Bull. Korean Math. Soc. **58** (2021), No. 3, pp. 637–657
https://doi.org/10.4134/BKMS.b200437
pISSN: 1015-8634 / eISSN: 2234-3016

DECOMPOSITION OF THE KRONECKER SUMS OF MATRICES INTO A DIRECT SUM OF IRREDUCIBLE MATRICES

Caixing Gu, Jaehui Park, Chase Peak, and Jordan Rowley

Abstract. In this paper, we decompose (under unitary similarity) the Kronecker sum $A \boxplus A$ $(= A \otimes I + I \otimes A)$ into a direct sum of irreducible matrices, when A is a 3×3 matrix. As a consequence we identify $\mathcal{K}(A \boxplus A)$ as the direct sum of several full matrix algebras as predicted by Artin–Wedderburn theorem, where $\mathcal{K}(T)$ is the unital algebra generated by T and T^*.

1. Introduction

Let H be a complex separable Hilbert space and $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. An operator $A \in \mathcal{B}(H)$ is said to be irreducible if A has no nontrivial reducing subspace. A reducing subspace of A is a closed subspace $\mathcal{M}$ of H which is invariant for both A and A^*, i.e., $A\mathcal{M} \subseteq \mathcal{M}$ and $A^*\mathcal{M} \subseteq \mathcal{M}$. Halmos [17] has shown that the set of irreducible operators is a dense G_δ set of $\mathcal{B}(H)$ in the operator norm. The set of reducible operator is also dense in $\mathcal{B}(H)$ [33]. Radjavi [24] has shown that any $A \in \mathcal{B}(H)$ is a sum of two irreducible operators, improving the result of [10] that A is a sum of four irreducible operators. See also [7,25], and the books [18,26] for more references on the study of irreducible operators.

Let I denote the identity operator on some Hilbert space which the context will make it clear. If $A, B \in \mathcal{B}(H)$ and either A or B is reducible, then it is easy to show that the tensor products $A \otimes B$ and $A \otimes I + I \otimes B$ are reducible

Received May 15, 2020; Revised August 14, 2020; Accepted December 23, 2020.
2010 *Mathematics Subject Classification.* Primary 47A15, 15A21; Secondary 47L40, 15A69.
Key words and phrases. Irreducible operator, unitary similarity, Kronecker sum, symmetric tensor.
We thank Bill and Linda Frost Fund for the Frost Student Research Award to fund this project. Chase Peak and Jordan Rowley are Frost Research Fellows and Caixing Gu is the Faculty Advisor. Jaehui Park was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (Grant No. NRF-2018R1A2B6004116). Jaehui Park would like to thank Department of Mathematics at Cal Poly for the hospitality while he was visiting and working on this project.

operators in $\mathcal{B}(H\otimes H)$. If both A and B are irreducible, $A\otimes B$ and $A\otimes I+I\otimes B$ could be reducible [19]. In particular, let

$$\begin{aligned} T(A) &:= A\otimes I + I\otimes A, \\ H_s &:= \operatorname{Span}\{h\otimes h : h\in H\}, \\ H_{as} &:= \operatorname{Span}\{h\otimes g - g\otimes h : g,h\in H\}, \end{aligned}$$

where "Span" means the closed linear span in $H\otimes H$. Then $H\otimes H = H_s\oplus H_{as}$, and H_s and H_{as} are two reducing subspaces of $T(A)$.

The operator $T(A)$ is sometimes called the tensor sum. If A is a matrix, then $T(A)$ is the Kronecker sum $A\boxplus A$. The Kronecker sum of two matrices A and B is defined by $A\boxplus B = A\otimes I + I\otimes B$. The Kronecker products $(A\otimes B)$ and the Kronecker sums have been studied in a long time and has many applications. Several applications of Kronecker product and Kronecker sum can be found in the book [2]. Brewer [3] has dealt with Kronecker product and sum in the study of system theory. See also [5, 20].

