf{H})$ has the *invariant* $\mathbf{H}$-*mvp*(=*mean value property*) if

$$f(z) = \int_{\partial E_r(\mathbf{e})} (f \circ \phi_z)\, d\omega_r \tag{1.5}$$

for $z \in \mathbf{H}$ and $0 < r < 1$. For the motivation of this definition, see Proposition 3.1 below. Also, we say that a function $f \in L^1(\mu)$ has the *invariant* $\mathbf{H}$-*vmvp*(=*volume mvp*) if

$$\int_{\mathbf{H}} (f \circ \phi_z)\, d\mu = f(z) \tag{1.6}$$

for $z \in \mathbf{H}$. For the motivation of this definition, see Proposition 4.1. Note $\phi_{\phi_w(z)} = \phi_w \circ \phi_z$. Thus the mean value properties defined above are invariant under composition with the maps ϕ_z. This is why we use the term "invariant" in the above definitions.

In this paper we obtain the following results:

(a) For continuous functions on $\mathbf{H}$, the invariant $\mathbf{H}$-mvp coincides with the $\mathcal{M}$-harmonicity;
(b) For essentially bounded functions on $\mathbf{H}$, the invariant $\mathbf{H}$-vmvp implies the $\mathcal{M}$-harmonicity;
(c) For μ-integrable functions on $\mathbf{H}$, the invariant $\mathbf{H}$-vmvp implies the $\mathcal{M}$-harmonicity if and only if $n \le 12$.

In Section 2 we collect some well-known facts and auxiliary results. In Section 3 we show Assertion (a). In Section 4 we prove results that contain Assertions (b) and (c) as special cases.

2. Preliminaries

In this section we recall some basic facts and collect some auxiliary results. The notation $x \cdot y$ will stand for the inner product of $x, y \in \mathbf{R}^n$.

2.1. Inversion relative to the unit sphere

Let $\Lambda(x) = x^*$ be the inversion of x relative to $\mathbf{S}$, i.e.,

$$\Lambda(x) = x^* := \begin{cases} \frac{x}{|x|^2} & \text{if } x \neq 0, \infty, \\ 0 & \text{if } x = \infty, \\ \infty & \text{if } x = 0. \end{cases} \tag{2.1}$$

For $x \in \mathbf{R}^n \setminus \{0\}$, let $Q(x)$ be the $n \times n$ matrix with the entries

$$Q(x)_{ij} = \frac{x_i x_j}{|x|^2}, \qquad i, j = 1, 2, \ldots, n$$

and put

$$U(x) := I - 2Q(x),$$

where I is the $n \times n$ identity matrix. Note $Q(x)^2 = Q(x)$ and thus $U(x)^2 = I$. Accordingly, $U(x)$ is an orthogonal matrix. By a straightforward calculation we have

$$\Lambda'(x) = \frac{1}{|x|^2} U(x). \tag{2.2}$$

Thus $\Lambda'(x)$ is a conformal matrix with scaling factor $|x|^{-2}$. We refer to [2, p. 18] for details.

2.2. Möbius transformations on B

We recall the canonical Möbius transformations on $\mathbf{B}$. All relevant details and related results can be found in [2, pp. 17–30].

Given $a \in \mathbf{B}$, the canonical Möbius transformation λ_a on $\mathbf{B}$ that exchanges a and 0 is given by

$$\lambda_a(x) = a + (1 - |a|^2)(a - x^*)^*, \qquad x \in \mathbf{B}; \tag{2.3}$$

note $\lambda_a = -T_a$ in the notation of [2]. Avoiding the $*$-notation, we have

$$\lambda_a(x) = \frac{(1 - |a|^2)(a - x) + |a - x|^2 a}{[a, x]^2},$$

where

$$[a, x] := \sqrt{1 - 2a \cdot x + |a|^2 |x|^2}.$$

The map λ_a is an involution of $\mathbf{B}$, i.e., $\lambda_a^{-1} = \lambda_a$. As is well known, these maps and orthogonal transformations on $\mathbf{R}^n$ generate all Möbius transformations on $\mathbf{B}$.

