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A BOUND ON HÖLDER REGULARITY FOR $\overline{\partial}$-EQUATION ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$ WITH SOME COMPARABLE EIGENVALUES OF THE LEVI-FORM

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Abstract. Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and assume that the $(n-2)$-eigenvalues of the Levi-form are comparable in a neighborhood of $z_0 \in b\Omega$. Also, assume that there is a smooth 1-dimensional analytic variety V whose order of contact with $b\Omega$ at z_0 is equal to η and $\Delta_{n-2}(z_0) < \infty$. We show that the maximal gain in Hölder regularity for solutions of the $\overline{\partial}$-equation is at most $\frac{1}{\eta}$.

1. Introduction

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and assume that $z_0 \in b\Omega$. Suppose that there exist a neighborhood U of z_0 and a constant $C > 0$ so that for each $v \in L^{0,1}_{\infty}(\Omega)$ with $\overline{\partial} v = 0$, there is a $u \in L^2(\Omega) \cap \Lambda_\kappa(U \cap \overline{\Omega})$ such that $\overline{\partial} u = v$ in Ω and

$$\|u\|_{\Lambda_\kappa(U\cap\overline{\Omega})} \leq C\|v\|_{L_\infty(\Omega)}, \tag{1.1}$$

for some $\kappa > 0$, where $\Lambda_\kappa(S)$ denotes the Hölder space of order κ on S. In this event, we say the Hölder estimates of order $\kappa > 0$ for $\overline{\partial}$-equation hold on U.

When Ω is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$, (1.1) holds for $\kappa = \frac{1}{2}$ [10]. For weakly pseudoconvex domain in $\mathbb{C}^n$, however, (1.1) is known only for some special cases. Namely, pseudoconvex domains of finite type in $\mathbb{C}^2$ [12, 13], convex finite type domains in $\mathbb{C}^n$ [9], etc. Therefore, the Hölder estimate for general pseudocovex domains in $\mathbb{C}^n$ is one of the big questions in several complex variables.

Meanwhile, it is of great interest to find a necessary condition or optimal possible gain of $\kappa > 0$ in (1.1). Normally this question depends on the boundary geometry of Ω near $z_0 \in b\Omega$. Several authors have obtained necessary conditions for Hölder regularity of $\overline{\partial}$ on restricted classes of domains [11–14].

Let $\Delta_q(z)$ denote the D'Angelo's finite q-type at z, and let $\Delta_q^{Reg}(z)$ be the "regular finite q-type", which is defined by the maximum order of contact

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of non-singular q-dimensional varieties [8]. Note that $\Delta_p(z) \leq \Delta_q(z)$ (and $\Delta_p^{Reg}(z) \leq \Delta_q^{Reg}(z)$) if $p \geq q$, $\Delta_q^{Reg}(z) \leq \Delta_q(z)$, and $\Delta_q^{Reg}(z)$ is a positive integer.

When $\Delta_{n-1}(z_0) := m_{n-1} < \infty$, Krantz [11] showed that $\kappa \leq \frac{1}{m_{n-1}}$. Krantz's result is sharp for $\Omega \subset \mathbb{C}^2$, and when α is a $(0, n-1)$-form. In [12], McNeal proved sharp Hölder estimates for $(0,1)$-form α under the condition that Ω has a holomorphic support function at $z_0 \in \Omega$. Note that the existence of holomorphic support function is satisfied for restricted domains and it is often the first step to prove the Hölder estimates for the $\overline{\partial}$-equation [13]. In the rest of this section, we let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r, that is, $\Omega = \{z : r(z) < 0\} \Subset \mathbb{C}^n$.

Definition 1.1. Let $\lambda_1(z), \ldots, \lambda_{n-1}(z)$ be the nonnegative eigenvalues of the Levi-form, $\partial\overline{\partial}r(z)$. We say the eigenvalues $\{\lambda_k : k = s, \ldots, s+l\}$ are comparable in a neighborhood U of $z_0 \in b\Omega$ if there are constants $c, C > 0$ such that

$$c\lambda_j(z) \leq \lambda_k(z) \leq C\lambda_j(z), \quad j,k = s, \ldots, s+l, \quad z \in U.$$

Definition 1.2. We say that a 1-dimensional analytic variety V has order of contact η at $z_0 \in b\Omega$ if there are constants $c, C > 0$ such that

$$c|z - z_0|^\eta \leq |r(z)| \leq C|z - z_0|^\eta$$

for all $z \in V$ sufficiently close to z_0.

Example. Let $\Omega \subset \mathbb{C}^4$ be a domain defined by

$$\Omega = \{z : r(z) = 2Rez_4 + |z_1|^{10} + (|z_2|^2 + |z_3|^2)^{11/3} < 0\}.$$

Then, $\Delta_1(0) = 10 = \Delta_1^{Reg}(0)$, $\Delta_2(0) = \frac{22}{3}$, and $V = \{(t,0,0,0) : |t| \leq a\}$ is a smooth variety whose order of contact with $b\Omega$ at 0 is 10. Set $L_j = \frac{\partial}{\partial z_j} - (\frac{\partial r}{\partial z_4})^{-1}\frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_4}$, $j = 1,2,3$. Then, the eigenvalues $\lambda_k(z) \approx \partial\overline{\partial}r(z)(L_k, \overline{L}_k)$, $k = 2,3$, are comparable near 0.

In this paper, we want to study a necessary condition for the Hölder estimates of the $\overline{\partial}$ equation when $(n-2)$-eigenvalues of the Levi-form are comparable and $\Delta_{n-2}(z_0) < \infty$:

Theorem 1.3. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 3$, and assume that there is a smooth 1-dimensional variety whose order of contact at $z_0 \in b\Omega$ is $\eta < \infty$. Also, assume that the $(n-2)$-eigenvalues of the Levi-forms are comparable in a neighborhood of $z_0 \in b\Omega$ and $\Delta_{n-2}(z_0) < \infty$. If there exist a neighborhood U of z_0 and a constant $C > 0$ so that for each $v \in L^{0,1}_\infty(\Omega)$ with $\overline{\partial}v = 0$, there is a $u \in L^2(\Omega) \cap \Lambda_\kappa(U \cap \overline{\Omega})$ such that $\overline{\partial}u = v$ on Ω and*

$$\|u\|_{\Lambda_\kappa(U\cap\overline{\Omega})} \leq C\|v\|_{L_\infty(\Omega)}, \tag{1.2}$$

then $\kappa \leq \frac{1}{\eta}$.