Note that H_s is the subspace of symmetric tensors and H_{as} is the subspace of anti-symmetric tensors. If $E = \{e_n : n\geq 1\}$ is an orthonormal basis for H, then H_s and H_{as} have the following orthonormal bases:

$$\begin{aligned} H_s &= \operatorname{Span}\left\{e_n\otimes e_n, \frac{1}{\sqrt{2}}(e_n\otimes e_m + e_m\otimes e_n) : m>n\geq 1\right\}, \\ H_{as} &= \operatorname{Span}\left\{\frac{1}{\sqrt{2}}(e_n\otimes e_m - e_m\otimes e_n) : m>n\geq 1\right\}. \end{aligned}$$

Let

$$T_s(A) := T(A)|H_s, \quad T_{as}(A) := T(A)|H_{as}.$$

We record the above observation as a lemma.

Lemma 1.1. *Let $A\in\mathcal{B}(H)$. Then $T(A) = T_s(A)\oplus T_{as}(A)$ on $H_s\oplus H_{as}$.*

Proof. We include a more abstract proof which indicates more general results hold for operators invariant under the permutation group on the tensor product $H\otimes\cdots\otimes H$ (cf. [15, 32]). Let σ denote the permutation of $\{1,2\}$. Let U_σ be the unitary operator on $H\otimes H$ defined by $U_\sigma(h\otimes g) = g\otimes h$. Then $U_\sigma^2 = I$. The eigenvalues of U_σ are 1 and -1, and their eigenspaces are

$$\ker(U_\sigma - I) = H_s \quad \text{and} \quad \ker(U_\sigma + I) = H_{as}.$$

Since $T(A)U_\sigma = U_\sigma T(A)$, it follows that both H_s and H_{as} are reducing subspaces of $T(A)$. □

The above lemma motivates the following question.

Problem 1.2. *For which irreducible operator A are both $T_s(A)$ and $T_{as}(A)$ irreducible?*

Besides the general question of decomposing $T(A)$ into a direct sum of irreducible operators, another motivation for studying $T(A)$ is that they represent multiplication operators on function spaces. For example, the multiplication operator by $z_1 + z_2$ on the Hardy space of the bidisk is unitarily equivalent to $T(A)$ where A is the unilateral shift. See [13, 14, 16] for the study of reducing subspaces of weighted shifts and related multiplication operators on holomorphic Hilbert spaces of several variables. Recently, Fang and the first named author [6] have answered the above question topologically by proving the following result which generalizes Halmos' result mentioned above.

Theorem 1.3 ([6]). *The set of operators A such that both $T_s(A)$ and $T_{as}(A)$ are irreducible is a dense G_δ set in $\mathcal{B}(H)$.*

On a finite dimensional Hilbert space H, the study of the Jordan structures of tensor products $A \otimes B$ and $A \otimes I + I \otimes B$ and related maps on $\mathcal{B}(H)$ given by $X \mapsto \tau_{A,B}(X) = AXB$ and $X \mapsto \delta_{A,B}(X) = AX + XB$ have a long history. See [4, 11, 30, 31], and references therein. On an infinite dimensional Hilbert space H, the spectra of $\tau_{A,B}$ and $\delta_{A,B}$ are known [27]. The reflexivity of $A \otimes B$ is studied in [21]. The canonical forms of these tensor products under unitary similarity are unknown. Problem 1.2 can be seen as a first step toward finding the canonical form of $T(A)$ under unitary similarity. The questions of finding a complete set of unitary invariants of a matrix, finding the canonical form of a matrix and the related question of when two matrices are unitarily equivalent have always been of central importance in linear algebra and operator theory, see some classical papers [22, 23, 29] and more recent papers [9, 12, 28].

In this paper we resolve Problem 1.2 when A is an arbitrary 3×3 matrix. Let

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We obtain the following main theorem of this paper.

