Bull. Korean Math. Soc. **58** (2021), No. 3, pp. 573–591
https://doi.org/10.4134/BKMS.b200236
pISSN: 1015-8634 / eISSN: 2234-3016

POLYNOMIALITY OF THE EQUIVARIANT GROMOV-WITTEN THEORY OF $\mathbb{P}^{r-1}$

Hyenho Lho

Abstract. We study the equivariant Gromov-Witten theory of $\mathbb{P}^{r-1}$ for all $r \geq 2$. We prove a polynomiality property in r of the Gromov-Witten classes of $\mathbb{P}^{r-1}$. Using this polynomiality property, we define a set of polynomial valued classes in $H^*(\overline{M}_{g,n})$ which generalize the limit of Witten's s-spin classes studied by Pandharipande, Pixton and Zvonkine.

1. Introduction

1.1. Overview

Since the study of relations in the cohomology of the moduli space of curves by Mumford in the 1980s ([11]), there has been substantial progress in the study of the structure of the tautological rings

$$RH^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n})\,.$$

We refer the reader to [1] for an introduction to the tautological rings.

Recently, certain polynomiality properties were proved in [7, 13] for sets of classes in $RH^*(\overline{M}_{g,n})$. Our main result is the proof of a polynomiality property in r for a set of equivariant Gromov-Witten classes of $\mathbb{P}^{r-1}$. Using the polynomiality, we define a new set of classes

$$\Omega^{\mathbb{P}^\infty,d}_{g,A} \in H^{2d}(\overline{M}_{g,n}) \otimes \mathbb{Q}[u,r]$$

for $g, d \geq 0$ and $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ satisfying

$$g - 1 + d - \sum_i a_i = 0\,.$$

For $d = g-1$, the new class, after restriction to $u = 0$, recovers the Witten's s-spin class ([13]) with $r = s - 1$. Finding a geometric interpretation of the new class $\Omega^{\mathbb{P}^\infty,d}_{g,A}$ is an interesting question.

Received March 13, 2020; Revised January 25, 2021; Accepted February 9, 2021.
2010 *Mathematics Subject Classification.* 14D23.
Key words and phrases. Gromov-Witten theory, tautological class.

1.2. Equivariant Gromov-Witten theory of $\mathbb{P}^{r-1}$

For $r \in \mathbb{N}$, the cohomological field theory (CohFT) associated to $\mathbb{P}^{r-1}$ can be constructed as follows. Let the algebraic torus

$$\mathsf{T}_r = (\mathbb{C}^*)^r$$

act with the standard linearization on $\mathbb{P}^{r-1}$ with weights $\lambda_0, \ldots, \lambda_{r-1}$ on the vector space $H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(1))$.

Let $\overline{M}_{g,n}(\mathbb{P}^{r-1}, d)$ be the moduli space of stable maps to $\mathbb{P}^{r-1}$ equipped with the canonical T_r-action, and let

$$\pi : \overline{M}_{g,n}(\mathbb{P}^{r-1}, d) \to \overline{M}_{g,n}$$

be the natural morphism forgetting the map. Let $A = (a_1, \ldots, a_n) \in \{0, \ldots, r-1\}^n$. The Gromov-Witten classes of the $\mathbb{P}^{r-1}$ are defined via the equivariant push-forward

$$\Omega^{\mathbb{P}^{r-1}}_{g,n}(a_1, \ldots, a_n) = \sum_{d\geq 0} q^d \pi_* \Big(\prod_{i=1}^{n} \mathrm{ev}_i^*(H^{a_i}) \cap [\overline{M}_{g,n}(\mathbb{P}^{r-1}, d)]^{\mathrm{vir}} \Big) . \tag{1}$$

The sum (1) defines a polynomial valued class

$$\Omega^{\mathbb{P}^{r-1}}_{g,A}(q) := \Omega^{\mathbb{P}^{r-1}}_{g,n}(a_1, \ldots, a_n) \in H^*(\overline{M}_{g,n}) \otimes \mathbb{C}[q]$$

after the specialization

$$\lambda_i = \zeta_r^i$$

for a primitive rth root of unity ζ_r.

Let $V := H^*(\mathbb{P}^{r-1}, \mathbb{C})$ be the cohomology ring of $\mathbb{P}^{r-1}$ with basis $H^0, H^1, \ldots, H^{r-1}$ and bilinear form

$$\eta_{ab} = \eta(H^a, H^b) = \delta_{a+b,r-1} ,$$

and unit vector $\mathbf{1} = H^0$. The Gromov-Witten classes (1) define a CohFT by

$$\Omega^{\mathbb{P}^{r-1}}_{g,n} : V^{\otimes n} \to H^*(\overline{M}_{g,n}) \otimes \mathbb{C}[q] , \ \Omega^{\mathbb{P}^{r-1}}_{g,n}(H^{a_1} \otimes \cdots \otimes H^{a_n}) = \Omega^{\mathbb{P}^{r-1}}_{g,n}(a_1, \ldots, a_n) .$$

The genus 0 sector defines a quantum product $\bullet$ on V with unit $\mathbf{1}$,

$$\eta(H^a \bullet H^b, H^c) = \Omega^{\mathbb{P}^{r-1}}_{0,3}(a, b, c) .$$

The resulting algebra is semisimple if and only if $q \neq -1$.

1.3. Tautological class via $\mathbb{P}^\infty$

Here we state a polynomiality property in r of the class $\Omega^{\mathbb{P}^{r-1}}_{g,A}$. Denote by

$$\Omega^{\mathbb{P}^{r-1},d}_{g,A} \in H^{2d}(\overline{M}_{g,n}) \otimes \mathbb{C}[q]$$

the degree $2d$ part of the class $\Omega^{\mathbb{P}^{r-1}}_{g,A} \in H^*(\overline{M}_{g,n}) \otimes \mathbb{C}[q]$.

Theorem 1. *For $\sum_{i=1}^n a_i = g - 1 - d$ with $a_i \geq 0$, we have*

(i) $\Omega^{\mathbb{P}^{r-1},d}_{g,A} \in H^{2d}(\overline{M}_{g,n}) \otimes \mathbb{C}[q]$ *is a polynomial in q of degree $g - 1$.*

(ii) *The coefficient of* q^k *for* $0 \leq k \leq g-1$ *in*

$$\Omega_{g,A}^{\mathbb{P}^{r-1},d} \in H^{2d}(\overline{M}_{g,n}) \otimes \mathbb{C}[q]$$

is a polynomial in r *for all sufficiently large* r.

For $A = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$ satisfying $\sum_{i=1}^n a_i = g - 1 - d$, we denote by

$$\Omega_{g,A}^{\mathbb{P}^{\infty},d}(q,r) \in H^{2d}(\overline{M}_{g,n}) \otimes \mathbb{C}[q,r]$$

the polynomial valued class associated to $\Omega_{g,A}^{\mathbb{P}^{r-1},d}$ by Theorem 1. Via the change of the variable

$$u = q + 1\,,$$

we define the *kth polynomial class*

$$\Omega_{g,A,k}^{\mathbb{P}^{\infty},d} \in H^{2d}(\overline{M}_{g,n}) \otimes \mathbb{C}[r]$$

for $0 \leq k \leq g-1$ to be the coefficient of u^k in

$$\Omega_{g,A}^{\mathbb{P}^{\infty},d}(u,r) \in H^{2d}(\overline{M}_{g,n}) \otimes \mathbb{C}[u,r]\,.$$

In [12], the authors proved a polynomiality property in s for Witten's s-spin class $W_{g,n}^s(a_1, \ldots, a_n)$.