Let $z = (z_1, \dots, z_n)$ be local coordinates about z_0. In the rest of this paper, we set $z' = (z_2, \dots, z_n)$, $z'' = (z_2, \dots, z_{n-1})$, and the same notations will be used for other coordinates or multi-indices, $\alpha = (\alpha_1, \dots, \alpha_n)$, that is, $\alpha' = (\alpha_2, \dots, \alpha_n)$, and $\alpha'' = (\alpha_2, \dots, \alpha_{n-1})$, etc.

Remark 1.4. (1) Since V is a smooth analytic variety, we note that η is a positive integer and $\Delta_{n-1}(z_0) := m_{n-1} \le \eta$. Thus, we have $\kappa \le \frac{1}{\eta} \le \frac{1}{m_{n-1}}$ in (1.2) which improves Krantz's result.

(2) In following, we will fix z_1 and consider the z_1 slice of Ω:

$$\Omega_{z_1} := \{(z_1, z') : (z_1, z') \in \Omega\}. \tag{1.3}$$

Then, Ω_{z_1} can be regarded as a bounded pseudoconvex domain in $\mathbb{C}^{n-1}$. Since the $(n-2)$-eigenvalues of the Levi-form are comparable, the condition $\Delta_{n-2}(z_0) < \infty$ will play as the role of the condition $\Delta_1(z_0) < \infty$ on each Ω_{z_1}.

(3) If $n = 3$, the comparable eigenvalues condition of the Levi form holds vacuously. In this case, You [14] proved Theorem 1.3. Note that $\Delta_2(z_0) \le \Delta_1^{Reg}(z_0)$ when $n = 3$. Consider the domain in $\mathbb{C}^3$ (see [8]) defined by

$$r(z) = Rez_3 + |z_1^2 - z_2^3|^2.$$

Then $\Delta_1^{Reg}(0) = 6$, and $\Delta_2(0) = 4$ while $\Delta_1(0) = \infty$ as the complex analytic curve $\gamma(t) = (t^3, t^2, 0)$ lies in the boundary. Note that $\gamma(t)$ is not a smooth curve.

(4) Whenever we have $(n-2)$-positive eigenvalues, these eigenvalues are comparable and hence Theorem 1.3 implies the results in [7] where we assumed that we have $(n-2)$-positive eigenvalues and $\Delta_1(z_0) < \infty$.

In Section 2, we construct special coordinates at each reference point and show that the z_1-coordinate represents the given variety V, and the z''-directions represent the comparable Levi-form directions. Let $C_b(z_0, \delta_0)$ denote the curve close to the z_1-direction as defined in (2.8). To prove the main theorem (Theorem 1.3), for each small $\delta > 0$, we need to construct a uniformly bounded holomorphic function f_δ on Ω that satisfies

$$\left| \frac{\partial f_\delta}{\partial z_n}(z_\delta) \right| \ge \frac{1}{\delta} \tag{1.4}$$

for each $z_\delta \in C_b(z_0, \delta_0)$.

In Section 2, we fix $z_1 = \check{z}_1$ near $z_1 = \delta^{\frac{1}{\eta}}$ and consider the sliced domain $\Omega_{\check{z}_1}$. Then, we construct a family of plurisubharmonic functions with maximal Hessian on each thin neighborhood of $b\Omega_{\check{z}_1}$ as in [1] for $n = 2$ case, and then show a bumping theorem. In Section 3, we push out the boundary of the domain $\Omega_{\check{z}_1}$ as far as possible at each reference point $\tilde{z}_\delta \in b\Omega_{\check{z}_1}$. These are some of the main ingredients for a construction of f_δ in (1.4). Section 4 is devoted to proving Theorem 1.3.

Remark 1.5. Note that the bumping theorem or pushing out the domains are done for the domains with $\Delta_1(z_0) < \infty$ [2, 3, 5]. In this paper, the condition

$\Delta_1(z_0) < \infty$ is replaced by the conditions $\Delta_{n-2}(z_0) < \infty$ and the compatibility of the $(n-2)$-eigenvalues.

2. Special coordinates and polydiscs

In the sequel, we assume that Ω is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 3$, with smooth defining function r_0 and that there is a smooth 1-dimensional holomorphic curve V whose order of contact with $b\Omega$ at $z_0 \in b\Omega$ is equal to η and $\Delta_{n-2}(z_0) < \infty$. We also assume that the $(n-2)$-eigenvalues of the Levi-form are comparable in a neighborhood W of z_0. We may assume that there are coordinate functions $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n)$ near z_0 such that $\tilde{z}(z_0) = 0$ and $|\partial r_0 / \partial \tilde{z}_n| \geq c_0$ in W for some fixed constant $c_0 > 0$.

Using these $\tilde{z}$-coordinates, set

$$L_n = \frac{\partial}{\partial \tilde{z}_n} \quad \text{and}$$
$$L_k = \frac{\partial}{\partial \tilde{z}_k} - \left(\frac{\partial r_0}{\partial \tilde{z}_n}\right)^{-1} \frac{\partial r_0}{\partial \tilde{z}_k} \frac{\partial}{\partial \tilde{z}_n}, \quad k = 1, \ldots, n-1,$$

set

$$c_{ij}(\tilde{z}) := \partial\overline{\partial} r_0(L_i, \overline{L}_j)(\tilde{z}), \quad i, j = 1, \ldots, n-1,$$

and assume that the eigenvalues of the matrix $A := (c_{ij})_{2 \leq i,j \leq n-1}$ are comparable. Let m be the smallest integer bigger than or equal to $\Delta_{n-2}(z_0)$ ($\Delta_{n-2}(z_0)$ could be a rational number). Here we may also assume that $\eta \geq m$. As in Proposition 2.3 in [6], we can prove that there are coordinate functions $z = (z_1, \ldots, z_n)$ near $z_0 = 0$ such that the given smooth one dimensional variety V can be regarded as the z_1-axis:

Proposition 2.1. *Let Ω, r_0, $z_0 \in b\Omega$ and $W \ni z_0$ be as above. There is a biholomorphism $\Phi_0 : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $\Phi_0(z) = \tilde{z}$, $\Phi_0(0) = z_0$ such that in terms of z coordinates, $r(z) := r_0 \circ \Phi_0(z)$ can be written as*

$$\begin{aligned} r(z) = Re z_n &+ \sum_{\substack{j+k=\eta \\ j,k>0}} a_{j,k} z_1^j \overline{z}_1^k + \sum_{\substack{|\alpha''+\beta''| \leq m \\ |\alpha''|,|\beta''|>0}} b_{\alpha''\beta''} z^{\alpha''} \overline{z}^{\beta''} \\ &+ \sum_{\substack{1 \leq j+k \leq \eta \\ 1 \leq |\alpha''+\beta''| \leq m}} c^{j,k}_{\alpha''\beta''} z_1^j \overline{z}_1^k z^{\alpha''} \overline{z}^{\beta''} + \mathcal{O}\left(E_{m,\eta}(z)\right), \end{aligned} \tag{2.1}$$

where $E_{m,\eta}(z) = |z||z_n| + |z_1|^{\eta+1} + |z''|^{m+1}$, and $r(z)$ satisfies

$$c|t|^\eta \leq |r(t, 0, \ldots, 0, 0)| \leq C|t|^\eta \tag{2.2}$$

for some constants $c, C > 0$.