Main Theorem. *Let A be a 3×3 irreducible matrix. Then*

(a) *$T_s(A)$ is reducible if A is unitarily equivalent to $\alpha I + dD + aJ$ for some $\alpha, d, a \in \mathbb{C}$. In this case, $T_s(A)$ has two minimal reducing subspaces H_1 and H_2, where $\dim H_1 = 5$ and $\dim H_2 = 1$.*

(b) *$T_s(A)$ is irreducible if A is not unitarily equivalent to $\alpha I + dD + aJ$.*

(c) *$T_{as}(A)$ is irreducible.*

The above result has an algebra formulation. For a square matrix T, let $\mathcal{K}(T)$ be the unital algebra generated by T and T^*. Thus $\mathcal{K}(T)$ is a finite dimensional unital C^*-algebra, and is semisimple. By Artin–Wedderburn theorem, $\mathcal{K}(T)$ is $*$-isomorphic to the direct sum of several full matrix algebras. Let M_n denote the full matrix algebra consisting of all $n \times n$ complex matrices. By Burnside's theorem, if T is an irreducible $n \times n$ matrix, then $\mathcal{K}(T) = M_n$. The following corollary then follows from our Main Theorem.

Corollary 1.4. *Let A be a 3×3 irreducible matrix.*

(a) *If A is unitarily equivalent to $\alpha I + dD + aJ$, then $\mathcal{K}(T(A))$ is $*$-isomorphic to $M_5 \oplus M_1 \oplus M_3$.*

(b) *If A is not unitarily equivalent to $\alpha I + dD + aJ$, then $\mathcal{K}(T(A))$ is $*$-isomorphic to $M_6 \oplus M_3$.*

Here is the outline of this paper. In Section 2, we establish the matrix representations of $T_s(A)$ and $T_{as}(A)$. We also make several observations and prove several lemmas including the lemma that characterizes when a 3×3 matrix A is irreducible. Section 3 is devoted to the proof of Main Theorem. The proof is computational and relies on several simple yet crucial observations which greatly reduce the computation and the number of the cases needed to be covered. In Section 4, we ask when the operator $A \otimes I + I \otimes B$ is irreducible and present several examples which lead to an interesting open question.

2. Preliminaries

It is clear that $T_s(A)$ (respectively, $T_{as}(A)$) is irreducible if and only if $T_s(U^*AU)$ (respectively, $T_{as}(U^*AU)$) is irreducible, when U is unitary. Therefore, by Schur's unitary triangularization, we may assume that A is an upper triangular irreducible matrix. The following irreducible invariance of $T_s(A)$ will simplify our proofs.

Lemma 2.1. $T_s(A)$ (*respectively,* $T_{as}(A)$) *is irreducible if and only if* $T_s(\alpha A + \beta I)$ (*respectively,* $T_{as}(\alpha A + \beta I)$) *is irreducible, for any $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$.*

Proof. The result follows from the calculation

$$\begin{aligned} T_s(\alpha A + \beta I) &= \alpha T_s(A) + 2\beta(I \otimes I), \\ T_{as}(\alpha A + \beta I) &= \alpha T_{as}(A) + 2\beta(I \otimes I). \end{aligned} \qquad \square$$

The above simple observation allows us to assume that one of the eigenvalues of A is 0, and another eigenvalue of A is 1 if A has more than one distinct eigenvalue. We introduce some notation. Let

$$A = \begin{bmatrix} \beta & a & b \\ 0 & \gamma & c \\ 0 & 0 & \delta \end{bmatrix}, \quad T = A \otimes I + I \otimes A \simeq \begin{bmatrix} T_s & 0 \\ 0 & T_{as} \end{bmatrix},$$

$$f_1 = e_1 \otimes e_1, \quad f_2 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1), \quad f_3 = \frac{1}{\sqrt{2}}(e_1 \otimes e_3 + e_3 \otimes e_1),$$

$$f_4 = e_2 \otimes e_2, \quad f_5 = \frac{1}{\sqrt{2}}(e_2 \otimes e_3 + e_3 \otimes e_2), \quad f_6 = e_3 \otimes e_3,$$

$$g_1 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1), \quad g_2 = \frac{1}{\sqrt{2}}(e_1 \otimes e_3 - e_3 \otimes e_1),$$

$$g_3 = \frac{1}{\sqrt{2}}(e_2 \otimes e_3 - e_3 \otimes e_2),$$

where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{C}^3$. (We use T, T_s, and T_{as} to denote $T(A)$, $T_s(A)$, and $T_{as}(A)$, respectively, when the context is clear.) Then $H_s = \text{Span}\{f_1, f_2, f_3, f_4, f_5, f_6\}$ and $H_{as} = \text{Span}\{g_1, g_2, g_3\}$. By a direct computation, we have the following matrix representations of T_s and T_{as} under these orthonormal bases.

