Bull. Korean Math. Soc. **58** (2021), No. 3, pp. 537–557
https://doi.org/10.4134/BKMS.b190805
pISSN: 1015-8634 / eISSN: 2234-3016

# CURVATURE ESTIMATES FOR GRADIENT EXPANDING RICCI SOLITONS

Liangdi Zhang

Abstract. In this paper, we investigate the curvature behavior of complete noncompact gradient expanding Ricci solitons with nonnegative Ricci curvature. For such a soliton in dimension four, it is shown that the Riemann curvature tensor and its covariant derivatives are bounded. Moreover, the Ricci curvature is controlled by the scalar curvature. In higher dimensions, we prove that the Riemann curvature tensor grows at most polynomially in the distance function.

## 1. Introduction

A complete Riemannian manifold $(M^n, g)$ is called a gradient expanding Ricci soliton if there exists a smooth function $f$ on $M^n$ such that the Ricci tensor $Ric$ of the metric $g$ satisfies the equation

$$Ric + Hess\ f = \lambda g \tag{1}$$

for some negative constant $\lambda$. The function $f$ is called a potential function of the expanding soliton. By scaling the metric $g$, one customarily normalizes $\lambda = -\frac{1}{2}$ so that

$$Ric + Hess\ f = -\frac{1}{2}g. \tag{2}$$

It is well-known that a compact gradient expanding Ricci soliton is necessarily an Einstein metric (see [8]). In this paper, we shall focus our attention on complete noncompact gradient expanding Ricci solitons.

In recent years, much effort has been devoted to study gradient expanding Ricci solitons. In dimension 3, P. Peterson and W. Wylie [12] proved that such a soliton with constant scalar curvature is a finite quotient of $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, or $\mathbb{H}^3$. For a 3-dimensional gradient expanding Ricci soliton with nonnegative Ricci curvature and integrable scalar curvature, i.e., $R \in L^1(M^3)$, G. Catino,

Received August 30, 2019; Revised August 12, 2020; Accepted December 9, 2020.
2010 *Mathematics Subject Classification.* Primary 53C21, 53C25.
*Key words and phrases.* Curvature estimate, gradient expanding Ricci soliton, nonnegative Ricci curvature.

P. Mastrolia and D. D. Monticelli [4] showed that it is isometric to a quotient of the Gaussian soliton $\mathbb{R}^3$.

Moreover, H. D. Cao et al. [1] proved that a 3-dimensional complete expanding gradient Ricci soliton with nonnegative Ricci curvature and divergence-free Bach tensor, i.e., $divB = 0$ is rotationally symmetric. In higher dimensions, they also obtained a classification theorem that a complete Bach-flat gradient expanding Ricci soliton with nonnegative Ricci curvature is rotationally symmetric. In 2017, G. Catino, P. Mastrolia and D. D. Monticelli [5] proved that a gradient expanding Ricci soliton with nonnegative Ricci curvature and fourth order divergence-free Weyl tensor, i.e., $div^4 W = 0$ has harmonic Weyl curvature.

For a complete noncompact expanding Ricci soliton with nonnegative Ricci curvature, Y. Deng and X. Zhu [6] proved that the scalar curvature is bounded and it attains the maximum at the unique equilibrium point. It is obvious that the Ricci curvature must be bounded.

Motivated by the work of Munteanu-Wang [10], Cao-Cui [2] and Munteanu-Wang [11], we study curvature estimates of complete noncompact gradient expanding Ricci solitons. In [10], O. Munteanu and J. Wang derived several curvature estimates for 4-dimensional complete noncompact gradient shrinking Ricci solitons with bounded scalar curvature. Under some conditions on the Ricci curvature and the scalar curvature, H. D. Cao and X. Cui [2] proved certain curvature estimates for 4-dimensional complete noncompact gradient steady Ricci solitons. In general dimensions, O. Munteanu and M. T. Wang [11] showed that a complete noncompact gradient shrinking Ricci soliton with bounded Ricci curvature satisfies that the Riemann curvature tensor grows at most polynomially in the distance function. The main theorems of this paper are following.

For 4-dimensional complete noncompact gradient expanding Ricci solitons with nonnegative Ricci curvature, the first theorem concerns the boundness of the Riemann curvature tensor and its covariant derivatives.

**Theorem 1.1.** *Let $(M^4, g, f)$ be a four-dimensional complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then the Riemann curvature tensor and its covariant derivatives are bounded.*

The second theorem provides that the Riemann curvature tensor can be controlled by the scalar curvature.

**Theorem 1.2.** *Let $(M^4, g, f)$ be a four-dimensional complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then for each $0 < a < 1$, there exists a universal constant $c > 0$ such that*

$$|Ric|^2 \leq cR^a. \tag{3}$$

In dimension $n$ $(n \geq 5)$, we prove that the Riemann curvature tensor of a complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature grows at most polynomially in the distance function.

**Theorem 1.3.** *Let $(M^n, g, f)$ be an $n$-dimensional $(n \geq 5)$ complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then the Riemann curvature tensor is at most polynomial growth in the distance function, i.e., there exist positive constants $b$ and $K$ so that*

$$|Rm|(x) \leq K(r(x)+1)^b, \tag{4}$$

*where $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$.*

*Remark* 1.4. From the proof of Theorem 1.3, we will see that $b$ depends only on $n$ and the upper bound of $Ric$, while $K$ depends only on $n$ and the volume of unit geodesic ball centered at $x$, i.e., $Vol(B_x(1))$.

The rest of this paper is organized as follows. In Section 2, we fix our notations and present some formulas needed in the proof of main theorems. In Section 3, we prove Theorem 1.1 and Theorem 1.2. We finish the proof of Theorem 1.3 in Section 4.

## 2. Preliminaries

Here is a well-known identity for expanding Ricci solitons by tracing (3) (see e.g. [3,8]).

$$R + \Delta f = -\frac{n}{2}. \tag{5}$$

Normalize the potential function $f$, up to an additive constant, by

$$R + |\nabla f|^2 + f = 0. \tag{6}$$

The following formula can be obtained by using the second Bianchi identity and the soliton equation (2) (see e.g. [7]).

$$\nabla_l R_{ijkl} = R_{ijkl} \nabla_l f. \tag{7}$$

Recall three elliptic equations for curvatures. We may refer to Peterson-Wylie [12] for detail proofs.

**Proposition 2.1.** *Let $(M^n, g_{ij}, f)$ $(n \geq 3)$ be a gradient expanding soliton. Then we have*

$$\Delta_f R = -R - 2|Ric|^2, \tag{8}$$

$$\Delta_f R_{ij} = -R_{ij} - 2R_{ikjl}R_{kl}, \tag{9}$$

*and*

$$\Delta_f Rm = -Rm + Rm * Rm, \tag{10}$$

*where $\Delta_f := \Delta - \nabla_{\nabla f}$ and $Rm * Rm$ denotes a finite number of terms involving quadratics in Riemann curvature $Rm$.*

We need the asymptotic behavior of the potential function of complete noncompact gradient expanding Ricci solitons.