Theorem 2 (Pandharipande, Pixton and Zvonkine [12])**.** *For* $\sum_{i=1}^n a_i = 2g-2$,

$$s^{g-1} W_{g,n}^s(a_1, \ldots, a_n) \in H^{2(g-1)}(\overline{M}_{g,n})$$

is a polynomial in s *for all sufficiently large* s.

For $(d,k) = (g-1,0)$, the class $(-1)^{g-1} \cdot \Omega_{g,A,0}^{\mathbb{P}^{\infty},g-1}$ equals[1] the polynomial in Theorem 2 with $r = s-1$. In [12], the following was conjectured.

Conjecture 3. *For* $\sum_{i=0}^n a_i = 2g-2$, *we have*

$$\Omega_{g,A,0}^{\mathbb{P}^{\infty},g-1}(-1) = [\mathcal{H}_g(a_1, \ldots, a_n)] \in H^{2(g-1)}(\overline{M}_{g,n})\,.$$

Here, $\mathcal{H}_g(a_1, \ldots, a_n)$ is the class of the closure of the locus of holomorphic differentials with multiplicities of the zeroes given by $(a_1, \ldots, a_n)$. We refer the reader to [12, Appendix] for an introduction to the moduli space of holomorphic differentials. Finding a geometric interpretation of $\Omega_{g,A,k}^{\mathbb{P}^{\infty},d}$ for $(d,k) \neq (g-1,0)$ is an interesting question.

---

[1]The $(-1)^{g-1}$ factor is due to the fact that the R-matrix for Witten's s-spin class differs from the R-matrix for $\mathbb{P}^{s-2}$ by a factor $(-s)$.

1.4. Stable graphs and strata

1.4.1. *Summation over stable graphs.* The strata of $\overline{M}_{g,n}$ are the substacks parameterizing pointed curves of a fixed *topological type*. The moduli space $\overline{M}_{g,n}$ is a disjoint union of finitely many strata.

The main result of the paper is the proof of an explicit formula for $\Omega^{\mathbb{P}^\infty}_{g,A}(u,r)$ in the cohomology ring $H^*(\overline{M}_{g,n}) \otimes \mathbb{C}[u,r]$. The formula is written in terms of a summation over stable graphs Γ indexing the strata of $\overline{M}_{g,n}$. We review here the standard indexing of the strata of $\overline{M}_{g,n}$ by stable graphs.

1.4.2. *Stable graphs.* The strata of the moduli space of curves correspond to stable graphs

$$\Gamma = (V, H, L, g : V \to \mathbb{N}, v : H \to V, \iota : H \to H)$$

satisfying the following properties:

(i) V is a vertex set with a genus function $g : V \to \mathbb{N}$,
(ii) H is a half-edge set equipped with a vertex assignment $v : H \to V$ and an involution ι,
(iii) The edge set E of Γ is defined by the 2-cycle of ι in H (self-edges at vertices are allowed),
(iv) L, the set of legs, is defined by the fixed points of ι and is placed in bijective correspondence with a set of markings,
(v) the pair (V, E) defines a *connected* graph,
(vi) for each vertex v, the stability condition holds:
$$2g(v) - 2 + n(v) > 0\,,$$
where $n(v)$ is the valence of Γ at v including both half-edges and legs.

An automorphism of Γ consists of automorphisms of the sets V and H which leave the structures L, g, v and ι invariant. Denote by $\mathrm{Aut}(\Gamma)$ the automorphism group of Γ.

The genus of a stable graph Γ is defined by:

$$g(\Gamma) = \sum_{v \in V} g(v) + h^1(\Gamma)\,.$$

A stratum of $\overline{M}_{g,n}$ corresponding to Deligne-Mumford stable curves of fixed topological type naturally determines a stable graph of genus g with n legs by considering the dual graph of a generic pointed curve parameterized by the stratum.

Let $\mathsf{G}_{g,n}$ be the set of isomorphism classes of stable graphs of genus g with n legs. The set $\mathsf{G}_{g,n}$ is finite.

1.4.3. *Strata algebra.* To each stable graph $\Gamma \in \mathsf{G}_{g,n}$, we associate the moduli space

$$\overline{M}_\Gamma = \prod_{v \in V} \overline{M}_{g(v),n(v)}\,.$$

Let

$$\xi_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,n} \tag{2}$$

be the canonical morphism whose image is equal to the closure of the stratum associated to the stable graph Γ. We require a family of stable pointed curves over $\overline{M}_\Gamma$ to construct ξ_Γ. Such a family is easily constructed by attaching the pull-backs of the universal families over the $\overline{M}_{g(v),n(v)}$ along the sections corresponding to the two halves of each edge in $E(\Gamma)$.

Each half-edge $h \in H(\Gamma)$ determines a cotangent line

$$\mathcal{L}_h \to \overline{M}_\Gamma .$$

For $h \in L(\Gamma)$, $\mathcal{L}_h$ is the pull-back via ξ_Γ of the corresponding cotangent line of $\overline{M}_{g,n}$. If h is a side of an edge $e \in E(\Gamma)$, then $\mathcal{L}_h$ is the cotangent line of the corresponding side of a node. We write

$$\psi_h = c_1(\mathcal{L}_h) \in H^2(\overline{M}_\Gamma, \mathbb{Q}) .$$

Let Γ be a stable graph. A *basic class* on $\overline{M}_\Gamma$ is defined to be a product of monomials in κ classes at each vertex of the graph and powers of ψ classes at each half-edge (including the legs),

$$\gamma = \prod_{v\in V}\prod_{i\geq 0}\kappa_i[v]^{x_i[v]}\prod_{h\in H}\psi_h^{y[h]} \in H^*(\overline{M}_\Gamma, \mathbb{Q}) ,$$

where $\kappa_i[v]$ is the i^{th} kappa class on $\overline{M}_{g(v),n(v)}$. To avoid the trivial vanishing of γ, we impose the condition

$$\sum_{i\geq 0} ix_i[v] + \sum_{h\in H[v]} y[h] \leq \dim_{\mathbb{C}}\overline{M}_{g(v),n(v)} = 3g(v) - 3 + n(v)$$

at each vertex of Γ. Here, $H[v] \subset H$ is the set of half-edges (including the legs) incident at v.

Consider the $\mathbb{Q}$-vector space $\mathcal{S}_{g,n}$ whose basis consists of the isomorphism classes of pairs $[\Gamma,\gamma]$ for stable graphs Γ of genus g with n legs and a basic class γ on $\overline{M}_\Gamma$. $\mathcal{S}_{g,n}$ is finite dimensional, since there are only finitely many pairs Γ,γ up to isomorphism.