Differentiating (2.3) via (2.2), we have

$$\lambda_a'(x) = \frac{|a|^2 - 1}{[a, x]^2} U(x) U(a - x^*). \tag{2.4}$$

Thus $\lambda_a'(x)$ is a conformal matrix with scaling factor $\frac{|a|^2 - 1}{[a, x]^2}$.

2.3. Pseudohyperbolic distances

We recall the well-known pseudohyperbolic distances on the ball and the half-space.

In the setting of the ball $\mathbf{B}$, the pseudohyperbolic distance $\rho_{\mathbf{B}}$ is defined by

$$\rho_{\mathbf{B}}(x,y) := |\lambda_x(y)| = \frac{|x-y|}{[x,y]}, \qquad x,y \in \mathbf{B};$$

see [2, p. 27] for the second equality. As is well-known, $\rho_{\mathbf{B}}$ is Möbius invariant, i.e.,

$$\rho_{\mathbf{B}}\big(\lambda_a(x),\lambda_a(y)\big) = \rho_{\mathbf{B}}(x,y) \tag{2.5}$$

for all $a,x,y \in \mathbf{B}$. For $a \in \mathbf{B}$ and $0 < r < 1$, we denote by $D_r(a)$ the pseudohyperbolic ball with radius r and center a. Note

$$\lambda_a[D_r(b)] = D_r\big(\lambda_a(b)\big) \tag{2.6}$$

for all $a,b \in \mathbf{B}$ by the Möbius invariance (2.5). A straightforward calculation shows that the pseudohyperbolic ball $D_r(a)$ is a Euclidean ball with

$$(\text{center}) = \frac{(1-r^2)}{1-|a|^2r^2}a \quad \text{and} \quad (\text{radius}) = \frac{(1-|a|^2)r}{1-|a|^2r^2}. \tag{2.7}$$

In the setting of the half-space $\mathbf{H}$, the pseudohyperbolic distance $\rho_{\mathbf{H}}$ is defined by

$$\rho_{\mathbf{H}}(z,w) := \frac{|z-w|}{|z-\overline{w}|}, \qquad z,w \in \mathbf{H}.$$

Here, and in what follows, we use the notation

$$\overline{w} := (w', -w_n)$$

for $w \in \mathbf{H}$. For $z \in \mathbf{H}$ and $0 < r < 1$, we denote by $E_r(z)$ the pseudohyperbolic ball with radius r and center z. The pseudohyperbolic ball $E_r(z)$ is also a Euclidean ball with

$$(\text{center}) = \left(z', \frac{1+r^2}{1-r^2}z_n\right) \quad \text{and} \quad (\text{radius}) = \frac{2r}{1-r^2}z_n.$$

In particular, we have

$$\partial E_r(\mathbf{e}) = \frac{1+r^2}{1-r^2}\mathbf{e} + \frac{2r}{1-r^2}\mathbf{S}.$$

So, the surface area measure ν_r on $\partial E_r(\mathbf{e})$ can be describes in terms of the surface area measure, denoted by σ, on $\mathbf{S}$ normalized to have total mass 1. More explicitly, ν_r is determined by the equation

$$\begin{aligned} &\int_{\partial E_r(\mathbf{e})} \psi(\zeta)\, d\nu_r(\zeta) \\ &= |\mathbf{S}| \left(\frac{2r}{1-r^2}\right)^{n-1} \int_{\mathbf{S}} \psi\left(\frac{1+r^2}{1-r^2}\mathbf{e} + \frac{2r}{1-r^2}\zeta\right) d\sigma(\zeta) \end{aligned} \tag{2.8}$$

for functions ψ continuous on $\partial E_r(\mathbf{e})$. Note

$$\left(\frac{1+r^2}{1-r^2}\mathbf{e}+\frac{2r}{1-r^2}\zeta\right)+\mathbf{e}=\frac{2}{1-r^2}(r\zeta+\mathbf{e}).$$

It follows from (1.5) and (2.8) that $f\in C(\mathbf{H})$ has the invariant $\mathbf{H}$-mvp if

$$f(z)=\int_{\mathbf{S}}(f\circ\phi_z)\left(\frac{1+r^2}{1-r^2}\mathbf{e}+\frac{2r}{1-r^2}\zeta\right)\left(\frac{1-r^2}{|r\zeta+\mathbf{e}|^2}\right)^{n-1}d\sigma(\zeta)$$

for $z\in\mathbf{H}$ and $0<r<1$.