Remark 2.2. (1) Let $d_0(z_1) := \sum_{j+k=\eta} a_{j,k} z_1^j \overline{z}_1^k$ be the first sum in (2.1). Then it follows from (2.1) and (2.2) that

$$|d_0(z_1)| \approx |r(z_1, 0')| \approx |z_1|^\eta. \tag{2.3}$$

(2) The coordinate change in Proposition 2.1 is about $z_0 = 0 \in b\Omega$, but not about arbitrary point $\tilde{z} \in W$.

In the rest of this section, we fix $\delta > 0$ and assume that $\check{z} = (\check{z}_1, \check{z}'', \check{z}_n) \in W$ satisfies

$$|\check{z}_1 - \delta^{\frac{1}{\eta}}| < \gamma \delta^{\frac{1}{\eta}} \tag{2.4}$$

for a sufficiently small $\gamma > 0$. Let us fix $\check{z}_1$ satisfying (2.4) and consider the $\check{z}_1$-slice defined in (1.3). Then for each $\check{z}'$ with $(\check{z}_1, \check{z}') \in W$, we can remove the pure terms in the z'' (or $\bar{z}''$) variables inductively in the Taylor series expansion of $r_{\check{z}_1} = r(\check{z}_1, \cdot)$ as in the proof of Proposition 1.1 in [1]:

Proposition 2.3. *For each fixed $\check{z} = (\check{z}_1, \check{z}') \in W$, where $\check{z}_1$ satisfies (2.4), there exist numbers $d_{\alpha''}(\check{z})$, depending smoothly on $\check{z}$, such that in the new coordinates $\zeta = (\check{z}_1, \zeta')$ defined by*

$$z = (z_1, \Phi_{\check{z}}(\zeta')) = (\check{z}_1, \check{z}'' + \zeta'', \check{z}_n + \Phi_n(\zeta')),$$

where

$$\Phi_n(\zeta') = \left(\frac{\partial r}{\partial \check{z}_n}(\check{z})\right)^{-1} \left(\frac{\zeta_n}{2} - \sum_{l=1}^{m} \sum_{|\alpha''|=l} d_{\alpha''}(\check{z})\zeta^{\alpha''}\right),$$

and the function $\rho(\check{z}_1, \zeta') := r \circ (\check{z}_1, \Phi_{\check{z}}(\zeta'))$ satisfies

$$\rho(\check{z}_1, \zeta') = r(\check{z}) + Re\zeta_n + \sum_{\substack{|\alpha''+\beta''| \le m \\ |\alpha''|, |\beta''| > 0}} c_{\alpha''\beta''}(\check{z})\zeta^{\alpha''}\bar{\zeta}^{\beta''} + \mathcal{O}\left(E(\check{z}_1, \zeta')\right), \tag{2.5}$$

where $E(\check{z}_1, \zeta') = |\zeta_n||\zeta| + |\check{z}_1|^{\eta+1} + |\zeta''|^{m+1}$.

Remark 2.4. (1) Set $2\kappa_0 := \max_{\alpha'', \beta''} |c_{\alpha''\beta''}(z_0)|$. Since $\Delta_{n-2}(z_0) \le m$, we have $\kappa_0 > 0$. Therefore it follows that

$$\max_{\alpha'', \beta''} |c_{\alpha''\beta''}(\check{z})| \ge \kappa_0 > 0, \tag{2.6}$$

independent of $\check{z}$ provided W is sufficiently small because $c_{\alpha''\beta''}(\check{z})$ are smooth in $\check{z}$.

(2) By setting $\zeta_1 = \check{z}_1$ and $\zeta = (\check{z}_1, \zeta')$, we may regard that $\Phi_{\check{z}} : \mathbb{C}^n \to \mathbb{C}^n$, that is,

$$\Phi_{\check{z}}(\zeta) = (\check{z}_1, z').$$

(3) For each $z = (z_1, z'', z_n) \in W$, define $\pi(z) := (z_1, z'', \pi_n(z)) \in b\Omega$, where $\pi_n(z)$ is the projection onto $b\Omega$ along the z_n direction. For each $\check{z}_1$ satisfying (2.4), set $\tilde{z} = (\check{z}_1, 0')$ and set $\check{z} = \pi(\tilde{z}) = (\check{z}_1, 0'', \pi_n(\tilde{z})) \in b\Omega$. Using a Taylor series in the variable z_n about $\pi_n(\tilde{z})$, we see that

$$r(\check{z}_1, 0') = 2Re\left[\frac{\partial r(\check{z})}{\partial z_n}[-\pi_n(\tilde{z})]\right] + \mathcal{O}(\pi_n(\tilde{z})^2).$$

Since $|\pi_n(\tilde{z})| \ll 1$ and $2|\frac{\partial r}{\partial z_n}| = 1 + \mathcal{O}(|z|) \ge \frac{1}{2}$ on W, it follows from (2.3) that

$$|\pi_n(\tilde{z})| \approx |r(\check{z}_1, 0')| \approx |d_0(\check{z}_1)| \approx |\check{z}_1|^{\eta}.$$

For each small $\delta > 0$, set $\tilde{z}_\delta = (\delta^{\frac{1}{\eta}}, 0')$ (i.e., $\check{z}_1 = \delta^{\frac{1}{\eta}}$) and set

$$\check{z}_\delta := \pi(\tilde{z}_\delta) := (\delta^{\frac{1}{\eta}}, 0'', \pi_n(\tilde{z}_\delta)) \in b\Omega. \tag{2.7}$$

For a sufficiently small $b > 0$, set $z_\delta := (\delta^{\frac{1}{\eta}}, 0'', \pi_n(\tilde{z}_\delta) - b\delta) \in \Omega$, and set

$$C_b(z_0, \delta_0) := \{z_\delta : 0 \leq \delta \leq \delta_0\} \cup \{z_0\} \subset \Omega \cup \{z_0\}, \tag{2.8}$$

where $\delta_0 > 0$ is a sufficiently small number such that $z_\delta \in W$ for all $0 \leq \delta \leq \delta_0$.