Via the product on $\mathcal{S}_{g,n}$ defined by intersection theory with respect to the morphism (2), $\mathcal{S}_{g,n}$ is a finite dimensional $\mathbb{Q}$-algebra, called the *strata algebra* [12]. Push-forward along ξ_Γ defines a canonical ring homomorphism

$$\mathsf{q} : \mathcal{S}_{g,n} \to H^*(\overline{M}_{g,n}, \mathbb{Q}) , \quad \mathsf{q}([\Gamma,\gamma]) = \xi_{\Gamma*}(\gamma)$$

from the strata algebra to the cohomology ring of the moduli space of curves.

1.5. Generalization of Pixton's formula

The series

$$B_{ra}(u,z) = \sum_{k\geq 0} B_{rak}(u) z^k \tag{3}$$

will be defined in Section 3.2.

Let $f(T)$ be a power series with vanishing constant and linear terms,

$$f(T) \in T^2\mathbb{Q}[[T]]\,.$$

We define

$$\kappa(f) = \sum_{m\geq 0} \frac{1}{m!} p_{m*}\Big(f(\psi_{n+1}\cdots\psi_{n+m})\Big) \in H^*(\overline{M}_{g,n},\mathbb{Q})\,, \tag{4}$$

where p_m is the canonical map which forgets the last m markings.

$$p_m : \overline{M}_{g,n+m} \to \overline{M}_{g,n}\,.$$

Due to the vanishing in degree 0 and 1 of f, the sum (4) is finite.

Definition 4. For $k \in \mathbb{Z}$, define the class $\Omega^{r-1}_{g,A,k}(u)$ by the term of degree k (in variable u) of the mixed degree cohomology class

$$\mathsf{q}\Big(\sum_{\Gamma\in\mathsf{G}_{g,n}} \frac{1}{|Aut(\Gamma)|}[\Gamma,[\prod_{v\in V}\kappa_v\prod_{l\in L}\eta_l\prod_{e\in E}\Delta_e]_x]\Big) \in H^*(\overline{M}_{g,n},\mathbb{Q})\otimes\mathbb{C}[u^{\frac{1}{r}},u^{-\frac{1}{r}}],$$

where

- For $v \in V$, let $\kappa_v = \Big(r\,u^{\frac{r-1}{r}}\Big)^{g_v-1}\kappa\Big(T - TB_{r0}(u,-x_vT)\Big)$.
- For $l \in L$, let $\eta_l = x_{v_l}^{-a_l}B_{ra_l}(u,-x_{v_l}\psi_l)$, where $v_l \in V$ is the vertex to which the leg l is assigned.
- For $e \in E$, let

$$\Delta_e = \frac{\sum_{i=0}^{r-1}(x')^{-i}(x'')^{-r+1+i} - \sum_{i=0}^{r-1}(x')^{-i}B_{ri}(u,-x'\psi')(x'')^{-r+1+i}B_{r\,r-1-i}(u,-x''\psi'')}{\psi'+\psi''},$$

  where x', x'' are the x-variables assigned to the vertices adjacent to the edge e and ψ', ψ'' are the ψ-classes corresponding to the half-edges.

For a polynomial $\prod$ in variables x_v, the notation $[\prod]_x$ means the term of degree 0 in all variables x_v. The numerator of Δ_e is divisible by the denominator due to the identity

$$\sum_{i=0}^{r-1} B_{ri}(u,T)B_{r\,r-1-i}(u,-T) = r\,.$$

We write

$$\Omega^{\mathbb{P}^{r-1}}_{g,A}(u) := \sum_{k\in\mathbb{Z}} u^k\,\Omega^{\mathbb{P}^{r-1}}_{g,A,k}\,.$$

The following fundamental polynomiality property of $\Omega^{r-1}_{g,A}$ can be proven by the argument of [7, Section 4.6].

Proposition 5. *For fixed g and A, the class*

$$\Omega^{r-1}_{g,A}(u) \in H^*(\overline{M}_{g,n})\otimes\mathbb{C}[u,u^{-1}]$$

is a polynomial in r for all sufficiently large r.

Via the torus localization technique, we obtain the following result.

Theorem 6. *For $g \geq 0$ and $A \in \{0, \dots, r-1\}^n$, we have*

$$\Omega_{g,A}^{\mathbb{P}^{r-1}} = \Omega_{g,A}^{r-1} \in H^*(\overline{M}_{g,n}) \otimes \mathbb{C}[u, u^{-1}]\,.$$

By Proposition 5, Theorem 1 follows from Theorem 6. Since

$$\Omega_{g,A}^{\mathbb{P}^{r-1}} \in H^*(\overline{M}_{g,n}) \otimes \mathbb{C}[u]\,,$$

the coefficients of u^k for $k < 0$ in $\Omega_{g,A}^{r-1}$ give us tautological relations in $H^*(\overline{M}_{g,n})$ whose coefficients are rational functions in r.

1.6. Plan of the paper

After a review of the localization formula for $\mathbb{P}^{r-1}$ in the precise form required for the proof of the polynomiality property in Sections 2 and 3, Proposition 5 and Theorem 6 are proven in Section 4.

Acknowledgments. I thank Y. Bae, H. Fan, F. Janda, R. Pandharipande, J. Schmitt and L. Wu for discussions over the years about the moduli space of curves and the tautological classes. This work was supported by research fund of Chungnam National University.

2. Localization graphs

2.1. Torus action

Let $\mathsf{T} = (\mathbb{C}^*)^r$ act diagonally on the vector space $\mathbb{C}^r$ with weights

$$-\lambda_0, \dots, \lambda_{r-1}\,.$$

Let

$$p_0, \dots, p_{r-1}$$

be the T-fixed points of the induced T-action on $\mathbb{P}^{r-1}$. The weights of T on the tangent space $T_{p_j}(\mathbb{P}^{r-1})$ are given by

$$\lambda_j - \lambda_0, \dots, \widehat{\lambda_j - \lambda_j}, \dots, \lambda_j - \lambda_{r-1}\,.$$

There is an induced T-action on the moduli space $\overline{M}_{g,n}(\mathbb{P}^{r-1}, d)$ of stable maps. The localization formula of [5] will play a fundamental role in our paper. The T-fixed loci are represented in terms of dual graphs, and the contributions of the T-fixed loci are given by tautological classes. The formulas here are standard, see [4, 9].

2.2. Graphs

Let the genus g and the number of markings n for the moduli space be in the stable range

$$2g - 2 + n > 0\,.$$

We organize the T-fixed loci of $\overline{M}_{g,n}(\mathbb{P}^{r-1}, d)$ according to decorated graphs. A *decorated graph* $\Gamma \in \mathsf{G}_{g,n}(\mathbb{P}^{r-1})$ consists of the data $(\mathsf{V}, \mathsf{E}, \mathsf{N}, \mathsf{g}, \mathsf{p})$ where

(i) V is the vertex set,

(ii) E is the edge set (including possible self-edges),
(iii) $\mathsf{N} : \{1, \ldots, n\} \to \mathsf{V}$ is the marking assignment,
(iv) $\mathsf{g} : \mathsf{V} \to \mathbb{Z}_{\geq 0}$ is a genus assignment satisfying

$$g = \sum_{v \in \mathsf{V}} \mathsf{g}(v) + h^1(\Gamma)$$

and for which $(\mathsf{V}, \mathsf{E}, \mathsf{N}, \mathsf{g})$ is a stable graph,
(v) $\mathsf{p} : \mathsf{V} \to (\mathbb{P}^{r-1})^{\mathsf{T}}$ is an assignment of a T-fixed point $\mathsf{p}(v)$ to each vertex $v \in \mathsf{V}$.