2.4. A Möbius transformation

Consider a Möbius transformation T defined by

$$T(z):=2\Lambda(\overline{z+\mathbf{e}})+\mathbf{e},\qquad z\in\mathbf{H}. \tag{2.9}$$

Differentiating this, we have

$$T'(z)=2\Lambda'(\overline{z+\mathbf{e}})R, \tag{2.10}$$

where R is the orthogonal matrix representing the reflection $z\mapsto\overline{z}$. Thus, we see from (2.2) that $T'(z)$ is a conformal matrix with scaling factor $\frac{2}{|z+\mathbf{e}|^2}$.

Meanwhile, we have by straightforward calculation

$$|T(z)|=\frac{|z-\mathbf{e}|}{|z+\mathbf{e}|} \tag{2.11}$$

so that

$$1-|T(z)|^2=\frac{4z_n}{|z+\mathbf{e}|^2}. \tag{2.12}$$

This shows that T takes $\mathbf{H}$ onto $\mathbf{B}$. Being a Möbius transformation, one may expect $T:\mathbf{H}\to\mathbf{B}$ to preserve pseudohyperbolic distances, which is actually the case as in the next lemma.

Lemma 2.1. *The identity*

$$\rho_{\mathbf{H}}(z,w)=\rho_{\mathbf{B}}\big(T(z),T(w)\big)$$

holds for $z,w\in\mathbf{H}$.

Proof. It suffices to check the identities

$$|T(z)-T(w)|=\frac{2|z-w|}{|z+\mathbf{e}||w+\mathbf{e}|} \tag{2.13}$$

and

$$[T(z),T(w)]=\frac{2|z-\overline{w}|}{|z+\mathbf{e}||w+\mathbf{e}|} \tag{2.14}$$

for $z,w\in\mathbf{H}$.

Note from the definition of T

$$T(z)\cdot T(w)=\frac{4(z\cdot w-z_nw_n)+(|z|^2-1)(|w|^2-1)}{|z+\mathbf{e}|^2|w+\mathbf{e}|^2}.$$

Also, note from (2.11)

$$\begin{aligned}|T(z)|^2 + |T(w)|^2 &= \frac{|z-\mathbf{e}|^2}{|z+\mathbf{e}|^2} + \frac{|w-\mathbf{e}|^2}{|w+\mathbf{e}|^2} \\ &= 2 \cdot \frac{(|z|^2+1)(|w|^2+1) - 4z_n w_n}{|z+\mathbf{e}|^2|w+\mathbf{e}|^2}.\end{aligned}$$

Now, (2.13) holds by the above two identities.

Next, using (2.12) and (2.13), we obtain

$$\begin{aligned}[T(z), T(w)]^2 &= |T(z) - T(w)|^2 + (1 - |T(z)|^2)(1 - |T(w)|^2) \\ &= \frac{4|z-w|^2 + 16 z_n w_n}{|z+\mathbf{e}|^2|w+\mathbf{e}|^2} \\ &= \frac{4|z-\overline{w}|^2}{|z+\mathbf{e}|^2|w+\mathbf{e}|^2}\end{aligned}$$

and thus (2.14) holds. The proof is complete. □

2.5. $\mathcal{M}$-harmonic functions on B

The *invariant* Laplacian $\widetilde{\Delta}_{\mathbf{B}}$ on $\mathbf{B}$ is defined by

$$\widetilde{\Delta}_{\mathbf{B}} = \frac{(1-|x|^2)^2}{4}\left[\Delta + \frac{2(n-2)}{1-|x|^2}\mathcal{R}\right], \qquad x \in \mathbf{B},$$

where $\mathcal{R} := \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$ denotes the radial differentiation. We say that a function $g \in C^2(\mathbf{B})$ is $\mathcal{M}$-*harmonic* on $\mathbf{B}$ if g is annihilated by $\widetilde{\Delta}_{\mathbf{B}}$. As in the case of the half-space, $\widetilde{\Delta}_{\mathbf{B}}$ commutes with composition with the maps λ_a