**Proposition 2.2** (H. D. Cao et al. [1]). *Let $(M^n, g_{ij}, f)$ $(n \geq 3)$ be a complete noncompact gradient expanding soliton with nonnegative Ricci curvature. Then, there exist some constants $c_1 > 0$ and $c_2 > 0$ such that the potential function $f$ satisfies the estimates*

$$\frac{1}{4}(r(x) - c_1)^2 - c_2 \leq -f(x) \leq \frac{1}{4}(r(x) + 2\sqrt{-f(O)})^2, \tag{11}$$

*where $r(x)$ is the distance function from any fixed base point in $M^n$. In particular, $f$ is a strictly concave exhaustion function achieving its maximum at some interior point $O$, which we take as the base point, and the underlying manifold $M^n$ is diffeomorphic to $\mathbb{R}^n$.*

According to the result of Y. Deng and X. Zhu [6] mentioned in the introduction, we set $0 \leq Ric \leq c_0 g$ for some positive constant $c_0$ throughout the paper. Therefore, the scalar curvature $R$ satisfies $0 \leq R \leq nc_0$.

Define the set

$$D(r) := \{x \in M : -f(x) \leq r\}.$$

Let $\phi$ be a smooth nonnegative function defined on $\mathbb{R}^+$ so that $\phi(t) = 1$ on $[0, s]$ and $\phi(t) = 0$ on $[2s, \infty)$. We may choose $\phi$ so that

$$t^2(|\phi'(t)|^2 + |\phi''(t)|) \leq c$$

for some universal constant $c > 0$.

For $s \geq 1$, $D(s)$ is compact since $-f$ is of quadratic growth (see Proposition 2.2). We use $\phi(-f(x))$ as a cut-off function with support in $D(2s)\backslash D(s)$.

Since $0 \leq R \leq nc_0$, it follows from (5) that $|\Delta f| \leq \frac{n}{2} + nc_0$. Moreover, (6) implies that $|\nabla f| \leq \sqrt{-f} \leq \sqrt{2s}$ on $D(2s)\backslash D(s)$. It is obviously that

$$|\nabla \phi(-f)| \leq |\phi'||\nabla f| \leq \frac{c|\nabla f|}{s} \leq \frac{c}{\sqrt{s}} \leq c, \tag{12}$$

and

$$|\Delta_f \phi(-f)| = |\phi''|\nabla f|^2 - \phi' \Delta_f f| \leq \frac{c|\nabla f|^2}{s^2} + \frac{c}{s} \cdot |\frac{n}{2} - f| \leq c \tag{13}$$

on $D(2s)\backslash D(s)$.

## 3. The four-dimensional case

In this section, we derive certain curvature estimates for 4-dimensional gradient expanding Ricci solitons with nonnegative Ricci curvature. Throughout the section, $c > 0$ denotes some universal constant depending only on $c_0$.

First of all, we present the following key fact due to Munteanu-Wang [10] and Cao-Cui [2].

**Proposition 3.1.** *Let $(M^4, g, f)$ be a four-dimensional complete noncompact gradient expanding Ricci soliton. Then, there exists some universal constant*

*$c > 0$ such that*

$$|Rm| \le c\left(\frac{|\nabla Ric|}{|\nabla f|} + \frac{|Ric|^2+1}{|\nabla f|^2} + |Ric|\right). \tag{14}$$

*Proof.* This result follows from the same arguments as in the proof of Proposition 1.1 of [10] but without replacing $|\nabla f|^2$ by $f$ in their proof. □

**Lemma 3.2.** *Let $(M^4, g, f)$ be a four-dimensional complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then, outside a compact set, there exists some universal constant $c > 0$ such that*

$$|Rm| \le c\left(\frac{|\nabla Ric|}{|\nabla f|} + 1\right). \tag{15}$$

*Furthermore, we have*

$$|Rm| \le c(|\nabla Ric| + 1). \tag{16}$$

*Proof.* Since $0 \le R \le 4c_0$, it follows from (6) and Proposition 2.2 that

$$|\nabla f| \ge C_0 \tag{17}$$

for some constant $C_0 > 0$ outside a compact set.

Applying (17) and $0 \le Ric \le c_0 g$ to Proposition 3.1, we obtained (15) and (16) immediately. □

**Lemma 3.3.** *Let $(M^4, g, f)$ be a four-dimensional complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then, outside a compact set, there exist a constant $C_1 > 0$ and a universal constant $c > 0$ such that*

$$\Delta_f(|Rm| + C_1|Ric|^2) \ge \frac{1}{2}(|Rm| + C_1|Ric|^2)^2 - c. \tag{18}$$

*Proof.* From (10), we know that

$$\begin{aligned}\Delta_f|Rm|^2 &= 2|\nabla Rm|^2 + 2\langle Rm, \Delta_f Rm\rangle \\ &= 2|\nabla Rm|^2 - 2|Rm|^2 - Rm * Rm * Rm \\ &\ge 2|\nabla Rm|^2 - 2|Rm|^2 - c|Rm|^3.\end{aligned}$$

It follows from Kato's inequality immediately that

$$\Delta_f|Rm| \ge -|Rm| - c|Rm|^2 \ge |Rm|^2 - c(|Rm|^2+1). \tag{19}$$

Applying (16) to (19), we have

$$\Delta_f|Rm| \ge |Rm|^2 - c(|\nabla Ric|^2+1). \tag{20}$$

By direct computations, we obtain

$$\begin{aligned}\Delta_f|Ric|^2 &= 2|\nabla Ric|^2 + 2R_{ij}\Delta_f R_{ij} \\ &= 2|\nabla Ric|^2 - 2|Ric|^2 - 4R_{ikjl}R_{ij}R_{kl} \\ &\ge 2|\nabla Ric|^2 - c(|\nabla Ric|+1),\end{aligned} \tag{21}$$

where we used (9) in the second equality. Moreover, we used the fact of $0 \leq Ric \leq c_0 g$ and (16) in the last.

Combining (20) and (21), we can find a constant $C_1 > 0$ such that

$$\begin{aligned}\Delta_f(|Rm| + C_1|Ric|^2) &\geq |Rm|^2 + 2C_1|\nabla Ric|^2 - c(|\nabla Ric|^2 + |\nabla Ric| + 1)\\ &\geq |Rm|^2 - c\\ &\geq \frac{1}{2}(|Rm| + C_1|Ric|^2)^2 - c.\end{aligned}$$

□

Now we are ready to prove Theorem 1.1.