We will use the methods developed in [4–6] on each domain $\Omega_{\check{z}_1}$ keeping track of the dependence of the $\check{z}_1$ variable. For each $\check{z} = (\check{z}_1, \check{z}') \in W$, set

$$C_{s_2}(\check{z}) = \max\{|c_{\alpha''\beta''}(\check{z})| : |\alpha'' + \beta''| = s_2\}, \tag{2.9}$$

where $c_{\alpha''\beta''}(\check{z})$ is defined in (2.5), and for each $\epsilon > 0$, define

$$\tau(\check{z}, \epsilon) = \min_{2 \leq s_2 \leq m} \{(\epsilon / C_{s_2}(\check{z}))^{1/s_2}\}. \tag{2.10}$$

Note that $\tau(\check{z}, \epsilon)$ is well defined by (2.6) and it follows from (2.9) and (2.10) that

$$\epsilon^{1/2} \lesssim \tau(\check{z}, \epsilon) \lesssim \epsilon^{1/m}, \text{ and}$$
$$(\epsilon'/\epsilon)^{\frac{1}{2}} \tau(\check{z}, \epsilon) \leq \tau(\check{z}, \epsilon') \leq (\epsilon'/\epsilon)^{\frac{1}{m}} \tau(\check{z}, \epsilon), \text{ if } \epsilon' < \epsilon.$$

In the sequel, set $\check{\zeta} = (\check{z}_1, 0')$. Note that $\Phi_{\check{z}}(\check{\zeta}) = \check{z}$. For each $c > 0$ and $\epsilon > 0$, define

$$R^\delta_{c\epsilon}(\check{z}) = \{\zeta : |\zeta_1 - \check{z}_1| < c\delta^{\frac{1}{\eta}},\ |\zeta_k| < c\tau(\check{z}, \epsilon),\ k = 2, \ldots, n-1,\ |\zeta_n| < c\epsilon\},$$

and set

$$Q^\delta_{c\epsilon}(\check{z}) = \{(\zeta_1, \Phi_{\check{z}}(\zeta')); (\zeta_1, \zeta') \in R^\delta_{c\epsilon}(\check{z})\}.$$

Also, we set

$$R'_{c\epsilon}(\check{z}) = \{(\check{z}_1, \zeta_2, \ldots, \zeta_n) : |\zeta_k| < c\tau(\check{z}, \epsilon),\ k = 2, \ldots, n-1,\ |\zeta_n| < c\epsilon\}, \tag{2.11}$$

a polydisc in the ζ' variables, and

$$Q'_{c\epsilon}(\check{z}) = \{(\check{z}_1, \Phi_{\check{z}}(\zeta')) : (\check{z}_1, \zeta') \in R'_{c\epsilon}(\check{z})\}.$$

As in Proposition 1.7 in [1], there exists an independent constant $C > 0$ such that if $z = (\check{z}_1, z') \in Q'_\epsilon(\check{z})$, then

$$Q'_\epsilon(z) \subset Q'_{C\epsilon}(\check{z}), \quad \text{and} \quad Q'_\epsilon(\check{z}) \subset Q'_{C\epsilon}(z).$$

In view of (2.6), we note that the same inclusion relations hold if we fix $\check{z}'$ and vary $\check{z}_1$. Thus, we obtain that

$$Q^\delta_\epsilon(z) \subset Q^\delta_{C\epsilon}(\check{z}), \quad \text{and} \quad Q^\delta_\epsilon(\check{z}) \subset Q^\delta_{C\epsilon}(z), \quad \text{if} \quad z \in Q^\delta_\epsilon(\check{z}).$$

Again, by (2.6), we also have the following equivalence relations for $\tau(z, \epsilon)$ (Proposition 2.14 in [6]).

Proposition 2.5. *Assume $z = (\check{z}_1, z') \in Q^\delta_{c\epsilon}(\check{z})$. Then*

$$\tau(z, \epsilon) \approx \tau(\check{z}, \epsilon) \tag{2.12}$$

for all sufficiently small $c > 0$, independent of $\delta > 0$ and $\epsilon > 0$.

In the sequel, set $D_k = \frac{\partial}{\partial \zeta_k}$ or $\frac{\partial}{\partial \bar{\zeta}_k}$, $1 \le k \le n$, and set $\tau_1 = \delta^{\frac{1}{\eta}}$. Recall that $\check{\zeta} = (\check{z}_1, 0')$. Combining (2.4), (2.9) and (2.10), the error term $E(\check{z}_1, \zeta')$ in (2.5) satisfies

$$\begin{aligned} |D_1^{l_1} E(\check{\zeta})| &\lesssim \tau_1^{\eta+1-l_1} = \delta\tau_1^{-l_1+1}, \quad \text{and} \\ D_1^{l_1} D^{\nu''} E(\check{\zeta}) &= 0, \text{ if } 0 < |\nu''| \le m. \end{aligned} \tag{2.13}$$

Proposition 2.6. *Assume* $\check{z} = (\check{z}_1, \check{z}') \in W$ *satisfies* (2.4) *and assume that* $|r(\check{z})| \lesssim \delta$. *For each* l_1, *and for each multi index* $\nu'' = (\nu_2, \ldots, \nu_{n-1})$ *with* $0 < |\nu''| \le m$, *we have*

$$\begin{aligned} |D_1^{l_1} \rho(\check{\zeta})| &\lesssim \delta\tau_1^{-l_1}, \quad \textit{and} \\ |D^{\nu''} \rho(\check{\zeta})| &\lesssim \epsilon\tau(\check{z}, \epsilon)^{-|\nu''|}. \end{aligned} \tag{2.14}$$

Proof. From (2.1), (2.2) and (2.13), it follows that

$$|D_1^{l_1} \rho(\check{\zeta})| = |D_1^{l_1} r(\check{z})| \lesssim \delta\tau_1^{-l_1},$$

and the second estimates follows from (2.5), (2.9), (2.10) and (2.13) □

For each fixed $\delta > 0$, set $\check{z}_1 = \delta^{1/\eta}$ and consider $\delta^{1/\eta}$-slice of Ω, $\Omega_{\delta^{1/\eta}}$. For convenience of notation, set $\Omega_\delta = \Omega_{\delta^{1/\eta}}$. Then Ω_δ is a smoothly bounded pseudoconvex domain in $\mathbb{C}^{n-1}$ with comparable Levi-form near $\check{z}_\delta \in b\Omega_\delta$ where $\check{z}_\delta = \pi(\delta^{\frac{1}{\eta}}, 0')$ is defined in (2.7). Since $\Delta_{n-2}(\check{z}_\delta) \le m$, and the Levi-forms are comparable, it follows that $\Delta_1(\check{z}_\delta) \le m$ (Proposition 2.12 in [6]).