$$\widetilde{\Delta}_{\mathbf{B}}(g \circ \lambda_a) = (\widetilde{\Delta}_{\mathbf{B}} g) \circ \lambda_a;$$

see [4, Eq. (1.2)]. So, the $\mathcal{M}$-harmonicity on $\mathbf{B}$ is invariant under composition with Möbius transformations, which is the reason why we use the term "invariant" for the operator $\widetilde{\Delta}_{\mathbf{B}}$. Furthermore, the invariant Laplacians $\widetilde{\Delta}_{\mathbf{B}}$, $\widetilde{\Delta}_{\mathbf{H}}$ and the Möbius transformation T are intimately related by

$$\widetilde{\Delta}_{\mathbf{H}}(g \circ T) = (\widetilde{\Delta}_{\mathbf{B}} g) \circ T; \tag{2.15}$$

see [4, Eq. (1.5)]. This can be verified through straightforward calculation via the explicit formula for (2.9):

$$T(z) = \left(\frac{2z'}{|z+\mathbf{e}|^2}, \frac{|z|^2 - 1}{|z+\mathbf{e}|^2}\right)$$

for $z \in \mathbf{H}$.

2.6. Results of Liu and Shi

We say that a function $g \in C(\mathbf{B})$ has *invariant* $\mathbf{B}$-*mvp* if

$$g(a) = \int_{\mathbf{S}} (g \circ \lambda_a)(r\zeta)\, d\sigma(\zeta) \tag{2.16}$$

for $a \in \mathbf{B}$ and $0 < r < 1$. In this definition, one may actually require more generally that

$$(g \circ \lambda)(0) = \int_{\mathbf{S}} (g \circ \lambda)(r\zeta)\, d\sigma(\zeta)$$

for all Möbius transformations λ on $\mathbf{B}$ and $0 < r < 1$. Note that these two definitions coincide, because Möbius transformations on $\mathbf{B}$ are generated by orthogonal transformations and canonical Möbius transformations λ_a.

As in the case of ordinary harmonicity, it turns out that the notions for continuous functions on $\mathbf{B}$ of the $\mathcal{M}$-harmonicity and the invariant $\mathbf{B}$-mvp coincide. See Liu and Shi [5] for a proof.

Theorem 2.2. *Every $\mathcal{M}$-harmonic function on $\mathbf{B}$ has the invariant $\mathbf{B}$-mvp. Conversely, if a continuous function on $\mathbf{B}$ has the invariant $\mathbf{B}$-mvp, then it is $\mathcal{M}$-harmonic on $\mathbf{B}$.*

Given $\alpha > -1$, let v_α be the standard α-weighted measure on $\mathbf{B}$ given by

$$dv_\alpha(x) := c_\alpha (1 - |x|^2)^\alpha\, dx, \qquad x \in \mathbf{B},$$

where the constant $c_\alpha := \frac{1}{|\mathbf{B}|} \cdot \frac{\Gamma(n/2+\alpha+1)}{\Gamma(n/2+1)\Gamma(\alpha+1)}$ is chosen so that v_α has total mass 1.

Multiply by $r^{n-1}(1-r^2)^\alpha$ the both sides of (2.16) and then integrate against the measure dr on $[0, 1)$. The resulting equality then reduces to

$$g(a) = \int_{\mathbf{B}} (g \circ \lambda_a)\, dv_\alpha =: B_\alpha^{\mathbf{B}} g(a) \tag{2.17}$$

for $a \in \mathbf{B}$. In other words, g is fixed by the transform $B_\alpha^{\mathbf{B}}$, often called the *α-Berezin transform* on $\mathbf{B}$. We see from (2.17) and Theorem 2.2 that

$$\mathcal{M}\text{-harmonicity on } \mathbf{B} \implies \alpha\text{-invariant } \mathbf{B}\text{-vmvp} \tag{2.18}$$

for each $\alpha > -1$.

Note that α-Berezin transform $B_\alpha^{\mathbf{B}}$ can be applied to v_α-integrable functions. We say that a function $g \in L^1(v_\alpha)$ have the *α-invariant* $\mathbf{B}$-*vmvp*, if it is fixed by $B_\alpha^{\mathbf{B}}$. The 0-invariant $\mathbf{B}$-vmvp is simply called the *invariant* $\mathbf{B}$-*vmvp*.