**Theorem 3.4.** *Let $(M^4, g, f)$ be a four-dimensional complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then the Riemann curvature tensor and its covariant derivatives are bounded.*

*Proof.* Define the nonnegative smooth function $v := |Rm| + C_1|Ric|^2$, where $C_1$ is the constant in Lemma 3.3. It follows that

$$\Delta_f v = \frac{1}{2}v^2 - c. \tag{22}$$

By direct computations, we have

$$\begin{aligned}&\phi^2(-f)\Delta_f(v\phi^2(-f))\\ &= \phi^4(-f)\Delta_f v + v\phi^2(-f)\Delta_f\big(\phi^2(-f)\big) + 2\phi^2(-f)\langle\nabla v, \nabla\phi^2(-f)\rangle\\ &= \phi^4(-f)\Delta_f v + 2v\phi^2(-f)\big(\phi(-f)\Delta_f\big(\phi(-f)\big) + |\nabla\phi(-f)|^2\big)\\ &\quad + 2\langle\nabla\big(v\phi^2(-f)\big), \nabla\phi^2(-f)\rangle - 8v\phi^2(-f)|\nabla\phi(-f)|^2\\ &\geq \frac{1}{2}\big(v\phi^2(-f)\big)^2 - cv\phi^2(-f) + 2\langle\nabla\big(v\phi^2(-f)\big), \nabla\phi^2(-f)\rangle - c, \qquad (23)\end{aligned}$$

where we used (22), (12) and (13).

The maximum principle implies that on $D(2s)\backslash D(s)$

$$v\phi^2(-f) \leq c.$$

Note that $c$ is independent of $s$. Taking $s \to +\infty$, we have

$$v = |Rm| + C_1|Ric|^2 \leq c.$$

Since the Ricci curvature is bounded, we conclude that

$$|Rm| \leq c. \tag{24}$$

Furthermore, we use Shi's estimates (see [14] or [10] for details) to prove that $|\nabla Rm| \leq c$.

From (10) and (2), we can derive that

$$\Delta_f \nabla Rm = -\frac{1}{2}\nabla Rm + Rm * \nabla Rm.$$

Moreover, we have

$$\Delta_f|\nabla Rm|^2 = 2|\nabla^2 Rm|^2 + 2\langle\nabla Rm, \Delta_f \nabla Rm\rangle$$

$$\geq 2|\nabla^2 Rm|^2 - |\nabla Rm|^2 - c|Rm||\nabla Rm|^2$$
$$\geq 2|\nabla|\nabla Rm||^2 - c|\nabla Rm|^2,$$

where we used Kato's inequality and (24). Therefore, we get

$$\Delta_f|\nabla Rm| \geq -c|\nabla Rm|. \tag{25}$$

It follows from (10) and (24) that

$$\begin{aligned}\Delta_f|Rm|^2 &= 2|\nabla Rm|^2 + 2\langle Rm, \Delta_f Rm\rangle \\ &\geq 2|\nabla Rm|^2 - c. \end{aligned}\tag{26}$$

Hence, (24), (25) and (26) imply that

$$\Delta_f(|\nabla Rm| + |Rm|^2) \geq (|\nabla Rm| + |Rm|^2)^2 - c.$$

The maximum principle argument as above shows that $|\nabla Rm| + |Rm|^2$ is bounded on $M^4$. Therefore, we obtain

$$|\nabla Rm| \leq c. \tag{27}$$

Using the same method, we conclude that higher order derivatives of the Riemann curvature $|\nabla^l Rm|$ ($l \in \{2, 3, 4, \dots\}$) are bounded.

This completes the proof of this theorem. □

**Lemma 3.5.** *Let $(M^4, g, f)$ be a four-dimensional complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then, outside a compact set, we have*

$$\Delta_f\left(\frac{|Ric|^2}{R^a}\right) \geq \left(2a - \frac{cR}{(1-a)|\nabla f|^2}\right)\frac{|Ric|^4}{R^{a+1}} - c\frac{|Ric|^2}{R^a} \tag{28}$$

*for each constant $a \in (0, 1)$.*

*Proof.* By direct computations, we have

$$\begin{aligned}\Delta_f|Ric|^2 &= 2|\nabla Ric|^2 + 2R_{ij}\Delta_f R_{ij} \\ &= 2|\nabla Ric|^2 - 2|Ric|^2 - 4R_{ijkl}R_{ik}R_{jl} \\ &\geq 2|\nabla Ric|^2 - 2|Ric|^2 - 4|Rm||Ric|^2 \\ &\geq 2|\nabla Ric|^2 - c|Ric|^2 - c\frac{|\nabla Ric||Ric|^2}{|\nabla f|}, \end{aligned}\tag{29}$$

where we used (9) in the second equality and (15) in the last.

From (8), we can derive that

$$\begin{aligned}\Delta_f(R^{-a}) &= -aR^{-a-1}\Delta_f R + a(a+1)R^{-a-2}|\nabla R|^2 \\ &= aR^{-a} + 2a|Ric|^2R^{-a-1} + a(a+1)R^{-a-2}|\nabla R|^2. \end{aligned}\tag{30}$$

Using (29) and (30), we get

$$\Delta_f\left(\frac{|Ric|^2}{R^a}\right) = R^{-a}\Delta_f|Ric|^2 + |Ric|^2\Delta_f(R^{-a}) + 2\langle\nabla|Ric|^2, \nabla R^{-a}\rangle$$

$$
\begin{aligned}
&\geq \frac{1}{R^a}\left(2|\nabla Ric|^2 - c|Ric|^2 - c\frac{|\nabla Ric||Ric|^2}{|\nabla f|}\right)\\
&\quad + a\frac{|Ric|^2}{R^a}\left(1 + 2\frac{|Ric|^2}{R} + (a+1)\frac{|\nabla R|^2}{R^2}\right)\\
&\quad - 4a\frac{|\nabla R||Ric||\nabla Ric|}{R^{a+1}}\\
&= 2\frac{|\nabla Ric|^2}{R^a} - 4a\frac{|\nabla R||Ric||\nabla Ric|}{R^{a+1}} + a(a+1)\frac{|Ric|^2|\nabla R|^2}{R^{a+2}}\\
&\quad - c\frac{|\nabla Ric||Ric|^2}{|\nabla f|R^a} + 2a\frac{|Ric|^4}{R^{a+1}} + (a-c)\frac{|Ric|^2}{R^a}\\
&\geq \frac{2(1-a)}{1+a}\frac{|\nabla Ric|^2}{R^a} - c\frac{|\nabla Ric||Ric|^2}{|\nabla f|R^a} + 2a\frac{|Ric|^4}{R^{a+1}} - c\frac{|Ric|^2}{R^a}\\
&\geq \left(2a - \frac{cR}{(1-a)|\nabla f|^2}\right)\frac{|Ric|^4}{R^{a+1}} - c\frac{|Ric|^2}{R^a}. \qquad (31)
\end{aligned}
$$

□

Next, we finish the proof of Theorem 1.2.