We will often call the markings $\mathsf{L} = \{1, \ldots, n\}$ *legs*. We write the localization formula as

$$\sum_{d \geq 0} [\overline{M}_{g,n}(\mathbb{P}^{r-1}, d)]^{\mathrm{vir}} q^d = \sum_{\Gamma \in \mathsf{G}_{g,n}(\mathbb{P}^{r-1})} \mathrm{Cont}_\Gamma \,.$$

While $\mathsf{G}_{g,n}(\mathbb{P}^{r-1})$ is a finite set, each contribution Cont_Γ is a series in q obtained from an infinite sum over all edge possibilities.

2.3. Basic correlators

2.3.1. *Overview.* We review here basic series in q which arise in the genus 0 theory of Gromov-Witten invariants of $\mathbb{P}^{r-1}$. We fix a torus action $\mathsf{T} = (\mathbb{C}^*)^r$ on $\mathbb{P}^{r-1}$ with weights

$$-\lambda_0, \ldots, -\lambda_{r-1}$$

on the vector space $\mathbb{C}^r$. The following specialization

$$\lambda_i = \zeta_r^i \tag{5}$$

will be imposed for our *entire* study of $\mathbb{P}^{r-1}$. Here ζ_r is a primitive rth root of unity.

2.4. First correlators

We require several correlators defined via Gromov-Witten invariants of $\mathbb{P}^{r-1}$. The first two are obtained from standard Gromov-Witten invariants. For $\gamma_i \in H^*_{\mathsf{T}}(\mathbb{P}^{r-1})$, define

$$\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle_{g,n,d} = \pi_* \Big([\overline{M}_{g,n}(\mathbb{P}^{r-1}, d)]^{\mathrm{vir}} \cap \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) \psi^{a_i} \Big),$$

$$\langle\langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle\rangle_{g,n} = \sum_{d \geq 0} \frac{q^d}{d!} \langle \gamma_1 \psi^{a_1}, \ldots, \gamma_n \psi^{a_n} \rangle_{g,n,d},$$

where

$$\pi : \overline{M}_{g,n}(\mathbb{P}^{r-1}, d) \to \overline{M}_{g,n}$$

is the canonical morphism which forgets the map. For each T-fixed point $p_i \in \mathbb{P}^{r-1}$, let

$$e_i = e(T_{p_i} \mathbb{P}^{r-1})$$

be the equivariant Euler class of the tangent space of $\mathbb{P}^{r-1}$ at p_i. Let

$$\phi_i = \frac{\prod_{j\neq i}(H-\lambda_j)}{e_i}\,, \quad \phi^i = e_i\phi_i \,\in H^*_{\mathsf{T}}(\mathbb{P}^{r-1})$$

be cycle classes.

The following series will play a fundamental role in our paper.

$$\mathbb{S}_i(\gamma) = e_i \langle\langle \frac{\phi_i}{z-\psi}, \gamma \rangle\rangle_{0,2}\,,$$
$$\mathbb{V}_{ij} = \langle\langle \frac{\phi_i}{x-\psi}, \frac{\phi_j}{y-\psi} \rangle\rangle_{0,2}\,.$$

Unstable degree 0 terms are included by hand in the above formulas. The unstable degree 0 term for $\mathbb{S}_i(\gamma)$ (resp. $\mathbb{V}_{ij}$) is $\gamma|_{p_i}$ (resp. $\frac{\delta_{ij}}{e_i(x+y)}$). We write

$$\mathbb{S}(\gamma) = \sum_{i=0}^{r-1} \phi_i \mathbb{S}_i(\gamma)\,.$$

The series $\mathbb{S}_i$ and $\mathbb{V}_{ij}$ satisfy the basic relation

$$e_i \mathbb{V}_{ij}(x,y) e_j = \frac{\sum_{k=0}^{r-1} \mathbb{S}_i(\phi_k)|_{z=x} \mathbb{S}_j(\phi^k)|_{z=y}}{x+y} \tag{6}$$

which follows from the WDVV equation ([4]).

2.5. Further calculations

Define the I-function by

$$\mathbb{I}(q) = \sum_{d=0}^{\infty} \frac{q^d}{\prod_{i=0}^{r-1}\prod_{k=1}^{d}(H-\lambda_i+kz)} \in H^*_{\mathsf{T}}(\mathbb{P}^{r-1}) \otimes \mathbb{C}[[q, \frac{1}{z}]]\,.$$

Define differential operators

$$\mathsf{D} = q\frac{d}{dq}\,, \quad M = H + z\mathsf{D}\,. \tag{7}$$

Using Birkhoff factorization, an evaluation of the series $\mathbb{S}(H^j)$ can be obtained from the I-function, see [8, 10]:

$$\begin{aligned} \mathbb{S}(1) =& \mathbb{I}\,, \\ \mathbb{S}(H^j) =& M\mathbb{S}(H^{j-1}) \;\text{ for } 1 \leq j \leq r-1\,, \\ \mathbb{S}(H^r) =& \frac{M\mathbb{S}(H^{r-1})}{L_r^r}\,. \end{aligned} \tag{8}$$

Here, $L_r(q) = (1+q)^{\frac{1}{r}}$. The function $\mathbb{I}$ satisfies the following Picard-Fuchs equation

$$\Big(M^r - 1 - q\Big)\,\mathbb{I} = 0\,. \tag{9}$$

The restriction $\mathbb{I}|_{H=\lambda_i}$ admits the following asymptotic form

$$\mathbb{I}|_{H=\lambda_i} = e^{\mu\lambda_i/z}\Big(R_0 + R_1\Big(\frac{z}{\lambda_i}\Big) + R_2\Big(\frac{z}{\lambda_i}\Big)^2 + \cdots\Big) \tag{10}$$

with series $\mu, R_k \in \mathbb{C}[[q]]$.

A derivation of (10) can be obtained via the Picard-Fuchs equation (9) for $\mathbb{I}|_{H=\lambda_i}$. The series μ and R_k are found by solving differential equations obtained from the coefficient of z^k. For example,

$$\begin{aligned}
&1 + \mathsf{D}\mu = L_r\,,\\
&\quad R_0 = L_r^{\frac{1-r}{2}}\,,\\
&\quad R_1 = \frac{L_r^{-\frac{1+3r}{2}}}{24\,r}(r-1)(-2r-1+(1+r)L_r^r + rL_r^{1+r})\,,\\
&\quad R_2 = \frac{L_r^{-\frac{3+5r}{2}}}{1152\,r^2}(r-1)(23+69r+48r^2+4r^3 - 2L_r^r(23+46r+25r^2+2r^3)\\
&\qquad - 2L_r^{1+r}r(-1-r+2r^2) + L_r^{2r}(23+23r+r^2+r^3)\\
&\qquad + 2L_r^{2r+1}r(-1+r^2) + L_r^{2r+2}(r-1)r^2)\,.
\end{aligned}$$

The specialization (5) is used for these results.