The following two theorems are due to Liu and Shi [5].

Theorem 2.3. *Let $\alpha > -1$. If $g \in L^\infty(v_\alpha)$ has the α-invariant $\mathbf{B}$-vmvp, then g is $\mathcal{M}$-harmonic on $\mathbf{B}$.*

Theorem 2.4. *Let m be a nonnegative integer. Then the m-invariant $\mathbf{B}$-vmvp of an arbitrary function $g \in L^1(v_m)$ implies the $\mathcal{M}$-harmonicity of g on $\mathbf{B}$ if and only if $n + 2m \leq 12$.*

The special case $m = 0$ of the above theorem might be of independent interest: *the invariant* $\mathbf{B}$*-vmvp of an arbitrary function* $g \in L^1(v_0)$ *implies the* $\mathcal{M}$*-harmonicity of* g *on* $\mathbf{B}$ *if and only if* $n \leq 12$. This is the real analogues of results of Ahern, Flores and Rudin [1] mentioned in the Introduction.

3. Invariant mean value property

In this section, we show that the notions for continuous functions on $\mathbf{H}$ of the $\mathcal{M}$-harmonicity and the invariant $\mathbf{H}$-mvp coincide, which is the half-space analogue of Theorem 2.2.

We set some notation for simplicity. For $0 < r < 1$, put

$$M_r^{\mathbf{B}} g(a) := \int_{\mathbf{S}} (g \circ \lambda_a)(r\zeta)\, d\sigma(\zeta), \qquad a \in \mathbf{B}$$

for $g \in C(\mathbf{B})$ and

$$M_r^{\mathbf{H}} f(z) := \int_{\partial E_r(\mathbf{e})} (f \circ \phi_z)(\zeta)\, d\omega_r(\zeta), \qquad z \in \mathbf{H}$$

for $f \in C(\mathbf{H})$; recall that ω_r is the measure introduced in (1.4).

With the notation introduced above, we see that a function $g \in C(\mathbf{B})$ has the invariant $\mathbf{B}$-mvp if and only if $M_r^{\mathbf{B}} g = g$ for each $0 < r < 1$. Similarly, a function $f \in C(\mathbf{H})$ has the invariant $\mathbf{H}$-mvp if and only if $M_r^{\mathbf{H}} f = f$ for each $0 < r < 1$.

Proposition 3.1. *The identity*

$$M_r^{\mathbf{H}}(g \circ T) = (M_r^{\mathbf{B}} g) \circ T$$

holds for $g \in C(\mathbf{B})$ *and* $0 < r < 1$.

Proof. Let $z \in \mathbf{H}$ and $0 < r < 1$. Put $a := T(z) \in \mathbf{B}$. Note from (2.6)

$$\lambda_a(r\mathbf{B}) = \lambda_a(D_r(0)) = D_r(a)$$

and hence

$$\lambda_a(r\mathbf{S}) = \partial D_r(a).$$

With this in mind consider the function $F : \partial D_r(a) \to \mathbf{S}$ defined by $F(\eta) := r^{-1}\lambda_a(\eta)$. Note from (2.4) that $F'(\eta)$ is a conformal matrix with scaling factor $\frac{|a|^2-1}{r[a,\eta]^2}$. Thus, given $g \in C(\mathbf{B})$, we obtain by the change of variables $\xi = F(\eta)$

$$\begin{aligned} M_r^{\mathbf{B}} g(a) &= \int_{\mathbf{S}} (g \circ \lambda_a)(r\xi)\, d\sigma(\xi) \\ &= \frac{1}{|\mathbf{S}|} \int_{\partial D_r(a)} (g \circ \lambda_a)\big(rF(\eta)\big) \left(\frac{1-|a|^2}{r[a,\eta]^2} \right)^{n-1} d\tau_{r,a}(\eta) \\ &= \frac{1}{|\mathbf{S}|} \int_{\partial D_r(a)} g(\eta) \left(\frac{1-|a|^2}{r[a,\eta]^2} \right)^{n-1} d\tau_{r,a}(\eta), \end{aligned} \tag{3.1}$$

where $\tau_{r,a}$ is the surface area measure on $\partial D_r(a)$.