**Theorem 3.6.** *Let $(M^4, g, f)$ be a four-dimensional complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then for each $0 < a < 1$, there exists a universal constant $c > 0$ such that*

$$|Ric|^2 \leq cR^a. \qquad (32)$$

*Proof.* Since $0 \leq R \leq 4c_0$, it follows from (6) and Proposition 2.2 that $|\nabla f|^2$ is of quadratic growth. Therefore, outside a compact set, we can obtain that

$$2a - \frac{cR}{(1-a)|\nabla f|^2} \geq a.$$

Define the smooth function $u := \frac{|Ric|^2}{R^a}$. By Lemma 3.5, we have

$$\Delta_f u \geq \frac{a}{R^{1-a}}u^2 - cu \geq \frac{a}{(4c_0)^{1-a}}u^2 - cu. \qquad (33)$$

By direct computations, we obtain that

$$
\begin{aligned}
\phi^2\Delta_f(u\phi^2) &= \phi^4\Delta_f u + \phi^2 u\Delta_f\phi^2 + 2\phi^2\langle\nabla u, \nabla\phi^2\rangle\\
&\geq \frac{a}{(4c_0)^{1-a}}(u\phi^2)^2 - cu\phi^4 + 2u\phi^2(\phi\Delta_f\phi + |\nabla\phi|^2)\\
&\quad + 2\langle\nabla(u\phi^2), \nabla\phi^2\rangle - 8u\phi^2|\nabla\phi|^2\\
&\geq \frac{a}{(4c_0)^{1-a}}(u\phi^2)^2 - cu\phi^2 + 2\langle\nabla(u\phi^2), \nabla\phi^2\rangle. \qquad (34)
\end{aligned}
$$

The maximum principle implies that

$$u\phi^2(-f) \leq c$$

on $D(2s)\backslash D(s)$. Note that $c$ is independent of $s$. Taking $s\to+\infty$, we obtain that

$$u=\frac{|Ric|^2}{R^a}\le c.$$

This completes the proof. $\square$

## 4. The $n$-dimensional case

In this section, we estimate the curvature operator of $n$-dimensional ($n\ge 5$) complete noncompact gradient expanding Ricci solitons with nonnegative Ricci curvature. Let $b$ be a fixed number to be determined later and $C$ be a universal constant depending only on $p$, $q$, $n$ and $c_0$. For $s\ge 1$, we set $\phi$ be a smooth nonnegative function defined on $\mathbb{R}^+$ so that $\phi(t)=1$ on $[0,s]$, $\phi(t)=\frac{2s-t}{s}$ on $(s,2s)$, and $\phi(t)=0$ on $[2s,\infty)$. Then $\phi(-f(x))$ is still a cut-off function with support in $D(2s)\backslash D(s)$.

First of all, we prove the following proposition.

**Proposition 4.1.** *Let $(M^n,g,f)$ be an $n$-dimensional ($n\ge 5$) complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. For any integer $p\ge 3$ and integer $q\ge 2p+1$, there exist positive constants $C_2$ and $C_3$ such that*

$$\begin{aligned}
&(b-C)\int_M |Rm|^p(1-f)^{-b}\big(\phi(-f)\big)^q\\
\le\ &\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\big(\phi(-f)\big)^q\\
&+C\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\big(\phi(-f)\big)^q+C_2+C_3.
\end{aligned}\tag{35}$$

*Proof.* Integrating by parts, we obtain

$$\begin{aligned}
&b\int_M |Rm|^p|\nabla f|^2(1-f)^{-b-1}\phi^q\\
=\ &\int_M |Rm|^p\langle\nabla f,\nabla(1-f)^{-b}\rangle\phi^q\\
=\ &-\int_M |Rm|^p\Delta f(1-f)^{-b}\phi^q-\int_M |Rm|^p(1-f)^{-b}\langle\nabla f,\nabla\phi^q\rangle\\
&-\int_M \langle\nabla|Rm|^p,\nabla f\rangle(1-f)^{-b}\phi^q\\
\le\ &-\int_M |Rm|^p\Delta f(1-f)^{-b}\phi^q-\int_M \langle\nabla|Rm|^p,\nabla f\rangle(1-f)^{-b}\phi^q.
\end{aligned}\tag{36}$$

Here we used $\langle\nabla f,\nabla\phi^q(-f)\rangle=\frac{q\phi^{q-1}|\nabla f|^2}{s}\ge 0$ to get the inequality.

Note that $R\le nc_0$. Using (5) and (6), we obtain that

$$b|\nabla f|^2(1-f)^{-b-1}+\Delta f(1-f)^{-b}=\Big(\frac{b(-f-R)}{1-f}-\frac{n}{2}-R\Big)(1-f)^{-b}$$

$$\geq (b-n-nc_0)(1-f)^{-b} \tag{37}$$

on $M\backslash D\big(2b(1+c_0)\big)$.

Applying (37) to (36), we have

$$\begin{aligned}(b-n-nc_0)\int_M |Rm|^p(1-f)^{-b}\phi^q \\ \leq -\int_M \langle\nabla|Rm|^p,\nabla f\rangle(1-f)^{-b}\phi^q + C_2,\end{aligned}$$

where $C_2 := \int_{D(2b(1+c_0))}(b-n-nc_0-b|\nabla f|^2(1-f)^{-1}-\Delta f)|Rm|^p(1-f)^{-b}\phi^q$.

By direct computations, we have

$$\begin{aligned}&-\int_M \langle\nabla|Rm|^p,\nabla f\rangle(1-f)^{-b}\phi^q \\ &= -p\int_M \nabla_h R_{ijkl}R_{ijkl}\nabla_h f|Rm|^{p-2}(1-f)^{-b}\phi^q \\ &= -p\int_M (\nabla_k R_{ijhl}+\nabla_l R_{ijkh})R_{ijkl}\nabla_h f|Rm|^{p-2}(1-f)^{-b}\phi^q \\ &= -2p\int_M \nabla_l R_{ijkh}R_{ijkl}\nabla_h f|Rm|^{p-2}(1-f)^{-b}\phi^q \\ &= 2p\int_M R_{ijkh}\nabla_l(R_{ijkl}\nabla_h f|Rm|^{p-2}(1-f)^{-b}\phi^q) \\ &= I+II+III+IV+V, \end{aligned} \tag{38}$$

where we used the second Bianchi identity in the second equality. Moreover, we define

$$\begin{aligned}I &= 2p\int_M R_{ijkh}\nabla_l R_{ijkl}\nabla_h f|Rm|^{p-2}(1-f)^{-b}\phi^q, \\ II &= 2p\int_M R_{ijkh}R_{ijkl}\nabla_l\nabla_h f|Rm|^{p-2}(1-f)^{-b}\phi^q, \\ III &= 2p\int_M R_{ijkh}R_{ijkl}\nabla_h f\nabla_l|Rm|^{p-2}(1-f)^{-b}\phi^q,\end{aligned}$$

and

$$\begin{aligned}IV &= 2bp\int_M R_{ijkh}R_{ijkl}\nabla_h f|Rm|^{p-2}(1-f)^{-b-1}\nabla_l f\phi^q, \\ V &= 2pq\int_M R_{ijkh}R_{ijkl}\nabla_h f|Rm|^{p-2}(1-f)^{-b}\phi^{q-1}\nabla_l\phi(-f).\end{aligned}$$