From the equations (8) and (10), we can show the series $\mathbb{S}(H^j)$ have the following asymptotic expansions:

$$\mathbb{S}_i(H^j) = e^{\frac{\mu\lambda_i}{z}}\sum_{k\geq 0} R_{kj}\Big(\frac{z}{\lambda_i}\Big)^k \quad \text{for } 0 \leq j \leq r\,. \tag{11}$$

The following constraints play a fundamental role for the proof of polynomiality in Proposition 5.

Proposition 7. *For all $k \geq 0$, we have*

$$R_{ki} = P_k(i, r)$$

with $P_k(w,v) \in \mathbb{C}[L_r^{\pm\frac{1}{2}}][w, v, v^{-1}]$, where w, v are formal variables.

Proof. By induction on k, we prove that there exists a polynomial $P_k(w,v)$ such that $R_{ki} = P_k(i,r)$. By applying (11) to (8), we have

$$R_{ki} = L_r R_{k\,i-1} + \mathsf{D}R_{k-1\,i-1}\,. \tag{12}$$

By applying the above equation repeatedly, we obtain the following equation

$$R_{ki} = L_r^i R_k + \sum_{j=0}^{i-1} L_r^{i-1-j}\mathsf{D}R_{k-1\,j}\,. \tag{13}$$

Especially we have

$$R_{0i} = L_r^i R_0 = L_r^{\frac{1-r+2i}{2}}$$

and therefore the induction hypothesis is true for $k = 0$.

Now suppose the induction hypothesis is true for $l \le k-1$. Since $R_{lj} = P_l(j,r)$ for some $P_l(w,v) \in \mathbb{C}[L_r^{\pm\frac{1}{2}}][w,v,v^{-1}]$, we also have $\mathsf{D}R_{lj} = \widetilde{P}_l(j,r)$ for some $\widetilde{P}_l(w,v) \in \mathbb{C}[L_r^{\pm\frac{1}{2}}][w,v,v^{-1}]$. Therefore the sum in the equation (13)

$$\sum_{j=0}^{i-1} L_r^{i-1-j}\mathsf{D}R_{lj} = Q_{li}(S_0(i-1), S_1(i-1), \dots S_{n_{li}}(i-1), r)$$

for some $n_{li} \in \mathbb{N}$ and $Q_{li} \in \mathbb{C}[L_r^{\pm\frac{1}{2}}][w_1, w_2, \dots, w_{n_{li}}, v, v^{-1}]$. Here $S_a(b) = \sum_{s=0}^{b} s^a$ for $a, b \in \mathbb{Z}_{\ge 0}$. Since $S_a(i-1)$ is a polynomial in i for all $a \in \mathbb{Z}_{\ge 0}$, we conclude

$$\sum_{j=0}^{i-1} L_r^{i-1-j}\mathsf{D}R_{lj} = \widetilde{Q}_{li}(i,r) \tag{14}$$

for some $\widetilde{Q}_{li}(w,v) \in \mathbb{C}[L_r^{\pm\frac{1}{2}}][w,v,v^{-1}]$. By applying (9) to the case $(l,i) = (k+1,r)$ of (13), we obtain

$$\sum_{j=0}^{r-1} L_r^{r-1-j}\mathsf{D}R_{kj} = 0\,.$$

Applying the argument of (14) to the above equation, we have

$$\mathsf{D}(L_r^{\frac{r-1}{2}} R_k) + W_k(r)\cdot \mathsf{D}L_r = 0$$

for some $W_k(v) \in \mathbb{C}[L_r^{\pm\frac{1}{2}}][v,v^{-1}]$. By solving the above differential equation for R_k, we conclude

$$R_k = \widetilde{P}_k(r)$$

for some $\widetilde{P}_k(v) \in \mathbb{C}[L_r^{\pm\frac{1}{2}}][v,v^{-1}]$.

Finally using (13), we have

$$R_{ki} = P_k(i,r)$$

for some $P_k(w,v) \in \mathbb{C}[L_r^{\pm\frac{1}{2}}][w,v,v^{-1}]$ satisfying

$$P_k(0,v) = \widetilde{P}_k(v)\,.$$

□

3. Higher genus series on $\mathbb{P}^{r-1}$

3.1. Higher genus reconstruction theory

We review here the now standard method first used by Givental [4, 9, 10] to express genus g correlators in terms of genus 0 data.

Let $\Gamma \in \mathsf{G}_{g,n}(\mathbb{P}^{r-1})$ be a decorated graph defined in Section 2. The *flags* of Γ are the half-edges. Denote by F the set of flags. From the standard argument of the torus localization technique, we obtain the following result.

Proposition 8 (Givental [4]). *We have*

$$\langle\langle H^{a_1},\ldots,H^{a_n}\rangle\rangle_{g,n} = \sum_{\Gamma\in \mathsf{G}_{g,n}(\mathbb{P}^{r-1})} \frac{1}{|\mathrm{Aut}(\Gamma)|}[\Gamma,[\prod \kappa_v \prod \eta_l \prod \Delta_e]] \in H^*(\overline{M}_{g,n})\otimes \mathbb{C}[[q]]\,,$$

where

- *For* $v\in\mathsf{V}$, *let* $\kappa_v = \mathrm{Hodge}_v(\frac{1}{R_{00}})^{2\mathsf{g}(v)-2+\mathsf{n}(v)}\,\kappa\Big(T - T\,e^{\frac{\mu\lambda_{\mathsf{p}(v)}}{T}}\Big(\frac{\mathbb{S}_{\mathsf{p}(v)}(1)(-T)}{R_{00}}\Big)\Big)$, *where*
  $$\mathrm{Hodge}_v = \frac{\prod_{s=1}^{g}\prod_{j\neq p(v)}(\lambda_{\mathsf{p}(v)}-\lambda_j-\rho_s)}{e_{\mathsf{p}(v)}}$$
  with Chern roots $\rho_1\ldots\rho_g$ *of the* Hodge bundle *on* $\overline{M}_{g(v),n(v)}$.
- *For* $l\in\mathsf{L}$, *let* $\eta_l = e^{\frac{\mu\lambda_{v(l)}}{\psi_l}}\Big(\mathbb{S}_{\mathsf{p}(v(l))}(H^{a_l})(-\psi_l)\Big)$, *where* $v(l)\in V$ *is the vertex to which the leg is attached.*
- *For* $e\in\mathsf{E}$, *let*
  $$\Delta_e = e^{\frac{\mu\lambda_{\mathsf{p}(v')}}{\psi'}+\frac{\mu\lambda_{\mathsf{p}(v'')}}{\psi''}}\,\mathbb{V}_{\mathsf{p}(v'),\mathsf{p}(v'')}(-\psi',-\psi''),$$
  where ψ',ψ'' *are the* ψ*-classes corresponding to the half-edges assigned to* v',v''.