Lemma 2.2. *With respect to the orthonormal bases* $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ *and* $\{g_1, g_2, g_3\}$*, we have*

$$T_s = \begin{bmatrix} 2\beta & \sqrt{2}a & \sqrt{2}b & 0 & 0 & 0 \\ 0 & \beta+\gamma & c & \sqrt{2}a & b & 0 \\ 0 & 0 & \beta+\delta & 0 & a & \sqrt{2}b \\ 0 & 0 & 0 & 2\gamma & \sqrt{2}c & 0 \\ 0 & 0 & 0 & 0 & \gamma+\delta & \sqrt{2}c \\ 0 & 0 & 0 & 0 & 0 & 2\delta \end{bmatrix},$$

$$T_{as} = \begin{bmatrix} \beta+\gamma & c & -b \\ 0 & \beta+\delta & a \\ 0 & 0 & \gamma+\delta \end{bmatrix}.$$

Since we are dealing with a linear transformation acting on H_s and $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ is an orthonormal basis for H_s, we will denote a vector $v = \sum_{i=1}^{6} x_i f_i$ in H_s by $(x_1, \ldots, x_6)$. In other words, we will directly work with the matrix represented by $T_s(A)$. For example, when we say $e_1 - e_5$ is in $\ker T_s$, it actually means $f_1 - f_5$ is in $\ker T_s$.

We divide the proof of Main Theorem into three big cases pertaining to whether A has one or two, or three distinct eigenvalues. In each big case we further divide the proof into several small cases, according to whether some off-diagonal entries of A are zero or not. We have spent much time to consolidate and unify different cases, but we still have a number of cases to discuss to ensure the completeness and accuracy of our results. The following simple observation will be used repeatedly, sometimes without explicit mentioning. Let $\sigma(B)$ denote the set of (distinct) eigenvalues of B. For several subspaces H_i, $1 \le i \le k$, we denote by $\bigvee_{i=1}^{k} H_i$ the smallest subspace containing all H_i. An alternative notation is $\bigvee_{i=1}^{k} H_i = H_1 + H_2 + \cdots + H_k$.

Lemma 2.3. *Assume that* B *is reducible and* $B = B_1 \oplus B_2$ *on* $H_1 \oplus H_2$*. If* $\lambda \in \sigma(B)$ *and* $\ker(B - \lambda I)$ *is not orthogonal to* H_1*, then* $\lambda \in \sigma(B_1)$ *and* $\ker(B - \lambda I) \cap H_1 \neq \{0\}$*. In particular, if* $\ker(B - \lambda I)$ *is not orthogonal to* H_1 *and* $\dim \ker(B - \lambda I) = 1$*, then* $\ker(B - \lambda I) \subseteq H_1$*.*

Proof. By using the fact that H_i is invariant for B, we can show that

$$\begin{aligned} \ker(B - \lambda I) &= [\ker(B - \lambda I) \cap H_1] \oplus [\ker(B - \lambda I) \cap H_2] \\ &= [\ker(B_1 - \lambda I) \cap H_1] \oplus [\ker(B_2 - \lambda I) \cap H_2]. \end{aligned}$$

Thus if $\ker(B - \lambda I) \not\perp H_1$, then $\ker(B - \lambda I) \nsubseteq H_2$, and hence

$$\ker(B - \lambda I) \cap H_1 \neq \{0\}$$

and $\lambda \in \sigma(B_1)$. □

Our strategy is to look for orthogonal eigenspaces of the operator B to decompose H into a direct sum of reducing subspaces. When $\ker(B - \lambda I)$ (or $\ker(B^* - \overline{\lambda}I)$) is one-dimensional, then it is contained in one of the reducing subspaces. When two one-dimensional eigenspaces are not orthogonal, they have to be contained in the same reducing subspace. When the dimension of $\ker(B-\lambda I)$ is greater than one, then it needs to be decomposed into the sum of several orthogonal subspaces and the computation could be complicated. Once a vector v belongs to a reducing subspace, then Bv, B^*v, and so on, belong to the same reducing subspace. The orthogonality of reducing subspaces plays a key role in our approach. This is roughly the algorithm we will be using to decompose B into a direct sum of irreducible ones. When we failed to do the decomposition, we essentially prove that B is irreducible.