It follows from the second Bianchi identity and (7) that

$$\begin{aligned}I+III &\leq C\int_M |\nabla Ric||\nabla Rm||Rm|^{p-2}(1-f)^{-b}\phi^q \\ &\leq \frac{1}{2}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q \\ &\quad + C\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q.\end{aligned}$$

Note that the Ricci curvature is bounded. Using (2), we have

$$\begin{aligned} II &= 2p\int_M R_{ijkh}R_{ijkl}(-\frac{1}{2}g_{hl} - R_{hl})|Rm|^{p-2}(1-f)^{-b}\phi^q \\ &\leq C\int_M |Rm|^p(1-f)^{-b}\phi^q. \end{aligned}$$

Moreover, we get

$$\begin{aligned} IV &\leq 2bp\int_M |\nabla Ric||Rm|^{p-1}|\nabla f|(1-f)^{-b-1}\phi^q \\ &\leq \frac{1}{2}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q + \frac{b^2p^2}{2}\int_M |Rm|^{p-1}(1-f)^{-b-1}\phi^q \\ &\leq \frac{1}{2}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q + \int_M |Rm|^p(1-f)^{-b}\phi^q \\ &\quad + c(n,p)\int_M (1-f)^{-b-p}\phi^q, \end{aligned}$$

where we used Young's inequality in the last inequality.

Now we work on the term $V$ of the right-hand of (38).

$$\begin{aligned} V &= \frac{2pq}{s}\int_M R_{ijkh}\nabla_h f R_{ijkl}\nabla_l f|Rm|^{p-2}(1-f)^{-b}\phi^{q-1} \\ &= \frac{4pq}{s}\int_M \nabla_j R_{ik}R_{ijkl}\nabla_l f|Rm|^{p-2}(1-f)^{-b}\phi^{q-1} \\ &= -\frac{4pq}{s}\int_M R_{ik}\nabla_j(R_{ijkl}\nabla_l f|Rm|^{p-2}(1-f)^{-b}\phi^{q-1}) \\ &= i + ii + iii + iv + v, \end{aligned} \tag{39}$$

where we used the second Bianchi identity in the second equality. Moreover, we define

$$\begin{aligned} i &:= -\frac{4pq}{s}\int_M R_{ik}\nabla_j R_{ijkl}\nabla_l f|Rm|^{p-2}(1-f)^{-b}\phi^{q-1}, \\ ii &:= -\frac{4pq}{s}\int_M R_{ik}R_{ijkl}\nabla_j\nabla_l f|Rm|^{p-2}(1-f)^{-b}\phi^{q-1}, \\ iii &:= -\frac{4pq}{s}\int_M R_{ik}R_{ijkl}\nabla_l f\nabla_j|Rm|^{p-2}(1-f)^{-b}\phi^{q-1}, \end{aligned}$$

and

$$\begin{aligned} iv &:= -\frac{4pq}{s}\int_M R_{ik}R_{ijkl}\nabla_l f|Rm|^{p-2}\nabla_j(1-f)^{-b}\phi^{q-1}, \\ v &:= -\frac{4pq}{s}\int_M R_{ik}R_{ijkl}\nabla_l f|Rm|^{p-2}(1-f)^{-b}\nabla_j\phi^{q-1}. \end{aligned}$$

Next, we deal with $i$ to $v$.

It follows from $R \geq 0$ and (6) that $|\nabla f|^2 \leq -f$. Moreover, we have $|\nabla f| \leq \sqrt{2s}$ on $D(2s)\backslash D(s)$.

By direct computations, we obtain

$$\begin{aligned} i &= -\frac{4pq}{s}\int_M R_{ik}R_{ijkl}\nabla_j f\nabla_l f|Rm|^{p-2}(1-f)^{-b}\phi^{q-1} \\ &\le \frac{C}{s}\int_{D(2s)\setminus D(s)} |\nabla f|^2|Rm|^{p-1}(1-f)^{-b}\phi^{q-1} \\ &\le C\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-1}, \end{aligned} \tag{40}$$

where we used (7) in the first equality.

$$\begin{aligned} ii &= -\frac{4pq}{s}\int_M R_{ik}R_{ijkl}\nabla_j\nabla_l f|Rm|^{p-2}(1-f)^{-b}\phi^{q-1} \\ &= \frac{4pq}{s}\int_M R_{ik}R_{ijkl}(\frac{1}{2}g_{jl}+R_{jl})|Rm|^{p-2}(1-f)^{-b}\phi^{q-1} \\ &\le \frac{C}{s}\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-1}, \end{aligned} \tag{41}$$

where we used (2) in the second equality.

$$\begin{aligned} iii &= -\frac{4pq}{s}\int_M R_{ik}R_{ijkl}\nabla_l f\nabla_j|Rm|^{p-2}(1-f)^{-b}\phi^{q-1} \\ &\le \frac{C}{\sqrt{s}}\int_M |\nabla Rm||Rm|^{p-2}(1-f)^{-b}\phi^{q-1} \\ &\le \int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^{q-1} \\ &\quad + \frac{C}{s}\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-1}, \end{aligned} \tag{42}$$

$$\begin{aligned} iv &= -\frac{4pq}{s}\int_M R_{ik}R_{ijkl}\nabla_l f|Rm|^{p-2}\nabla_j(1-f)^{-b}\phi^{q-1} \\ &= -\frac{4pq}{s}\int_M R_{ik}R_{ijkl}\nabla_l f|Rm|^{p-2}\nabla_j f(1-f)^{-b-1}\phi^{q-1} \\ &\le \frac{C}{s}\int_M |\nabla f|^2|Rm|^{p-1}(1-f)^{-b-1}\phi^{q-1} \\ &\le \frac{C}{s}\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-1}, \end{aligned} \tag{43}$$

and

$$\begin{aligned} v &= -\frac{4pq}{s}\int_M R_{ik}R_{ijkl}\nabla_l f|Rm|^{p-2}(1-f)^{-b}\nabla_j\phi^{q-1} \\ &= \frac{4pq(q-1)}{s}\int_M R_{ik}R_{ijkl}\nabla_l f|Rm|^{p-2}(1-f)^{-b}\nabla_j f\phi^{q-2} \\ &\le \frac{C}{s}\int_{D(2s)\setminus D(s)} |\nabla f|^2|Rm|^{p-1}(1-f)^{-b}\phi^{q-2} \end{aligned}$$

$$\leq C\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-2}. \tag{44}$$

Note that $\phi \leq 1$ and $s \geq 1$. Plugging (40) to (44) into (39), we have

$$V \leq \int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q + C\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-2}.$$