3.2. Grothendieck-Riemann-Roch formula

Using Mumford's Grothendieck-Riemann-Roch formula [11], we can remove the factor Hodge_v at each vertex v in the localization formula of

$$\langle\langle H^{a_1},\ldots,H^{a_n}\rangle\rangle_{g,n}$$

in Proposition 8 by modifying the edge terms.

Define a new series $\mathbb{B}^i_j(z)$ in z by

$$\mathbb{B}^i_j(z) := \mathrm{Exp}\Big(-\sum_{k\geq 0} z^{2k-1}\frac{\sum_{s\neq i}(\lambda_i-\lambda_s)^{1-2k}}{2k-1}\frac{B_{2k}}{2k}\Big)\Big(\sum_{k\geq 0}R_{kj}\Big(\frac{z}{\lambda_i}\Big)^k\Big)\,. \tag{15}$$

In Section 4, the polynomiality of the series

$$\mathbb{B}^i_j(z)\in\mathbb{C}[L_r^{\pm\frac{1}{2}},z]$$

will be proven. We define the series $B_{rj}(u,z)\in\mathbb{C}[u,z]$ by the following equation

$$\mathbb{B}^0_j(z) = L_r^{\frac{-1+r-2j}{2}}B_{rj}(L_r^r,z)\,. \tag{16}$$

Proposition 9. *We have*

$$\langle\langle H^{a_1},\ldots,H^{a_n}\rangle\rangle_{g,n} = \sum_{\Gamma\in \mathsf{G}_{g,n}(\mathbb{P}^{r-1})} \frac{1}{|\mathrm{Aut}(\Gamma)|}[\Gamma,[\prod \kappa_v \prod \eta_l \prod \Delta_e]] \in H^*(\overline{M}_{g,n},\mathbb{Q})\otimes \mathbb{C}[[q]]\,,$$

where

- *For* $v \in \mathsf{V}$, *let* $\kappa_v = e_{\mathsf{p}(v)}^{\mathsf{g}(v)-1}(\frac{1}{R_{00}})^{2\mathsf{g}(v)-2+\mathsf{n}(v)}\,\kappa\Big(T - \frac{1}{R_{00}}\mathbb{B}_0^{\mathsf{p}(v)}(-T)\Big)$.
- *For* $l \in \mathsf{L}$, *let* $\eta_l = \Big(\mathbb{B}^{a_l}_{\mathsf{p}(v(l))}(-\psi_l)\Big)$, *where* $v(l) \in \mathsf{V}$ *is the vertex to which the leg is attached.*
- *For* $e \in \mathsf{E}$, *let*

$$\Delta_e = \frac{\sum_{i=0}^{r-1}\mathbb{B}_i^{\mathsf{p}(v')}(0)\,\mathbb{B}_{r-1-i}^{\mathsf{p}(v'')}(0) - \sum_{i=0}^{r-1}\mathbb{B}_i^{\mathsf{p}(v')}(-\psi')\,\mathbb{B}_{r-1-i}^{\mathsf{p}(v'')}(-\psi'')}{\psi'+\psi''},$$

where ψ', ψ'' *are the* ψ*-classes corresponding to the half-edges assigned to* v', v''.

Proof. Using the Grothendieck-Riemann-Roch formula [11] at each vertex term, we can remove the factor Hodge_v at each vertex v in the localization formula of $\langle\langle H^{a_1},\ldots,H^{a_n}\rangle\rangle_{g,n}$ in Proposition 8 by modifying the half edge terms by (15). See [4, Section 2.3] for more explanations.

The proof of the proposition follows by applying (6), (11) to the localization formula of $\langle\langle H^{a_1},\ldots,H^{a_n}\rangle\rangle_{g,n}$ in Proposition 8 after the previous vertex-half edge modification. □

4. Polynomiality

4.1. *R*-matrix

Define the polynomial $P_{ka} \in \mathbb{C}[L_r]$ in L_r by the following normalization:

$$R_{ka} = L_r^{\frac{1-r-2k(1+r)+2a}{2}} P_{ka}\,.$$

Applying the equation (8) to the asymptotic expansions (11) of $\mathbb{S}_i(H^a)$, we obtain recursive relations for P_{ka}.

Lemma 10. *The polynomials* P_{ka} *satisfy the relations*

$$\begin{aligned} P_{ka} &= P_{k\,a-1} + \frac{(1-r)-2(k-1)(r+1)+2(a-1)}{2r}(L_r^r-1)P_{k-1\,a-1} \\ &\quad + L_r^r\mathsf{D}P_{k-1\,a-1}\,, \\ P_{k0} &= P_{kr}\,. \end{aligned}$$

Proof. Applying the equation (8) to the asymptotic expansions (11) of $\mathbb{S}_i(H^a)$, we obtain the first equation using

$$\mathsf{D}L = \frac{L_r^{1-r}}{r}(L_r^r-1)\,.$$

The second equation follows from the third equation in (8). □

Let

$$P_{ka}(0) = P_{ka}|_{L_r=0}\,.$$

From the constant term with respect to L_r in the equations of Lemma 10, we obtain:

Lemma 11. *The restrictions $P_{ka}(0)$ satisfy the relations*

$$P_{ka}(0) - P_{k\,a-1}(0) = \frac{1}{2r}((2k-1)(r+1) - 2a)P_{k-1\,a-1}(0)\,,$$
$$P_{k0}(0) = P_{kr}(0)\,.$$

Remark 12. The equations for $P_{ka}(0)$ in the above Lemma equal the equation for $P_k(r+1, a)$ in [12, Lemma 4.3] up to a factor $1/r$. Therefore the solutions of the equations in the above Lemma will differ from the solutions of the equation in [12, Lemma 4.3] by the factor $(1/r)^k$.

4.2. Equivariant mirror of $\mathbb{P}^{r-1}$

We review here an explicit description of the oscillating integrals on the mirror manifold of $\mathbb{P}^{r-1}$ in [2]. Givental introduced the mirror manifold for $\mathbb{P}^{r-1}$

$$\{(T_0, \dots, T_{r-1}) \,|\, e^{T_0} \cdots e^{T_{r-1} = q} \subset \mathbb{C}^r\}$$

with superpotential

$$\mathsf{F}(T) = \sum_{j=0}^{r-1} (e^{T_j} + \lambda_j T_j)\,.$$

Consider the integrals given by

$$\mathcal{I}_i = e^{-\ln(q)\,\lambda_i/z}(-2\pi z)^{\frac{-(r-1)}{2}} \int_{\Gamma_i \subset \{\sum T_j = \ln q\}} e^{\mathsf{F}(T)/z}\omega$$

along $(r-1)$-cycles Γ_i through a specific critical point of the superpotential F which can be constructed via the Morse theory of the real part of $\frac{\mathsf{F}(T)}{z}$. Here, ω is the restriction of $dT_0 \wedge \cdots \wedge dT_{r-1}$ to Γ_i.