To push out the domain Ω_δ as far as possible at the reference point $\check{z}_\delta \in b\Omega_\delta \cap W$, we need to construct bounded plurisubharmonic functions with maximal Hessian in a thin strip neighborhood of $b\Omega_\delta$ as in Theorem 3.1 in [1]. Set $r_\delta(z') = r(\delta^{\frac{1}{\eta}}, z')$, and for each small $\epsilon > 0$, define

$$\begin{aligned} \Omega_\delta^\epsilon &= \{(\delta^{\frac{1}{\eta}}, z') : r_\delta(z') < \epsilon\}, \\ S_\delta(\epsilon) &= \{(\delta^{\frac{1}{\eta}}, z') : -\epsilon < r_\delta(z') < \epsilon\}, \text{ and} \\ S_\delta^-(\epsilon) &= \{(\delta^{\frac{1}{\eta}}, z') : -\epsilon < r_\delta(z') \le 0\}. \end{aligned}$$

Using the estimates (2.12) and (2.14), we can prove the following theorem as in the proof of Theorem 3.1 in [5]:

Proposition 2.7. *For all small* $\epsilon > 0$, *there is a plurisubharmonic function* $\lambda_\delta^\epsilon \in C^\infty(W \cap \Omega_\delta)$ *with the following properties*:

(i) $|\lambda_\delta^\epsilon(z)| \le 1$, $z = (\delta^{\frac{1}{\eta}}, z') \in \Omega_\delta \cap W$,

(ii) *for all* $L' = \sum_{k=2}^n a_k L_k$ *at* $z = (\delta^{\frac{1}{\eta}}, z') \in S_\delta^-(\epsilon) \cap W$,

$$\partial\overline{\partial}\lambda_\delta^\epsilon(L', \overline{L}')(z) \approx \tau(z, \epsilon)^{-2} \sum_{k=2}^{n-1} |a_k|^2 + \epsilon^{-2}|a_n|^2, \ \textit{and}$$

(iii) *if* $\Phi_{\check z}(\zeta')$ *is the map associated with a given* $\check z = (\delta^{\frac{1}{\eta}}, \check z') \in S_\delta(\epsilon) \cap W$, *then*

$$|D^{\alpha'}(\lambda_\delta^\epsilon \circ \Phi_{\check z}(\zeta'))| \leq C'_\alpha \epsilon^{-\alpha_n} \tau(\check z, \epsilon)^{-|\alpha''|}$$

holds for all $\zeta' \in R'_\epsilon(\check z)$ *where* $\alpha' = (\alpha_2, \dots, \alpha_n)$*, and* $\alpha'' = (\alpha_2, \dots, \alpha_{n-1})$*, and* $R'_\epsilon(\check z)$ *is defined in* (2.11).

Remark 2.8. In Theorem 2.3 of [2], the author proved a bumping theorem near a point $z_0 \in \Omega$ of finite 1-type. All we need for that theorem is the existence of a family of plurisubharmonic functions with maximal Hessian on each thin strip $S_\delta(\epsilon)$ as stated above in Proposition 2.7. Since $\Delta_{n-2}(\check z_\delta) \leq m$ and the Levi-form is comparable, it follows that $\Delta_1(\check z_\delta) \leq m$ (Proposition 2.12 in [6]).

Recall that $\check z_\delta = \pi(\delta^{\frac{1}{\eta}}, 0') \in b\Omega_\delta$ defined in (2.7). In the sequel, for each $\check z = (\check z_1, \check z')$, set $B'_c(\check z) = \{(\check z_1, z') : |z' - \check z'| < c\}$, $c > 0$. Using the family of plurisubharmonic functions λ_δ^ϵ in Proposition 2.7, we have the following bumping theorem for each Ω_δ as in [2]:

Theorem 2.9. *Let* $V \subset\subset W$ *be a small neighborhood of* $z_0 \in b\Omega$*. There exists an independent constant* $r_0 > 0$ *such that for each* $\check z_\delta \in \overline{V} \cap b\Omega_\delta$*, we have* $B'_{2r_0}(\check z_\delta) \subset\subset W \cap \{(\delta^{\frac{1}{\eta}}, z') \in \mathbb{C}^n\}$*, and there is a smooth* 1*-parameter family of pseudoconvex domains* Ω_δ^t*,* $0 \leq t < t_0$*, called the bumping family of* Ω_δ *with front* $B'_{2r_0}(\check z_\delta)$*, each defined by* $\Omega_\delta^t = \{(\delta^{\frac{1}{\eta}}, z') : r_\delta^t(z') < 0\}$ *where* $r_\delta^t(z') = r^t(\delta, z')$ *has the following properties*;

(1) $r_\delta^t(z')$ *is smooth in* $z = (\delta, z') \in W$ *and in* t *for* $0 \leq t < t_0$.

(2) $r_\delta^t(z') = r_\delta(z')$ *for* $(\delta^{\frac{1}{\eta}}, z') \notin B'_{2r_0}(\check z_\delta)$.

(3) $\frac{\partial r_\delta^t}{\partial t}(z') \leq 0$.

(4) $r_\delta^0(z') = r_\delta(z') = r(\delta^{\frac{1}{\eta}}, z')$.

(5) *for* $z = (\delta^{\frac{1}{\eta}}, z') \in B'_{2r_0}(\check z_\delta)$*,* $\frac{\partial r_\delta^t}{\partial t}(z') < 0$.

3. A construction of special functions

In this section, we construct a family of uniformly bounded holomorphic functions $\{f_\delta\}_{\delta>0}$ with large derivatives in the z_n-direction along the curve $C_b(z_0, \delta_0) \subset \Omega$ defined in (2.8). Let us fix $\delta > 0$ for a while and concentrate on the point $\check z_\delta \in b\Omega_\delta$ defined in (2.7) where $\Omega_\delta := \Omega_{\delta^{\frac{1}{\eta}}} = \{(\delta^{\frac{1}{\eta}}, z') : (\delta^{\frac{1}{\eta}}, z') \in \Omega\}$. For a construction of $\{f_\delta\}_{\delta>0}$, we use "Bumping theorem" in Theorem 2.9 as well as pushing out $b\Omega_\delta$ as far as possible at each reference point $\check z_\delta$.