Furthermore, Young's inequality implies that

$$\begin{aligned}&\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-2}\\ \leq &\int_M |Rm|^p(1-f)^{-b}\phi^q + C\int_M (1-f)^{-b}\phi^{q-2p}.\end{aligned}$$

Finally, it results that

$$\begin{aligned}&(b-C)\int_M |Rm|^p(1-f)^{-b}\phi^q\\ \leq &\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q + C\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q\\ &+ C\int_M (1-f)^{-b}\phi^{q-2p} + c(n,p)\int_M (1-f)^{-b-p}\phi^q + C_2.\end{aligned} \tag{45}$$

Note that $Ric \geq 0$, the Bishop volume comparison theorem implies that each geodesic ball $B_x(r)$ of $M^n$ is still at most Euclidean growth. By Proposition 2.2, we can derive that for any $m > \frac{n}{2}+1$,

$$|\int_M (1-f)^{-m}| < +\infty. \tag{46}$$

Therefore, there exists a finite constant $C_3$ so that

$$\begin{aligned}&(b-C)\int_M |Rm|^p(1-f)^{-b}\phi^q\\ \leq &\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q + C\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q\\ &+ C_2 + C_3.\end{aligned}$$

□

**Lemma 4.2.** *Let $(M^n, g, f)$ be an $n$-dimensional $(n \geq 5)$ complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. For any integer $p \geq 2$ and integer $q \geq 2p+1$, there exist positive constants $b$ and $A$ depending only on $n$, $p$ and $c_0$ such that*

$$\int_M |Rm|^p(1-f)^{-b} \leq A. \tag{47}$$

*In particular, for any $x \in M$ we have*

$$\int_{B_x(1)} |Rm|^p \leq A(1+r(x))^{2b}, \tag{48}$$

*where $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$.*

*Proof.* We discuss the case of $p \geq 3$ first. From (9) and (10), we can derive the following inequalities respectively by using the condition of $0 \leq Ric \leq c_0 g$.

$$\begin{aligned}
\frac{1}{2}\Delta_f |Ric|^2 &= |\nabla Ric|^2 + R_{ij}\Delta_f R_{ij} \\
&= |\nabla Ric|^2 - |Ric|^2 - 2R_{ijkl}R_{ik}R_{jl} \\
&\geq |\nabla Ric|^2 - nc_0^2 - 2nc_0^2|Rm|, \qquad (49)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\Delta_f |Rm|^2 &= |\nabla Rm|^2 + R_{ijkl}\Delta_f R_{ijkl} \\
&\geq |\nabla Rm|^2 - |Rm|^2 - C|Rm|^3. \qquad (50)
\end{aligned}$$

By (49), we have

$$\begin{aligned}
&\int_M |\nabla Ric|^2 |Rm|^{p-1}(1-f)^{-b}\phi^q \\
\leq\ & \frac{1}{2}\int_M \Delta|Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q \\
&- \frac{1}{2}\int_M \nabla_{\nabla f}|Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q \\
&+ c_0^2\int_M |Rm|^{p-1}(1-f)^{-b}\phi^q + nc_0^2\int_M |Rm|^p(1-f)^{-b}\phi^q \\
=\ & I + II + III + IV, \qquad (51)
\end{aligned}$$

where

$$\begin{aligned}
I &:= \frac{1}{2}\int_M \Delta|Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q, \\
II &:= -\frac{1}{2}\int_M \nabla_{\nabla f}|Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q,
\end{aligned}$$

and

$$\begin{aligned}
III &:= nc_0^2\int_M |Rm|^{p-1}(1-f)^{-b}\phi^q, \\
IV &:= nc_0^2\int_M |Rm|^p(1-f)^{-b}\phi^q.
\end{aligned}$$

We observe that

$$\begin{aligned}
I &= \frac{1}{2}\int_M \Delta|Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q \\
&= -\frac{1}{2}\int_M \langle\nabla|Ric|^2, \nabla|Rm|^{p-1}\rangle(1-f)^{-b}\phi^q \\
&\quad - \frac{b}{2}\int_M \langle\nabla|Ric|^2, \nabla f\rangle|Rm|^{p-1}(1-f)^{-b-1}\phi^q \\
&\quad - \frac{q}{2s}\int_M \langle\nabla|Ric|^2, \nabla f\rangle|Rm|^{p-1}(1-f)^{-b}\phi^{q-1}
\end{aligned}$$

$$= i + ii + iii, \tag{52}$$

where

$$i := -\frac{1}{2}\int_M \langle \nabla |Ric|^2, \nabla |Rm|^{p-1}\rangle (1-f)^{-b}\phi^q,$$

$$ii := -\frac{b}{2}\int_M \langle \nabla |Ric|^2, \nabla f\rangle |Rm|^{p-1}(1-f)^{-b-1}\phi^q,$$

and

$$iii := -\frac{q}{2s}\int_M \langle \nabla |Ric|^2, \nabla f\rangle |Rm|^{p-1}(1-f)^{-b}\phi^{q-1}.$$

It is easy to see that

$$\begin{aligned}
i &= -\frac{1}{2}\int_M \langle \nabla |Ric|^2, \nabla |Rm|^{p-1}\rangle (1-f)^{-b}\phi^q \\
&\leq C\int_M |\nabla Ric||\nabla Rm||Rm|^{p-2}(1-f)^{-b}\phi^q \\
&\leq \frac{1}{8}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q + C\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q.
\end{aligned}$$