We also record the following obvious characterization of one dimensional reducing subspaces. It will be of interest to characterize higher dimensional reducing subspaces of B.

Lemma 2.4. *Let $v \neq 0 \in H$ and $B \in \mathcal{B}(H)$. Then* $\mathrm{Span}\{v\}$ *is a reducing subspace of B if and only if there exists $\lambda \in \sigma(B)$ such that*

$$Bv = \lambda v \quad \text{and} \quad B^* v = \overline{\lambda} v,$$

that is, B and B^* have a common eigenvector.

The following lemma is certainly known. We include a short proof using the concept of common eigenvector.

Lemma 2.5. *Let*

$$A = \begin{bmatrix} \beta & a & b \\ 0 & \gamma & c \\ 0 & 0 & \delta \end{bmatrix}.$$

Then the following statements hold.

(a) *If A has three distinct eigenvalues, then A is irreducible if and only if two of a, b, c are not zero.*

(b1) *If $\beta = \gamma \neq \delta$, then A is irreducible if and only if $a \neq 0$ and one of b, c is not zero.*

(b2) *If $\beta \neq \gamma = \delta$, then A is irreducible if and only if $c \neq 0$ and one of a, b is not zero.*

(b3) *If $\beta = \delta \neq \gamma$, then A is irreducible if and only if $(\gamma - \beta)b \neq ac$ and one of a, c is not zero.*

(c) *If A has one distinct eigenvalue, then A is irreducible if and only if $ac \neq 0$.*

Proof. We may assume that $\beta = 0$, and $\gamma = 1$ if $\beta \neq \gamma$.

(a) Note that $\delta \neq 0, 1$. Suppose that A is reducible. Then, by Lemma 2.3, A and A^* have a common eigenvector $v = (x, y, z)$. Then

$$Av = (ay + bz, y + cz, \delta z) \quad \text{and} \quad A^* v = (\overline{a}x + y, \overline{b}x + \overline{c}y + \overline{\delta}z).$$

If $Av = A^*v = 0v$, then $y = z = 0$, and so $a = b = 0$. If $Av = A^*v = 1v$, then $x = z = 0$, and so $a = c = 0$. If $Av = \delta v$ and $A^*v = \overline{\delta}v$, then $x = y = 0$, and so $b = c = 0$. Conversely, if two of a, b, c are zero, then by the above argument A and A^* have a common eigenvector, so A is reducible.

(b1) We may assume that $\delta = 1$. Suppose that A is reducible. Then A and A^* have a common eigenvector $v = (x, y, z)$. If $Av = A^*v = 0v$, then $z = 0$, and so $a = 0$. If $Av = A^*v = 1v$, then $x = y = 0$, and so $b = c = 0$. Conversely, if $a = 0$ or $b = c = 0$, then by the above argument A and A^* have a common eigenvector, so A is reducible.

The proof of (b2) is similar to that of (b1).

(b3) If $a = c = 0$, then $e_2 = (0, 1, 0)$ is a common eigenvector of A and A^*, so A is reducible. If one of a, c is not zero and $b = ac$, then the vector $(c, -\overline{a}c, \overline{a})$ is a common eigenvector of A and A^*, so A is reducible. Conversely, suppose that A is reducible. Then A and A^* have a common eigenvector $v = (x, y, z)$. If $Av = A^*v = v$, then $x = z = 0$, and so $a = c = 0$. If $Av = A^*v = 0$, then

$$\begin{bmatrix} a & b \\ 1 & c \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \overline{a} & 1 \\ \overline{b} & \overline{c} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $v \neq 0$, the matrices in the above equation cannot be invertible, i.e., $ac = b$.