There are r critical points of F at which the integral $\mathcal{I}_i$ admits a stationary phase expansion. Let Z_i be the solution to

$$\prod_{i=0}^{r-1} (X - \lambda_i) = q$$

with limit λ_i as $q \to 0$. For each i, if we choose the critical point $T_j = \ln(Z_i - \lambda_j)$, the factor

$$e^{\frac{u_i}{z}} := \mathrm{Exp}\Big(\Big(\sum_{j=0}^{r-1} (Z_i - \lambda_j + \lambda_j \ln(Z_i - \lambda_j)) - \lambda_i \ln q\Big)/z\Big)$$

is well defined in the limit as $q \to 0$. Via the shift of the integral to the critical point and re-scaling of coordinates by $\sqrt{z}$, we have

$$\mathcal{I}_i = e^{\frac{u_i}{z}} \int \mathrm{Exp}\Big(-\sum_j (Z_j - \lambda_j) \sum_{k=3}^{\infty} \frac{T_j^k (-z)^{(k-2)/2}}{k!}\Big) d\mu_i\,, \tag{17}$$

where $d\mu_i$ is the Gaussian distribution

$$(2\pi)^{\frac{r-1}{2}}\mathrm{Exp}\Big(-\sum_j (Z_i-\lambda_j)\frac{T_j^2}{2}\Big)\,.$$

In order to find the asymptotic expansion, we formally expand the exponential in (17) and integrate over the real part of the image of the mirror. The integrals are moments of μ_i which can be calculated via the covariance matrix

$$\sigma_i(T_k,T_l)=\left\{\begin{array}{ll} -\frac{1}{\Delta_i}\prod_{j\notin\{k,l\}}(Z_i-\lambda_j) & \text{for } \ k\neq l\,,\\ \frac{1}{\Delta_i}\sum_{m\neq k}\prod_{j\notin\{k,m\}}(Z_i-\lambda_j) & \text{for } \ k=l\,.\end{array}\right.$$

From the vanishing of odd moments of Gaussian distributions, we find that the asymptotic expansion of $e^{-\frac{u_i}{z}}\mathcal{I}_i$ is a power series in the variable z.

In conclusion, we obtain the following asymptotic expansion

$$\mathcal{I}_i=e^{\frac{u_i}{z}}\cdot L_r^{\frac{1-r}{2}}F_r\Big(\frac{z}{\lambda_i L_r^{r+1}},L_r^r\Big)\,,$$

with $F_r(x,y)\in\mathbb{C}[x,y]$.

From the mirror theorem for $\mathbb{P}^{r-1}$, we have the following result.

Theorem 13 (Givental [3, Section 10])**.** *We have the equality of power series in z,*

$$\mathbb{B}_0^i(z)=L_r^{\frac{1-r}{2}}F_r\Big(\frac{z}{\lambda_i L_r^{r+1}},L_r^r\Big)\,.$$

We do not know the closed form of $F_r(x,y)$. For $r=2$, we obtain the following result from the argument in [6, Section 3.3].

$$F_2(x,0)=\sum_{i\geq 0}\frac{(6i)!}{(3i)!(2i)!}\Big(\frac{-x}{576}\Big)^i\,.$$

4.3. Proof of Proposition 5

Definition 14. Consider a stable graph Γ of genus g with $n+k$ marked legs. A *weighting* $\mathbf{a}$ of Γ is a function on the set of half-edges

$$\mathsf{H}(\Gamma)\to\{0,\dots,r-1\}\,,\ \ h\mapsto a_h$$

satisfying the following conditions:

(i) If h and h' are the two half-edges of a edge, then $a_h+a_{h'}=r-1$,
(ii) If h corresponds to the leg i for $1\leq i\leq n$, then $a_h=a_i$,
(iii) If h is a κ-leg, then $a_h=0$.

Let Γ be a stable graph with $n+k$ legs. Let $\mathbf{m}$ be a function

$$\mathbf{m}:\mathsf{H}(\Gamma)\to\mathbb{N}\,,\ \ h\mapsto m_h$$

satisfying the conditions

(i) $\sum_{j\in\mathsf{H}(\Gamma)}m_h=d+|\mathsf{E}(\Gamma)|$,
(ii) If h and h' are the two half-edges of a edge, then $(m_h,m_{h'})\neq(0,0)$.

Define the sum

$$S_{\Gamma,\mathbf{m}} = \sum_{\text{weighting }\mathbf{a}} p_{k*}\{\prod_v x_v^{g_v-1} \prod_h B_{r\, a_h\, m_h} x_v^{m_h-a_h}\}_x \,,$$

where $p_k : \overline{M}_{g,n+k} \to \overline{M}_{g,n}$ is the map which forgets the last k markings. Here we recall the series $B_{ra}(z) := \sum_{k\geq 0} B_{rak} z^k$ in (3). Since we can write the coefficient of the formula of Definition 4 in terms of $\pm S_{\Gamma,\mathbf{m}}$, the polynomiality assertion of Proposition 5 follows from the following lemma.

Lemma 15. *The sum* $S_{\Gamma,\mathbf{m}}$ *is a polynomial in* r *for all sufficiently large* r.

Proof. The proof here follows closely the argument in [13, Section 4.6]. Let Γ' be the graph obtained from Γ by adding a vertex at the end of each leg and in the middle of each edge. Let M be the edge-vertex adjacency matrix of Γ'. The matrix M satisfies the assumptions of [7, Proposition A1]. The vertex x of [7, Proposition A1] assigns an integer x_h to each edge of Γ' or, equivalently to each half-edge h of Γ'. The vectors a and b of [7, Proposition A1] assign an integer to each vertex of Γ'. We summarizes what these integers are for each vertex of Γ' and what conditions are imposed by the equation

$$\mathsf{M}\mathsf{x} = \mathsf{a} + r\mathsf{b}\,. \tag{18}$$

type of vertex of Γ'	a	b	effect on x
midpoint of edge $h - h'$ in Γ	$r-1$	0	$x_h + x_{h'} = r-1$
endpoint of leg h in Γ	a_h	0	$x_h = a_h$
vertex v of Γ	$g_v - 1 + \sum_{h\mapsto v} m_h$	b_v	TopFT condition from the variable x_v

The conditions on x imply that x is a weighting of Γ'. For each weighting $\mathbf{a}$, we can find the unique solution (a,b) of the equation (18). For a given graph Γ and a given choice of integers m_h, there are only *finitely* many possible values b_v. Therefore, the sum $S_{\Gamma,\mathsf{m}}$ over all weightings can be decomposed into a finite number of sums of the form of [7, Proposition A1]. Hence, the polynomiality of $S_{\Gamma,\mathsf{m}}$ follows from [7, Proposition A1]. □

4.4. Proof of Theorem 6

The formula is essentially a reformulation of the localization formula of Proposition 9 using (16). We give a few more explanations.

TopFT conditions at the vertices. The powers of x_v keep track of the remainders modulo r. More precisely, x^{k-j} is assigned to the factor R_{kj}. Therefore the coefficients of ψ^m in the formulas come with an mth power of the corresponding vertex variable.