Note that $2s \geq -f \geq \sqrt{s} \geq 1$ on $D(2s)\backslash D(s)$ and $R \geq 0$. It follows from (6) that $|\nabla f| \leq \sqrt{-f} \leq 1-f$. Therefore,

$$\begin{aligned}
ii &= -\frac{b}{2}\int_M \langle \nabla |Ric|^2, \nabla f\rangle |Rm|^{p-1}(1-f)^{-b-1}\phi^q \\
&\leq Cb\int_M |\nabla Ric||Rm|^{p-1}(1-f)^{-b}\phi^q \\
&\leq \frac{1}{8}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q + Cb^2\int_M |Rm|^{p-1}(1-f)^{-b}\phi^q.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
iii &= -\frac{q}{2s}\int_M \langle \nabla |Ric|^2, \nabla f\rangle |Rm|^{p-1}(1-f)^{-b}\phi^{q-1} \\
&\leq \frac{C}{\sqrt{s}}\int_M |\nabla Ric||Rm|^{p-1}(1-f)^{-b}\phi^{q-1} \\
&\leq \frac{1}{8}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q + \frac{C}{s}\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
I &= \frac{1}{2}\int_M \Delta |Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q \\
&\leq \frac{3}{8}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q + C\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q \\
&\quad + Cb^2\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-2} \\
&\leq \frac{3}{8}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q + C\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{6}\int_M |Rm|^p(1-f)^{-b}\phi^q + Cb^{2p}\int_M (1-f)^{-b}\phi^{q-2p} \\
&\le \frac{3}{8}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q + C\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q \\
&+\frac{1}{6}\int_M |Rm|^p(1-f)^{-b}\phi^q + C_4, \qquad (53)
\end{aligned}
$$

where we used Young's inequality in the second inequality and

$$C_4 := Cb^{2p}|\int_M (1-f)^{-b}| < +\infty.$$

Similarly, we obtain that

$$
\begin{aligned}
II = &-\frac{1}{2}\int_M \nabla_{\nabla f}|Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q \\
\le\ & C\int_M |\nabla Ric||Rm|^{p-1}|\nabla f|(1-f)^{-b}\phi^q \\
\le\ & \frac{1}{8}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q \\
& + C\int_M |Rm|^{p-1}|\nabla f|^2(1-f)^{-b}\phi^q \\
\le\ & \frac{1}{8}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q \\
& + \frac{1}{6}\int_M |Rm|^p(1-f)^{-b}\phi^q + C\int_M (1-f)^{-b}|\nabla f|^{2p}\phi^q \\
\le\ & \frac{1}{8}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q \\
& + \frac{1}{6}\int_M |Rm|^p(1-f)^{-b}\phi^q + C\int_M (1-f)^{-b+p}\phi^q \\
\le\ & \frac{1}{8}\int_M |\nabla Ric|^2|Rm|^{p-1}(1-f)^{-b}\phi^q \\
& + \frac{1}{6}\int_M |Rm|^p(1-f)^{-b}\phi^q + C_5, \qquad (54)
\end{aligned}
$$

where we used Young's inequality in the second inequality and $C_5 := C|\int_M (1-f)^{-b+p}| < +\infty$.

It follows from Young's inequality that

$$
\begin{aligned}
III &= nc_0^2\int_M |Rm|^{p-1}(1-f)^{-b}\phi^q \\
&\le \frac{1}{6}\int_M |Rm|^p(1-f)^{-b}\phi^q + C\int_M (1-f)^{-b}\phi^q. \qquad (55)
\end{aligned}
$$

Applying (53), (54) and (55) to (51), we obtain that

$$\begin{aligned} &\int_M |\nabla Ric|^2 |Rm|^{p-1}(1-f)^{-b}\phi^q \\ \le\ & C\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q \\ & + C\int_M |Rm|^p(1-f)^{-b}\phi^q + C_4 + C_5. \end{aligned} \tag{56}$$

Furthermore, we can derive the following inequality by combining (56) and Proposition 4.1.

$$\begin{aligned} &(b-C)\int_M |Rm|^p(1-f)^{-b}\phi^q \\ \le\ & C\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q + C_4 + C_5. \end{aligned} \tag{57}$$

From (50), we can derive that

$$\begin{aligned} &\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q \\ \le\ & \frac{1}{2}\int_M (\Delta|Rm|^2)|Rm|^{p-3}(1-f)^{-b}\phi^q \\ & - \frac{1}{2}\int_M \langle \nabla|Rm|^2, \nabla f\rangle |Rm|^{p-3}(1-f)^{-b}\phi^q \\ & + \int_M |Rm|^{p-1}(1-f)^{-b}\phi^q + C\int_M |Rm|^p(1-f)^{-b}\phi^q. \end{aligned} \tag{58}$$

We finish the proof by estimating each term in the right-hand side of (58). Using the similar method in (52), we find that

$$\begin{aligned} &\int_M (\Delta|Rm|^2)|Rm|^{p-3}(1-f)^{-b}\phi^q \\ =\ & -\int_M \nabla_i|Rm|^2\nabla_i(|Rm|^{p-3}(1-f)^{-b}\phi^q) \\ =\ & -2(p-3)\int_M |Rm|^{p-3}|\nabla|Rm||^2(1-f)^{-b}\phi^q \\ & - b\int_M (\nabla_{\nabla f}|Rm|^2)|Rm|^{p-3}(1-f)^{-b-1}\phi^q \\ & - \frac{q}{s}\int_M (\nabla_{\nabla f}|Rm|^2)|Rm|^{p-3}(1-f)^{-b}\phi^{q-1} \\ \le\ & \frac{2b}{\sqrt{s}}\int_{D(2s)\setminus D(s)} |\nabla Rm||Rm|^{p-2}(1-f)^{-b}\phi^q \\ & + \frac{2q}{\sqrt{s}}\int_{D(2s)\setminus D(s)} |\nabla Rm||Rm|^{p-2}(1-f)^{-b}\phi^{q-1} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q + \frac{4b^2}{s}\int_M |Rm|^{p-1}(1-f)^{-b}\phi^q \\
&\quad + \frac{4q^2}{s}\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-2} \\
&\leq \frac{1}{2}\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q + 4b^2\int_M |Rm|^{p-1}(1-f)^{-b}\phi^q \\
&\quad + 4q^2\int_M |Rm|^{p-1}(1-f)^{-b}\phi^{q-2} \\
&\leq \frac{1}{2}\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q + \int_M |Rm|^{p}(1-f)^{-b}\phi^q \\
&\quad + C\int_M (1-f)^{-b}\phi^{q-2p} + Cb^{2p}\int_M (1-f)^{-b}\phi^q \\
&\leq \frac{1}{2}\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q + \int_M |Rm|^{p}(1-f)^{-b}\phi^q + C_4. \qquad (59)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&-\int_M \langle \nabla |Rm|^2, \nabla f\rangle |Rm|^{p-3}(1-f)^{-b}\phi^q \\
&\leq 2\int_M |\nabla Rm||\nabla f||Rm|^{p-2}(1-f)^{-b}\phi^q \\
&\leq \frac{1}{2}\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q \\
&\quad + 2\int_M |Rm|^{p-1}|\nabla f|^2(1-f)^{-b}\phi^q \\
&\leq \frac{1}{2}\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q \\
&\quad + \int_M |Rm|^{p}(1-f)^{-b}\phi^q + C\int_M (1-f)^{-b}|\nabla f|^{2p}\phi^q \\
&\leq \frac{1}{2}\int_M |\nabla Rm|^2|Rm|^{p-3}(1-f)^{-b}\phi^q \\
&\quad + \int_M |Rm|^{p}(1-f)^{-b}\phi^q + C_5, \qquad (60)
\end{aligned}
$$

where we used Young's inequality in the third inequality.