Recall from (2.10) that $T'(\omega)$ is a conformal matrix with scaling factor $\frac{2}{|\omega+\mathbf{e}|^2}$. So, denoting by $\nu_{r,z}$ the surface area measure on $\partial E_r(z)$, we see via the change of variables $\eta = T(\omega)$ and Lemma 2.1 that the expression in (3.1) is equal to

$$\frac{1}{|\mathbf{S}|}\int_{\partial E_r(z)} (g\circ T)(\omega)\left(\frac{1-|T(z)|^2}{r[T(z),T(\omega)]^2}\right)^{n-1}\left(\frac{2}{|\omega+\mathbf{e}|^2}\right)^{n-1} d\nu_{r,z}(\omega).$$

In conjunction with the integrand of this integral, we note from (2.11), (2.12) and (2.14)

$$\frac{1-|T(z)|^2}{[T(z),T(\omega)]^2|\omega+\mathbf{e}|^2} = \frac{z_n}{|z-\overline{\omega}|^2}.$$

Summarizing what have been observed so far, we have

$$(M_r^{\mathbf{B}} g\circ T)(z) = \frac{1}{|\mathbf{S}|}\int_{\partial E_r(z)} f(\omega)\left(\frac{2z_n}{r|z-\overline{\omega}|^2}\right)^{n-1} d\nu_{r,z}(\omega),$$

where $f = g\circ T$. Recall $E_r(z) = \phi_z\big(E_r(\mathbf{e})\big)$. Note for $\zeta \in \partial E_r(\mathbf{e})$

$$|z-\overline{\phi_z(\zeta)}| = z_n|\zeta+\mathbf{e}| \tag{3.2}$$

and $\phi_z'(\zeta) = z_n I$ where I is the $n\times n$ identity matrix. Also, note $\nu_{r,\mathbf{e}} = \nu_r$. So, the change of variables $\omega = \phi_z(\zeta)$ yields

$$\begin{aligned}(M_r^{\mathbf{B}} g\circ T)(z) &= \frac{1}{|\mathbf{S}|}\int_{\partial E_r(\mathbf{e})} (f\circ\phi_z)(\zeta)\left(\frac{2z_n^2}{r|z-\overline{\phi_z(\zeta)}|^2}\right)^{n-1} d\nu_r(\zeta)\\ &= \frac{1}{|r\mathbf{S}|}\int_{\partial E_r(\mathbf{e})} (f\circ\phi_z)(\zeta)\left(\frac{2}{|\zeta+\mathbf{e}|^2}\right)^{n-1} d\nu_r(\zeta),\end{aligned}$$

which is equal to $M_r^{\mathbf{H}} f(z)$ by definition (1.4) of the measure ω_r. The proof is complete. □

Proposition 3.2. *Let $f \in C(\mathbf{H})$. Then f has the invariant $\mathbf{H}$-mvp if and only if $f\circ T^{-1}$ has the invariant $\mathbf{B}$-mvp.*

Proof. This is immediate from Proposition 3.1. □

We now obtain the half-space analogue of Theorem 2.2.

Theorem 3.3. *Every $\mathcal{M}$-harmonic function on $\mathbf{H}$ has the invariant $\mathbf{H}$-mvp. Conversely, if a continuous function on $\mathbf{H}$ has the invariant $\mathbf{H}$-mvp, then it is $\mathcal{M}$-harmonic on $\mathbf{H}$.*

Proof. Note from (2.15) that $f \in C^2(\mathbf{H})$ is $\mathcal{M}$-harmonic on $\mathbf{H}$ if and only if $f\circ T^{-1}$ is $\mathcal{M}$-harmonic on $\mathbf{B}$. So, the theorem holds by Theorem 2.2 and Proposition 3.2. □

4. Invariant volume mean value property

In this section we obtain the half-space analogues of Theorems 2.3 and 2.4 concerning the volume version of the invariant mean value property.