Powers of $ru^{\frac{r-1}{r}}$ at the vertices. The factor r corresponds to e_{p_v} and $u^{(r-1)/r} = (1+q)^{(r-1)/r}$ corresponds to $R_{00} = L_r^{(1-r)/2}$ at each vertex. Here, we recall $L_r := (1+q)^{1/r}$. The factor $(R_{00})^{-\mathsf{n}(v)}$ at the vertex v in the formula of Proposition 9 is absorbed into the edge factors in the formula of Definition 4.

4.5. Examples

We give a few examples of $\Omega^{\mathbb{P}^\infty,d}_{g,A}$. For $g-1=d$ and $\sum_i a_i = 2(g-1)$, the results here coincide with Witten's classes calculated in [13, Section A.3]. More precisely, we have $\Omega^{\mathbb{P}^\infty,g-1}_{g,A}|_{u=0} = (-s)^{g-1}W^s_{g,A}$ with $r=s-1$. For the results here, Conjecture 3 was verified using classical results in the moduli of curves in [13, Section A.3].

Genus 1. For $A=(0,\dots,0)$, we have

$$\Omega^{\mathbb{P}^\infty,0}_{1,A} = r \in H^0(\overline{M}_{1,n}) \otimes \mathbb{C}[u]\,.$$

Genus 2, $n=1, a_1=1, d=0$. For $A=(1)$, we have

$$\Omega^{\mathbb{P}^\infty,0}_{2,A} = ru \in H^0(\overline{M}_{2,1}) \otimes \mathbb{C}[u]\,.$$

Genus 2, $n=2, a_1=2, d=1$. Let δ_{sep}, δ_{nonsep} be the classes in $H^2(\overline{M}_{2,1})$, where the indices sep and nonsep refer to the boundary divisors with a separating or a nonseparating node. The kappa class κ_1 satisfies

$$\kappa_1 = \psi_1 + \frac{7}{5}\delta_{\mathrm{sep}} + \frac{1}{5}\delta_{\mathrm{nonsep}}\,.$$

For $A=(2)$, we have

$$\begin{aligned}
\Omega^{\mathbb{P}^\infty,1}_{2,A} =& \Big(-\frac{1}{24}r(r-1)(2r+1) + \frac{1}{24}r(r^2-1)u\Big)\kappa_1 \\
&+ \Big(\frac{1}{24}r(2r^2-25r+47) - \frac{1}{24}r(r^2-24r+47)u\Big)\psi_1 \\
&+ \Big(\frac{1}{24}r(r-1) - \frac{1}{24}r(r-1)(r+1)u\Big)\delta_{\mathrm{nonsep}} \\
&+ \Big(-\frac{1}{24}r(r-1)(2r-11) + \frac{1}{24}r(r-1)(r-11)u\Big)\delta_{\mathrm{sep}} \\
&\in H^2(\overline{M}_{2,1},\mathbb{Q}) \otimes \mathbb{C}[u]\,.
\end{aligned}$$

Genus 2, $n=2, a_1=a_2=1, d=1$.

- Let α be the locus of curves with a rational component carrying both markings and a genus 2 component,
- Let β be the locus of curves with two elliptic components carrying one marking each,
- Let γ be the locus of curves with two elliptic components one of which carries both markings and the other one no markings,
- Let δ_{nonsep} be the locus of curves with a nonseparating node.

For $A = (1,1)$, we have

$$\begin{aligned}\Omega^{\mathbb{P}^\infty,1}_{2,A} = &\Big(-\frac{1}{24}r(r-1)(2r+1)+\frac{1}{24}r(r-1)(r+1)u\Big)\kappa_1\\ &+\Big(\frac{1}{24}r(r-1)(2r-11)-\frac{1}{24}r(r-1)(r-11)u\Big)(\psi_1+\psi_2)\\ &+\Big(-\frac{1}{24}r(2r^2-25r+47)+\frac{1}{24}r(r^2-24r+47)u\Big)\alpha\\ &+\Big(-\frac{1}{24}r(r-1)(2r+1)+\frac{1}{24}r(r-1)(r+1)u\Big)\beta\\ &+\Big(-\frac{1}{24}r(r-1)(2r-11)+\frac{1}{24}r(r-1)(r-11)u\Big)\gamma\\ &+\Big(\frac{1}{24}r(r-1)-\frac{1}{24}r(r-1)(r+1)u\Big)\delta_{\mathrm{nonsep}}\\ \in\; &H^2(\overline{M}_{2,2})\otimes\mathbb{C}[u]\,.\end{aligned}$$

References

[1] C. Faber and R. Pandharipande, *Tautological and non-tautological cohomology of the moduli space of curves*, in Handbook of moduli. Vol. I, 293–330, Adv. Lect. Math. (ALM), 24, Int. Press, Somerville, MA, 2013.

[2] A. B. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices **1996** (1996), no. 13, 613–663. https://doi.org/10.1155/S1073792896000414

[3] ———, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1** (2001), no. 4, 551–568, 645. https://doi.org/10.17323/1609-4514-2001-1-4-551-568

[4] ———, *Semisimple Frobenius structures at higher genus*, Internat. Math. Res. Notices **2001** (2001), no. 23, 1265–1286. https://doi.org/10.1155/S1073792801000605

[5] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518. https://doi.org/10.1007/s002220050293

[6] F. Janda, *Relations on $\overline{M}_{g,n}$ via equivariant Gromov-Witten theory of* $\mathbb{P}^1$, Algebr. Geom. **4** (2017), no. 3, 311–336. https://doi.org/10.14231/AG-2017-018

[7] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, Publ. Math. Inst. Hautes Études Sci. **125** (2017), 221–266. https://doi.org/10.1007/s10240-017-0088-x

[8] B. Kim and H. Lho, *Mirror theorem for elliptic quasimap invariants*, Geom. Topol. **22** (2018), no. 3, 1459–1481. https://doi.org/10.2140/gt.2018.22.1459

[9] Y.-P. Lee and R. Pandharipande, *Frobenius manifolds, Gromov-Witten theory and Virasoro constraints*, https://people.math.ethz.ch/~rahul/, 2004.

[10] H. Lho and R. Pandharipande, *Stable quotients and the holomorphic anomaly equation*, Adv. Math. **332** (2018), 349–402. https://doi.org/10.1016/j.aim.2018.05.020

[11] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and geometry, Vol. II, 271–328, Progr. Math., 36, Birkhäuser Boston, Boston, MA, 1983.

[12] R. Pandharipande, A. Pixton, and D. Zvonkine, *Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures*, J. Amer. Math. Soc. **28** (2015), no. 1, 279–309. https://doi.org/10.1090/S0894-0347-2014-00808-0

[13] ———, *Tautological relations via r-spin structures*, J. Algebraic Geom. **28** (2019), no. 3, 439–496. https://doi.org/10.1090/jag/736

Hyenho Lho
Department of Mathematics
Chungnam National University
Daejeon 34134, Korea
Email address: hyenho@cnu.ac.kr