It follows from Young's inequality that

$$
\begin{aligned}
&\int_M |Rm|^{p-1}(1-f)^{-b}\phi^q \\
&\leq \int_M |Rm|^{p}(1-f)^{-b}\phi^q + C\int_M (1-f)^{-b}\phi^q \\
&\leq \int_M |Rm|^{p}(1-f)^{-b}\phi^q + C_6, \qquad (61)
\end{aligned}
$$

where $C_6$ is a finite constant equals to $C|\int_M (1-f)^{-b}|$.

Applying (59), (60) and (61) to (58), we can derive that

$$\begin{aligned}&\int_M |\nabla Rm|^2 |Rm|^{p-3}(1-f)^{-b}\phi^q \\ &\leq C\int_M |Rm|^p (1-f)^{-b}\phi^q + C_4 + C_5 + C_6. \end{aligned} \tag{62}$$

Combining (57) and (62), we conclude there exists some constant $A$ which is independent of $s$, so that

$$\int_M |Rm|^p (1-f)^{-b}\phi^q \leq A. \tag{63}$$

Taking $s \to +\infty$, for $p \geq 3$, we have

$$\int_M |Rm|^p (1-f)^{-b} \leq A.$$

Consider the case of $q = 2$. Using Young's inequality, we obtain

$$\begin{aligned}&\int_M |Rm|^2 (1-f)^{-b} \\ &\leq \frac{2}{3}\int_M |Rm|^3 (1-f)^{-b} + \frac{1}{3}\int_M (1-f)^{-b} \\ &\leq A. \end{aligned} \tag{64}$$

Therefore, we (47) is proved.

Applying Proposition 2.2 to (47), we conclude (48).

This completes the proof. $\square$

Now we are ready to finish the proof of Theorem 1.3.

**Theorem 4.3.** *Let $(M^n, g, f)$ be an $n$-dimensional $(n \geq 5)$ complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then the Riemann curvature tensor is at most polynomial growth in the distance function i.e., there exist positive constants $b$ and $K$ so that*

$$|Rm|(x) \leq K(r(x)+1)^b, \tag{65}$$

*where $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$.*

*Proof.* From (50), we infer that

$$\begin{aligned}\Delta|Rm|^2 &\geq 2|\nabla Rm|^2 + \langle \nabla |Rm|^2, \nabla f\rangle - |Rm|^2 - c|Rm|^3 \\ &\geq -Cw|Rm|^2, \end{aligned} \tag{66}$$

where $w := C(|Rm| + |\nabla f|^2 + 1)$.

Since $Ric \geq 0$, by the Sobolev inequality in [13], there exists a constant $C_S$ depending only on $n$ so that for any $\psi$ with support in $B_x(1)$, we have

$$\Big(\int_{B_x(1)} \psi^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \leq \frac{C_S}{Vol(B_x(1))^{\frac{2}{n}}}\int_{B_x(1)} \big(|\nabla\psi|^2 + \psi^2\big).$$

Then the standard Moser iteration (see e.g. [9]) implies that

$$|Rm|^2(x) \leq C_M\Big(\int_{B_x(1)} w^n + 1\Big)\int_{B_x(1)} |Rm|^2, \tag{67}$$

where $C_M$ depends only on $n$, $C_S$ and $Vol(B_x(1))$.

Finally, this theorem follows immediately from Lemma 4.2 and (67). □

**Acknowledgements.** The author would like to thank Professor Kefeng Liu for his encouragement and guidance.

## References


[1] H.-D. Cao, G. Catino, Q. Chen, C. Mantegazza, and L. Mazzieri, *Bach-flat gradient steady Ricci solitons*, Calc. Var. Partial Differential Equations **49** (2014), no. 1-2, 125–138. https://doi.org/10.1007/s00526-012-0575-3

[2] H.-D. Cao and X. Cui, *Curvature estimates for four-dimensional gradient steady Ricci solitons*, J. Geom. Anal. **30** (2020), no. 1, 511–525. https://doi.org/10.1007/s12220-019-00152-z

[3] H.-D. Cao and X.-P. Zhu, *A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math. **10** (2006), no. 2, 165–492. https://doi.org/10.4310/AJM.2006.v10.n2.a2

[4] G. Catino, P. Mastrolia, and D. D. Monticelli, *Classification of expanding and steady Ricci solitons with integral curvature decay*, Geom. Topol. **20** (2016), no. 5, 2665–2685. https://doi.org/10.2140/gt.2016.20.2665

[5] ———, *Gradient Ricci solitons with vanishing conditions on Weyl*, J. Math. Pures Appl. (9) **108** (2017), no. 1, 1–13. https://doi.org/10.1016/j.matpur.2016.10.007

[6] Y. Deng and X. Zhu, *Complete non-compact gradient Ricci solitons with nonnegative Ricci curvature*, Math. Z. **279** (2015), no. 1-2, 211–226. https://doi.org/10.1007/s00209-014-1363-x

[7] M. Eminenti, G. La Nave, and C. Mantegazza, *Ricci solitons: the equation point of view*, Manuscripta Math. **127** (2008), no. 3, 345–367. https://doi.org/10.1007/s00229-008-0210-y

[8] R. S. Hamilton, *The formation of singularities in the Ricci flow*, in Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.

[9] P. Li, *Lecture notes on geometric analysis*, Lecture Notes Series, 6, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1993.

[10] O. Munteanu and J. Wang, *Geometry of shrinking Ricci solitons*, Compos. Math. **151** (2015), no. 12, 2273–2300. https://doi.org/10.1112/S0010437X15007496

[11] O. Munteanu and M.-T. Wang, *The curvature of gradient Ricci solitons*, Math. Res. Lett. **18** (2011), no. 6, 1051–1069. https://doi.org/10.4310/MRL.2011.v18.n6.a2

[12] P. Petersen and W. Wylie, *On the classification of gradient Ricci solitons*, Geom. Topol. **14** (2010), no. 4, 2277–2300. https://doi.org/10.2140/gt.2010.14.2277

[13] L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds*, J. Differential Geom. **36** (1992), no. 2, 417–450. http://projecteuclid.org/euclid.jdg/1214448748

[14] W.-X. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. **30** (1989), no. 1, 223–301. http://projecteuclid.org/euclid.jdg/1214443292

Liangdi Zhang
Center of Mathematical Sciences
Zhejiang University
Hangzhou 310027, P. R. China
and
Yanqi Lake Beijing Institute of Mathematical Sciences and Applications
Beijing 101408, P. R. China
*Email address*: zhangliangdi@zju.edu.cn