(c) In this case $\beta = \gamma = \delta = 0$. Suppose that A is reducible. Then A and A^* have a common eigenvector $v = (x, y, z)$, and $Av = A^*v = 0v$. If $a \neq 0$, then $x = 0$, and so $c = 0$. Conversely, if $a = 0$ and $c = 0$, then $e_2 = (0, 1, 0)$ is a common eigenvector of A and A^*, so A is reducible. $\square$

Remark 2.6. We note that the results (b1) and (b2) are symmetric and simple, and the result (b3) seems more complicated. But Schur's unitary triangularization allows one to arrange the eigenvalues of A in any order on the diagonal of A so that we can reduce the case (b3) to the cases (b1) and (b2).

Proof of Main Theorem (c). Since T_{as} is a 3×3 matrix, we can use the above lemma to prove Main Theorem (c). By Schur's unitary triangularization and the remark after Lemma 2.1, we need only to check the following four cases:

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & \lambda \end{bmatrix},$$

where $\lambda \neq 0, 1$. The details are omitted. $\square$

3. Proof of Main Theorem

This section is devoted to the proof of Main Theorem (a) and (b). We first deal with the case where A has three distinct eigenvalues.

3.1. The case when A has three distinct eigenvalues

Let A be a 3×3 upper triangular irreducible matrix which has three distinct eigenvalues. By Lemma 2.1, we can assume that $\sigma(A)=\{0,1,\lambda\}$, where $\lambda\neq 0,1$. By Lemma 2.2, the eigenvalues of T_s are

$$\sigma(T_s)=\{0,1,\lambda,2,1+\lambda,2\lambda\}.$$

If $\lambda=2,-1,\frac{1}{2}$, then T_s has an eigenvalue of multiplicity 2. When $\lambda=2$, the condition that T_s is reducible is simple. When $\lambda=-1,\frac{1}{2}$, the condition that T_s is reducible is more complicated. However, the cases $\lambda=-1$ and $\lambda=\frac{1}{2}$ can be reduced to the case $\lambda=2$ since in all cases, the eigenvalues of A form an arithmetic sequence. For example, if $\sigma(A)=\{0,1,-1\}$, then $A+I$ is unitarily equivalent to some B such that $\sigma(B)=\{0,1,2\}$ and the reducibility of $T(A)$ and the reducibility of $T(B)$ are the same.

Case 3.1. *Suppose that*

$$A=\begin{bmatrix}0 & a & b\\ 0 & 1 & c\\ 0 & 0 & 2\end{bmatrix}\quad \textit{is irreducible.}$$

Then T_s is reducible if and only if

$$b=0\quad \textit{and}\quad |a|=|c|,$$

in which case T_s has two minimal reducing subspaces H_1 and H_2, where $\dim H_1=5$ *and* $\dim H_2=1$.

Proof. Since A is irreducible, two of a,b,c are not zero. Note that

$$T_s=\begin{bmatrix}0 & \sqrt{2}a & \sqrt{2}b & 0 & 0 & 0\\ 0 & 1 & c & \sqrt{2}a & b & 0\\ 0 & 0 & 2 & 0 & a & \sqrt{2}b\\ 0 & 0 & 0 & 2 & \sqrt{2}c & 0\\ 0 & 0 & 0 & 0 & 3 & \sqrt{2}c\\ 0 & 0 & 0 & 0 & 0 & 4\end{bmatrix},$$

$$T_s^*=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ \sqrt{2}\overline{a} & 1 & 0 & 0 & 0 & 0\\ \sqrt{2}\overline{b} & \overline{c} & 2 & 0 & 0 & 0\\ 0 & \sqrt{2}\overline{a} & 0 & 2 & 0 & 0\\ 0 & \overline{b} & \overline{a} & \sqrt{2}\overline{c} & 3 & 0\\ 0 & 0 & \sqrt{2}\overline{b} & 0 & \sqrt{2}\overline{c} & 4\end{bmatrix}.$$