Before proceeding, we first introduce some notation and terminology. Given $\alpha > -1$, we denote by μ_α the positive finite measure on $\mathbf{H}$ given by

$$d\mu_\alpha(z) := c_\alpha 2^{n+2\alpha} \frac{z_n^\alpha}{|z+\mathbf{e}|^{2(n+\alpha)}}\, dz, \qquad z \in \mathbf{H};$$

recall that c_α is the normalizing constant for the measure v_α on $\mathbf{B}$. Using this measure, we define α-Berezin transform $B_\alpha^{\mathbf{H}}$ by

$$B_\alpha^{\mathbf{H}} f(z) := \int_{\mathbf{H}} (f \circ \phi_z)\, d\mu_\alpha, \qquad z \in \mathbf{H}$$

for $f \in L^1(\mu_\alpha)$. Also, we say that a function $f \in L^1(\mu_\alpha)$ has the *α-invariant* $\mathbf{H}$*-vmvp* on $\mathbf{H}$, if it is fixed by $B_\alpha^{\mathbf{H}}$. So, the invariant $\mathbf{H}$-vmvp defined in (1.6) is the 0-invariant $\mathbf{H}$-vmvp.

It turns out that the Berezin transforms and the Möbius transformation T are intimately related as follows.

Proposition 4.1. *The identity*

$$B_\alpha^{\mathbf{H}}(g \circ T) = (B_\alpha^{\mathbf{B}} g) \circ T$$

holds for $\alpha > -1$ and $g \in L^1(v_\alpha)$.

Proof. Let $z \in \mathbf{H}$ and put $a := T(z) \in \mathbf{B}$. We note

$$1 - |\lambda_a(x)|^2 = \frac{(1-|a|^2)(1-|x|^2)}{[a,x]^2}, \qquad x \in \mathbf{B};$$

see [2, p. 27] for a proof. Also, recall from (2.4) that $\lambda_a'(x)$ is a conformal matrix with scaling factor $\frac{|a|^2-1}{[a,x]^2}$. Thus, by the change of variables $y = \lambda_a(x)$, we have

$$\begin{aligned}
(B_\alpha^{\mathbf{B}} g \circ T)(z) &= B_\alpha^{\mathbf{B}} g(a) \\
&= c_\alpha \int_{\mathbf{B}} (g \circ \lambda_a)(y)(1-|y|^2)^\alpha\, dy \\
&= c_\alpha \int_{\mathbf{B}} g(x)(1-|\lambda_a(x)|^2)^\alpha \left(\frac{1-|a|^2}{[a,x]^2}\right)^n dx \\
&= c_\alpha \int_{\mathbf{B}} g(x) \left(\frac{1-|a|^2}{[a,x]^2}\right)^{n+\alpha} (1-|x|^2)^\alpha\, dx.
\end{aligned}$$

Recall from (2.10) that $T'(\xi)$ is a conformal matrix with scaling factor $\frac{2}{|\xi+\mathbf{e}|^2}$. Thus, by the change of variables $x = T(\xi)$, the expression above is the same as

$$c_\alpha \int_{\mathbf{H}} f(\xi) \left(\frac{1-|T(z)|^2}{[T(z),T(\xi)]^2}\right)^{n+\alpha} (1-|T(\xi)|^2)^\alpha \left(\frac{2}{|\xi+\mathbf{e}|^2}\right)^n d\xi, \tag{4.1}$$

where $f := g \circ T$. In conjunction with this, we note from (2.12) and (2.14) that

$$\frac{(1-|T(\xi)|^2)^\alpha}{[T(z),T(\xi)]^{2(n+\alpha)}}\left(\frac{2}{|\xi+\mathbf{e}|^2}\right)^n = \frac{1}{2^n}\left(\frac{|z+\mathbf{e}|^2}{|z-\bar{\xi}|^2}\right)^{n+\alpha}\xi_n^\alpha$$

and

$$(1-|T(z)|^2)^{n+\alpha} = \left(\frac{4z_n}{|z+\mathbf{e}|^2}\right)^{n+\alpha}.$$