Recall the function $\Phi_{\check z_\delta}(\zeta) = (\delta^{\frac{1}{\eta}}, \Phi_{\check z_\delta}(\zeta'))$ defined in Proposition 2.3. Set $\widetilde{W}'_\delta = W \cap \{(\delta^{1/\eta}, z') : z' \in \mathbb{C}^{n-1}\}$, $\Omega'_\delta = (\Phi_{\check z_\delta})^{-1}(\Omega_\delta)$ and set

$$W'_\delta = (\Phi_{\check z_\delta})^{-1}(\widetilde{W}'_\delta).$$

Then Ω'_δ is a smoothly bounded pseudoconvex domain in $\mathbb{C}^{n-1}$ and the $(n-2)$-eigenvalues are uniformly comparable, and the estimate (2.6) holds uniformly, independent of $\delta > 0$. We want to construct a domain $D'_\delta \subset \mathbb{C}^{n-1}$ which contains Ω'_δ such that the boundary of D'_δ is pushed out essentially as far as possible near $\zeta^\delta = (\delta^{\frac{1}{\eta}}, 0') = (\Phi_{\check{z}_\delta})^{-1}(\check{z}_\delta) \in b\Omega'_\delta$, so that bD'_δ is pseudoconvex.

Set

$$J_\delta(\zeta') = \left(\delta^2 + |\zeta_n|^2 + \sum_{2 \le s_2 \le m} C_{s_2}(\check{z}_\delta)^2 |\zeta''|^{2s_2}\right)^{\frac{1}{2}}, \tag{3.1}$$

where $C_{s_2}(\check{z}_\delta)$ is defined in (2.9), and let $r_0 > 0$ be the constant in Theorem 2.9. Note that $B'_{2r_0} \subset W'_\delta$. For each small $e > 0$, set

$$W'_{\delta,e} = \{(\delta^{1/\eta}, \zeta') \in W'_\delta : \rho(\delta^{\frac{1}{\eta}}, \zeta') < eJ_\delta(\zeta')\} \cap B'_{r_0}(\check{z}_\delta).$$

If we use the family $\{\lambda^\epsilon_\delta\}$ constructed in Proposition 2.7, and follow the methods in Section 4 of [1], we can show that $W'_{\delta,e}$ is the maximally pushed out domain of Ω'_δ near ζ^δ such that

$$bW'_{\delta,e} := \{(\delta^{1/\eta}, \zeta') \in W'_\delta : \rho(\delta^{\frac{1}{\eta}}, \zeta') = eJ_\delta(\zeta')\} \cap B'_{r_0}(\check{z}_\delta)$$

is pseudoconvex for all sufficiently small $e > 0$.

To connect the pushed out part $W'_{\delta,e}$ and Ω'_δ, we use the bumping family $\{\Omega^t_\delta\}$ with front $B'_{2r_0}(\check{z}_\delta)$ as in Theorem 2.9. Set

$$D'_{t,\delta,e} = (\Omega^t_\delta \setminus B'_{r_0}(\check{z}_\delta)) \cup (W'_{\delta,e} \cap \Omega^t_\delta).$$

Then $D'_{t,\delta,e}$ becomes a pseudoconvex domain which is pushed out near $\zeta^\delta = (\Phi_{\check{z}_\delta})^{-1}(\check{z}_\delta)$ provided $t > 0$ and $e > 0$ are sufficiently small. In the sequel, we fix these $t = t_0$ and $e = e_0$ and set $D'_\delta := D'_{t_0,\delta,e_0}$. Note that these choices of t_0 and $e_0 > 0$ are independent of $\delta > 0$. If we use the methods in Section 6 of [1] (or Section 3 of [4]), we see that there exists a $L^2(D'_\delta)$ holomorphic function f_δ satisfying

$$\left|\frac{\partial f_\delta}{\partial \zeta_n}(z_\delta)\right| \ge \frac{1}{\delta}, \tag{3.2}$$

independent of δ, where $z_\delta = (\delta^{\frac{1}{\eta}}, 0'', \pi_n(\tilde{z}_\delta) - b\delta) \in C_b(z_0, \delta_0)$, and where $b > 0$ is taken so that $C_b(z_0, \delta_0) \subset \Omega$. Note that f_δ is independent of z_1 variable. We will show that f_δ is holomorphic in a domain including the z_1 direction near $z_1 = \delta^{\frac{1}{\eta}}$.

Recall that Ω'_δ or D'_δ can be regarded as domains in $\mathbb{C}^{n-1}$ by fixing $\zeta_1 = \delta^{\frac{1}{\eta}}$. In terms of the special coordinates $\zeta = (\check{z}_1, \zeta')$ defined in Proposition 2.3, set

$$P_{c_1,\delta}(\check{z}_\delta) := \{\zeta : |\zeta_1 - \delta^{\frac{1}{\eta}}| < c_1\delta^{\frac{1}{\eta}},\ |\zeta_k| < \frac{r_0}{2n},\ k = 2, \dots, n\},$$

where r_0 is the constant fixed in Theorem 2.9, and set

$$\Omega_{c_1,\delta}(\check{z}_\delta) = P_{c_1,\delta}(\check{z}_\delta) \cap \{\zeta : \rho(\zeta) < 0\} \subset \Omega.$$

Also, for each $\delta > 0$, $e > 0$, and $c_1 > 0$, set

$$\Omega^e_{c_1,\delta}(\check{z}_\delta) = P_{c_1,\delta}(\check{z}_\delta) \cap \{(\zeta_1, \zeta') : \rho(\delta^{\frac{1}{\eta}}, \zeta') < eJ_\delta(\zeta')\} \subset \mathbb{C}^n.$$

Then $\Omega^e_{c_1,\delta}(\check{z}_\delta)$ is obtained by moving $W'_{\delta,e}$ along the ζ_1 direction.

Lemma 3.1. *For sufficiently small $c_1 > 0$, we have $\Omega_{c_1,\delta}(\check{z}_\delta) \subset\subset \Omega^{e/2}_{c_1,\delta}(\check{z}_\delta)$, or equivalently,*

$$\rho(\delta^{\frac{1}{\eta}}, \zeta') - \rho(\zeta) < \frac{e}{2} J_\delta(\zeta') \quad \textit{for} \quad \zeta = (\zeta_1, \zeta') \in \Omega_{c_1,\delta}(\check{z}_\delta). \tag{3.3}$$

Proof. Assume $\zeta = (\zeta_1, \zeta') \in \Omega_{c_1,\delta}(\check{z}_\delta)$. Then

$$|\rho(\zeta) - \rho(\delta^{\frac{1}{\eta}}, \zeta')| \le c_1 \delta^{\frac{1}{\eta}} \max_{|\tilde{\zeta}_1 - \delta^{\frac{1}{\eta}}| < c_1 \delta^{\frac{1}{\eta}}} |D_1 \rho(\tilde{\zeta}_1, \zeta')|. \tag{3.4}$$