By a direct computation,

$$\begin{aligned}\ker T_s &= \operatorname{Span}\{e_1\},\\ \ker T_s^* &= \operatorname{Span}\{(1,-\sqrt{2}\overline{a},\tfrac{(\overline{ac}-\overline{b})}{\sqrt{2}},\overline{a}^2,\tfrac{(\overline{b}-\overline{ac})\overline{a}}{\sqrt{2}},\tfrac{(\overline{b}-\overline{ac})^2}{4})\},\\ \ker(T_s-I) &= \operatorname{Span}\{(\sqrt{2}a,1,0,0,0,0)\},\end{aligned}$$

$$\begin{aligned}
\ker(T_s^*-I) &= \mathrm{Span}\{(0,1,-\overline{c},-\sqrt{2}\overline{a},\tfrac{3\overline{ac}-\overline{b}}{2},\tfrac{\overline{c}(\overline{b}-\overline{ac})}{\sqrt{2}})\},\\
\ker(T_s-3I) &= \mathrm{Span}\{(\tfrac{a(b+ac)}{\sqrt{2}},\tfrac{b+3ac}{2},a,\sqrt{2}c,1,0)\},\\
\ker(T_s^*-3I) &= \mathrm{Span}\{(0,0,0,0,1,-\sqrt{2}\overline{c})\},\\
\ker(T_s-4I) &= \mathrm{Span}\{(\tfrac{(ac+b)^2}{4},\tfrac{(ac+b)c}{\sqrt{2}},\tfrac{ac+b}{\sqrt{2}},c^2,\sqrt{2}c,1)\},\\
\ker(T_s^*-4I) &= \mathrm{Span}\{e_6\}.
\end{aligned}$$

Thus T_s and T_s^* has no common eigenvector corresponding to the eigenvalues $\{0,1,3,4\}$. However, it follows from

$$\begin{aligned}
\ker(T_s-2I) &= \mathrm{Span}\{(a^2,\sqrt{2}a,0,1,0,0),(b+ac,\sqrt{2}c,\sqrt{2},0,0,0)\},\\
\ker(T_s^*-2I) &= \mathrm{Span}\{(0,0,0,1,-\sqrt{2}\overline{c},\overline{c}^2),(0,0,\sqrt{2},0,-\sqrt{2}\overline{a},\overline{ac}-\overline{b})\}
\end{aligned}$$

that $\ker(T_s-2I)\cap\ker(T_s^*-2I)\neq\{0\}$ if and only if $b=0$ and $|a|=|c|$. In this case

$$v := (0,0,\sqrt{2}a,-c,0,0)\in\ker(T_s-2I)\cap\ker(T_s^*-2I).$$

Since two of a,b,c are not zero, we divide the proof into several cases according to which two of a,b,c are not zero.

Case 1: $b=0$ and $|a|=|c|\neq 0$. Then v as above is a common eigenvector of T_s and T_s^*. Hence T_s is reducible. We will show that $T_1 := T_s|\mathrm{Span}\{v\}^\perp$ is irreducible. Assume to the contrary that T_s is reducible so that $T_1 = T_3\oplus T_4$ on $\mathrm{Span}\{v\}^\perp = H_3\oplus H_4$, where $\dim H_i\geq 1$, $i=3,4$. Since $\ker T_s=\mathrm{Span}\{e_1\}$ is one-dimensional, by Lemma 2.3, we may assume that

$$\ker T_s = \mathrm{Span}\{e_1\}\subseteq H_3.$$

Observe that $\ker T_s^*$ is not orthogonal to $\ker T_s$, and $\ker(T_s^*-4I)$ is not orthogonal to $\ker T_s^*$. It follows from Lemma 2.3 that $\ker(T_s^*-4I)=\mathrm{Span}\{e_6\}\subseteq H_3$, so $\{e_1,e_6\}\subseteq H_3$. Since $b=0$, $T_se_6=(0,0,0,0,\sqrt{2}c,4)\in H_3$ and $T_s^*e_1=(0,\sqrt{2}\overline{a},0,0,0,0)\in H_3$. It follows that $\{e_1,e_2,e_5,e_6\}\subseteq H_3$. It is easy to see that $\{e_1,e_2,e_5,e_6,T_se_5\}$ is a linearly independent subset of H_3. Thus $\dim H_3=5$, which is a contradiction to $\dim H_4\geq 1$. Therefore, T_1 is irreducible. We conclude that T_s has two minimal reducing subspaces $H_1=\mathrm{Span}\{v\}^\perp$ and $H_2=\mathrm{Span}\{v\}$.