Also, note that (3.2) is valid with general $w \in \mathbf{H}$ in place of ζ. Thus the expression in (4.1) simplifies to

$$\begin{aligned} &c_\alpha 2^{n+2\alpha}\int_{\mathbf{H}} f(\xi)\left(\frac{z_n}{|z-\bar{\xi}|^2}\right)^{n+\alpha}\xi_n^\alpha\,d\xi \\ &= c_\alpha 2^{n+2\alpha}\int_{\mathbf{H}}(f\circ\phi_z)(w)\left(\frac{z_n}{|z-\overline{\phi_z(w)}|^2}\right)^{n+\alpha}(z_n w_n)^\alpha z_n^n\,dw \\ &= c_\alpha 2^{n+2\alpha}\int_{\mathbf{H}}(f\circ\phi_z)(w)\frac{w_n^\alpha}{|w+\mathbf{e}|^{2(n+\alpha)}}\,dw, \end{aligned}$$

which is equal to $B_\alpha^{\mathbf{H}} f(z)$. The proof is complete. □

Proposition 4.2. *Let $\alpha > -1$ and $f \in L^1(\mu_\alpha)$. Then f has the α-invariant $\mathbf{H}$-vmvp if and only if $f \circ T^{-1}$ has the α-invariant $\mathbf{B}$-vmvp.*

Proof. This is immediate from Proposition 4.1. □

As a consequence, we see that the $\mathcal{M}$-harmonicity on $\mathbf{H}$ implies the α-invariant $\mathbf{H}$-vmvp.

Theorem 4.3. *Let $\alpha > -1$ and assume that $f \in L^1(\mu_\alpha)$ is $\mathcal{M}$-harmonic on $\mathbf{H}$. Then f has the α-invariant $\mathbf{H}$-vmvp.*

Proof. Since f is $\mathcal{M}$-harmonic on $\mathbf{H}$, $f \circ T^{-1}$ is $\mathcal{M}$-harmonic on $\mathbf{B}$ by (2.15). So, $f \circ T^{-1}$ has the α-invariant $\mathbf{B}$-vmvp by (2.18) and thus has the α-invariant $\mathbf{H}$-vmvp by Proposition 4.2. The proof is complete. □

We are now ready to prove the half-space analogue of Theorem 2.3.

Theorem 4.4. *Let $\alpha > -1$. If $f \in L^\infty(\mu_\alpha)$ has the α-invariant $\mathbf{H}$-vmvp, then f is $\mathcal{M}$-harmonic on $\mathbf{H}$.*

Proof. Note $f \circ T^{-1} \in L^\infty(v_\alpha)$ for $f \in L^\infty(\mu_\alpha)$. Thus the theorem holds by Proposition 4.2, Theorem 2.3 and (2.15). □

For the half-space analogue of Theorem 2.4, we first note that composition with T has the following isometric property.

Lemma 4.5. *The identity*

$$\int_{\mathbf{B}} g\, dv_\alpha = \int_{\mathbf{H}} (g \circ T)\, d\mu_\alpha$$

for $\alpha > -1$ *and positive Borel functions* g *on* $\mathbf{B}$.

Proof. Making the change of variables $z = T^{-1}(x)$ (as in the proof of Proposition 4.1), we obtain by (2.12)

$$\int_{\mathbf{B}} g(x)\, dv_\alpha(x) = c_\alpha \int_{\mathbf{H}} (g \circ T)(z)(1 - |T(z)|^2)^\alpha \left(\frac{2}{|z + \mathbf{e}|^2} \right)^n dz$$

for positive Borel functions g on $\mathbf{B}$. So, the lemma holds by (2.12). The proof is complete. □

Theorem 4.6. *Let* m *be a nonnegative integer. Then the* m*-invariant* $\mathbf{H}$*-vmvp of an arbitrary function* $f \in L^1(\mu_m)$ *implies the* $\mathcal{M}$*-harmonicity of* f *on* $\mathbf{H}$ *if and only if* $n + 2m \leq 12$.

Proof. Note $f \circ T^{-1} \in L^1(v_m)$ for $f \in L^1(\mu_m)$ by Lemma 4.5. Thus the theorem holds by Proposition 4.2, Theorem 2.3 and (2.15). □

As is mentioned after Theorem 2.2, the special case $m = 0$ of the above theorem might be of independent interest: *the invariant* $\mathbf{H}$*-vmvp of an arbitrary function* $f \in L^1(\mu)$ *implies the* $\mathcal{M}$*-harmonicity of* f *on* $\mathbf{H}$ *if and only if* $n \leq 12$.

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