Note that $\Phi_{\check{z}_\delta}$ is independent of $\zeta_1 = z_1$. Since $\rho(\zeta) = r \circ (\zeta_1, \Phi_{\check{z}_\delta}(\zeta'))$, it follows from (2.5), (2.14), (3.1), and a Taylor series that

$$|D_1 \rho(\tilde{\zeta}_1, \zeta')| \lesssim \delta^{1-\frac{1}{\eta}} \lesssim \delta^{-\frac{1}{\eta}} J_\delta(\zeta'). \tag{3.5}$$

Combining (3.4) and (3.5), we obtain (3.3) provided $c_1 > 0$ is sufficiently small. □

If we use the standard inequality:

$$ab \le \theta a^p + \theta^{-q/p} b^q, \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{for all} \quad \theta, a, b > 0,$$

one obtains that

$$(a+b)^s \le 2a^s + (s!)^{s-1} b^s, \quad s \ge 1. \tag{3.6}$$

Since f_δ is independent of ζ_1, we see that f_δ is holomorphic on $\Omega^e_{c_1,\delta}(\check{z}_\delta)$. We will show that f_δ is bounded uniformly on $\overline{\Omega}^{e/4}_{a_1,\delta}$ for some a_1, $0 < a_1 < c_1 \le \frac{r_0}{2n}$, to be determined. For each $q = (q_1, q') \in \overline{\Omega}^{e/4}_{a_1,\delta}$, set $\tau_1 = \delta^{\frac{1}{\eta}}$, $\tau_k = \tau(\check{z}_\delta, J_\delta(q'))$, $2 \le k \le n-1$, $\tau_n = J_\delta(q')$, and define a non-isotropic polydisc $Q^\delta_{a_1}(q)$ by

$$Q^\delta_{a_1}(q) := \{\zeta : |\zeta_k - q_k| < a_1 \tau_k, \ 1 \le k \le n\}.$$

Lemma 3.2. *There is an independent constant $0 < a_1 < c_1$ such that*

$$Q^\delta_{a_1}(q) \subset \Omega^e_{a_1,\delta} \quad \textit{for} \quad q = (q_1, q') \in \overline{\Omega}^{e/4}_{a_1,\delta}. \tag{3.7}$$

Proof. Assume $\zeta \in Q^\delta_{a_1}(q)$. Then, it follows from (2.9), and (2.10) that

$$\begin{aligned} C_{s_2}(\check{z}_\delta)^2 |\zeta'' - q''|^{2s_2} &\le (n-2)^{s_2} a_1^{2s_2} C_{s_2}(\check{z}_\delta)^2 \tau(\check{z}_\delta, J_\delta(q'))^{2s_2} \\ &\le (n-2)^{s_2} a_1^{2s_2} J_\delta(q')^2, \end{aligned} \tag{3.8}$$

and $|\zeta_n - q_n|^2 \le a_1^2 J_\delta(q')^2$. Thus, it follows from (3.1), (3.6), and (3.8) that

$$\begin{aligned} J_\delta(q')^2 &= \delta^2 + |q_n|^2 + \sum_{s_2=2}^{m} C_{s_2}(\check{z}_\delta)^2 |q''|^{2s_2} \\ &\le \delta^2 + 2|\zeta_n|^2 + 2|\zeta_n - q_n|^2 \\ &\quad + \sum_{s_2=2}^{m} C_{s_2}(\check{z}_\delta)^2 \left(2|\zeta''|^{2s_2} + ((2s_2)!)^{2s_2-1}|\zeta'' - q''|^{2s_2}\right) \\ &\le 2J_\delta(\zeta')^2 + \left[2mn^m((2m)!)^{2m-1}a_1^2\right] J_\delta(q')^2. \end{aligned}$$

If we take $a_1 > 0$ so that $2mn^m((2m)!)^{2m-1}a_1^2 \le \frac{1}{2}$, we obtain that $J_\delta(q') \le 2J_\delta(\zeta')$. By the same argument, we have $J_\delta(\zeta') \le 2J_\delta(q')$. Therefore we obtain that

$$\frac{1}{2}J_\delta(q') \le J_\delta(\zeta') \le 2J_\delta(q') \quad \text{for } \zeta \in Q^\delta_{a_1}(q). \tag{3.9}$$

Assume $q = (q_1, q') \in \overline{\Omega}^{e/4}_{a_1,\delta}$ and $\zeta \in Q^\delta_{a_1}(q)$. Then, $\rho(\delta^{\frac{1}{\eta}}, q') \le \frac{e}{4}J_\delta(q')$. Thus, we have

$$\rho(\delta^{\frac{1}{\eta}}, \zeta') \le \frac{e}{4}J_\delta(q') + |\nabla'\rho(\delta^{\frac{1}{\eta}}, \tilde{\zeta}') \cdot (\zeta' - q')| \tag{3.10}$$

for some $(\delta^{\frac{1}{\eta}}, \tilde{\zeta}') \in Q^\delta_{a_1}(q)$ where ∇' denotes the gradient of the ζ' variables. From (2.9), (2.10), and (2.14) (with ϵ replaced by $J_\delta(q')$), we obtain that

$$|D_k\rho(\delta^{\frac{1}{\eta}}, \tilde{\zeta}')| \lesssim J_\delta(q')\tau(\check{z}_\delta, J_\delta(q'))^{-1}, \ (\delta^{\frac{1}{\eta}}, \tilde{\zeta}') \in Q^\delta_{a_1}(q), \tag{3.11}$$

for $2 \le k \le n-1$, and $|D_n\rho| \lesssim 1$. Combining (3.9)–(3.11), we obtain that

$$\rho(\delta^{\frac{1}{\eta}}, \zeta') \le \frac{e}{2}J_\delta(\zeta') + C_2 a_1 J_\delta(q') < eJ_\delta(\zeta'),$$

if we take $a_1 > 0$ so that $4C_2a_1 < e$. Therefore, $\zeta \in \Omega^e_{a_1,\delta}$ proving (3.7). □

Remark 3.3. In the above discussion, $e > 0$ is any number such that $0 < e \le e_0$. Thus, in particular, we can fix $e = e_0$ where e_0 is fixed before (3.2).

Theorem 3.4. *f_δ is a bounded holomorphic function in $\overline{\Omega}^{e/4}_{a_1,\delta}$ and satisfies*

$$\left|\frac{\partial f_\delta}{\partial \zeta_n}(z_\delta)\right| \ge \frac{1}{\delta}, \ z_\delta \in C_b(z_0, \delta_0), \tag{3.12}$$

independent of δ.

Proof. By (3.2) and (3.3), we already know that there is a L^2 holomorphic function f_δ on $\Omega^e_{c_1,\delta}(\check{z}_\delta)$ satisfying the estimate (3.12). We only need to show that f_δ is bounded in $\overline{\Omega}^{e/4}_{a_1,\delta}$. Assume $q \in \overline{\Omega}^{e/4}_{a_1,\delta} \subset \Omega^e_{c_1,\delta}$, where $0 < a_1 < c_1$. Then $Q^\delta_{a_1}(q) \subset \Omega^e_{a_1,\delta} \subset \Omega^e_{c_1,\delta}$ by Lemma 3.2. Now if we use the mean value theorem on polydisc $Q^\delta_{a_1}(q) \subset \Omega^e_{c_1,\delta}$ and the fact that $f_\delta \in L^2(\Omega^e_{c_1,\delta})$ is holomorphic, we will get the boundedness of f_δ on $\overline{\Omega}^{e/4}_{a_1,\delta}$. □

4. Proof of Theorem 1.3

The proof is similar to that in [7]. We will sketch the proof briefly here. Let $c_1 > 0$, and $a_1 > 0$ be the constants fixed in Lemma 3.1 and Lemma 3.2 respectively. We may assume that $0 < 2b < a_1 \le c_1$. For each $\delta > 0$, let f_δ be the function defined in Theorem 3.4. Therefore, f_δ is L^2 holomorphic on $\Omega^e_{c_1,\delta}(\check{z}_\delta)$, bounded on $\overline{\Omega}^{e/4}_{a_1,\delta}$, independent of ζ_1 variable, and satisfies the estimates in (3.12). Set

$$g_\delta = \phi\left(\frac{|\zeta_1 - \delta^{\frac{1}{\eta}}|}{c_1\delta^{\frac{1}{\eta}}}\right)\phi\left(\frac{|\zeta|}{a_1}\right)\phi\left(\frac{|\zeta_3|}{a_1}\right)\cdots\phi\left(\frac{|\zeta_n|}{a_1}\right)f_\delta(0,\zeta'),$$

where

$$\phi(t) = \begin{cases} 1, & |t| \le \frac{1}{2}, \\ 0, & |t| \ge \frac{3}{4}. \end{cases}$$

Note that

$$\|\overline{\partial} g_\delta\|_{L^\infty(\Omega)} \lesssim \delta^{-\frac{1}{\eta}}.$$

Assume that $u_\delta \in L^2(\Omega) \cap \Lambda_\kappa(U \cap \Omega)$ solves $\overline{\partial} u_\delta = \overline{\partial} g_\delta$ on Ω as in Theorem 1.3. Then we have

$$\|u\|_{\Lambda_\kappa(U\cap\overline{\Omega})} \le C\|\overline{\partial} g_\delta\|_{L_\infty(\Omega)} \lesssim \delta^{-\frac{1}{\eta}}. \tag{4.1}$$

Set $h_\delta = u_\delta - g_\delta$. Then h_δ is holomorphic in Ω. Set

$$q_1^\delta(\theta) = (\delta^{1/\eta} + \frac{4}{5}c_1\delta^{1/\eta}e^{i\theta}, 0, \ldots, 0, -\frac{b\delta}{2}), \text{ and}$$
$$q_2^\delta(\theta) = (\delta^{1/\eta} + \frac{4}{5}c_1\delta^{1/\eta}e^{i\theta}, 0, \ldots, 0, -b\delta), \ \theta \in \mathbb{R}.$$

Note that $g_\delta(q_1^\delta(\theta)) = g_\delta(q_2^\delta(\theta)) = 0$. From (1.2) and (4.1) we obtain that

$$H_\delta := \left|\frac{1}{2\pi}\int_0^{2\pi}[u_\delta(q_1^\delta(\theta)) - u_\delta(q_2^\delta(\theta))]d\theta\right| \lesssim \delta^\kappa\|\overline{\partial} g_\delta\|_{L^\infty} \lesssim \delta^{\kappa-\frac{1}{\eta}}. \tag{4.2}$$

For the lower bounds of H_δ, set $\zeta'_\delta = (0'', -\frac{b\delta}{2})$, $\tilde{\zeta}'_\delta = (0'', -b\delta)$, $\zeta_\delta = (\delta^{\frac{1}{\eta}}, \zeta'_\delta)$, and $\tilde{\zeta}_\delta = (\delta^{\frac{1}{\eta}}, \tilde{\zeta}'_\delta)$. Then a Taylor's series of f_δ in ζ_n variable shows that

$$f_\delta(0'', \zeta_n) = f_\delta(\zeta'_\delta) + \frac{\partial f_\delta}{\partial \zeta_n}(\zeta'_\delta)(\zeta_n + \frac{b\delta}{2}) + \mathcal{O}\left(\left|\zeta_n + \frac{b\delta}{2}\right|^2\right).$$

Especially, when $\zeta_n = -b\delta$, we have

$$\left|f_\delta(\tilde{\zeta}'_\delta) - f_\delta(\zeta'_\delta)\right| = \left|\frac{\partial f_\delta}{\partial \zeta_n}(\zeta'_\delta)(-\frac{b\delta}{2}) + \mathcal{O}(\delta^2)\right| \gtrsim 1, \tag{4.3}$$

because $|\frac{\partial f_\delta}{\partial \zeta_n}(\zeta'_\delta)| \ge \frac{1}{\delta}$ by (3.12).

Note that $g_\delta(\zeta_\delta) = f(\zeta'_\delta)$ and $g_\delta(\tilde{\zeta}_\delta) = f(\tilde{\zeta}'_\delta)$ because $0 < 2b < a_1 \le c_1$. Therefore, it follows from (1.2), (4.1), (4.3), and the Mean Value Property that

$$H_\delta = \left| \frac{1}{2\pi}\int_0^{2\pi} [h_\delta(q_1^\delta(\theta)) - h_\delta(q_2^\delta(\theta))]d\theta \right| = \left| h_\delta(\zeta_\delta) - h_\delta(\tilde{\zeta}_\delta) \right| \tag{4.4}$$
$$\ge \left| f_\delta(\tilde{\zeta}'_\delta) - f_\delta(\zeta'_\delta) \right| - \left| u_\delta(\tilde{\zeta}_\delta) - u_\delta(\zeta_\delta) \right| \ge c_0 - C_0\delta^{\kappa - \frac{1}{\eta}}$$

for some constants $0 < c_0 < 1 < C_0$. If we combine (4.2) and (4.4), we obtain that

$$1 \lesssim \delta^{\kappa - \frac{1}{\eta}}. \tag{4.5}$$

Now, if we assume $\kappa > \frac{1}{\eta}$ and take $\delta \to 0$, then (4.5) will be a contradiction. Therefore, $\kappa \le \frac{1}{\eta}$. □

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