In the remaining cases $b\neq 0$ or $|a|\neq|c|$, we need to prove that T_s is irreducible. Since T_s and T_s^* share no eigenvectors, if T_s were reducible, then $T_s=T_1\oplus T_2$ on $H_1\oplus H_2$ with $\dim H_i\geq 2$, $i=1,2$. We will show that this does not happen. Without loss of generality, assume

$$\ker T_s=\mathrm{Span}\{e_1\}\subseteq H_1.$$

Case 2: $b=0$, $ac\neq 0$ and $|a|\neq|c|$. If $b=0$, then by the same argument of Case 1, we have $\{e_1,e_2,e_5,e_6,T_se_5\}\subseteq H_1$, and $\dim H_1\geq 5$, which is a contradiction.

Case 3: $a=0$ and $bc\neq 0$. Since $e_1\in H_1$, it follows that $T_s^*e_1 = (0,0,\sqrt{2}\overline{b},0,0,0)\in H_1$. Thus $e_3\in H_1$. Since $a=0$, $T_s^*e_3=(0,0,2,0,0,\sqrt{2}\overline{b})\in$

H_1, we have $e_6 \in H_1$. Since $T_s e_6 = (0, 0, \sqrt{2}b, 0, \sqrt{2}c, 4) \in H_1$, we have $e_5 \in H_1$. Since $b \neq 0$, it is easy to see that $\{e_1, e_3, e_5, e_6, T_s e_5\}$ is a linearly independent subset of H_1. Thus $\dim H_1 \geq 5$, which is a contradiction.

Case 4: $ab \neq 0$. Since $a \neq 0$, $\ker(T_s - I)$ is not orthogonal to $\ker T_s$. Thus $\ker(T_s - I) \subseteq H_1$. It follows that $\{e_1, e_2\} \subseteq H_1$. Since

$$T_s^* e_1 = (0, \sqrt{2}\overline{a}, \sqrt{2}\overline{b}, 0, 0, 0) \in H_1,$$

we have $e_3 \in H_1$. Since $ab \neq 0$, it is easy to see that $\{e_1, e_2, e_3, T_s^* e_2, T_s^* e_3\}$ is a linearly independent subset of H_1. Thus $\dim H_1 \geq 5$, which is a contradiction. The proof is complete. $\square$

The following numerical example illustrates the above result.

Example 3.2. Consider the irreducible matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Then $T(A) = A \otimes I + I \otimes A$ is unitarily equivalent to

$$\begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 3 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \oplus [2] \oplus \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix},$$

and each block is irreducible.

Now we ready to prove one part of Main Theorem.

Proof of Main Theorem when A has three distinct eigenvalues. We first prove (a). Suppose that A is a 3×3 irreducible matrix which is unitarily equivalent to $\alpha I + dD + aJ$ for some α, d, a. Since A is irreducible, we have $d \neq 0$. Let $B := \frac{1}{d}(A - \alpha I)$. Then B is unitarily equivalent to $D + \frac{a}{d}J$. Then, by Case 3.1, $T_s(B)$ is reducible and has two minimal reducing subspaces whose dimensions are 5 and 1. It follows from Lemma 2.1 that $T_s(A)$ is reducible and has two minimal reducing subspaces whose dimensions are 5 and 1. This completes the proof of (a).

Now we prove (b). Suppose that A is a 3×3 irreducible matrix which is not unitarily equivalent to $\alpha I + dD + aJ$ for any α, d, a. We need to prove that $T_s(A)$ is irreducible. There are two cases according to the eigenvalues of A.

First assume that $\sigma(A) = \{\alpha, \alpha{+}d, \alpha{+}2d\}$ for some α, d. Then A is unitarily equivalent to $\alpha I + dD + J(a, b, c)$ for some a, b, c, where

$$J(a, b, c) = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

Put $B := \frac{1}{d}(A - \alpha I)$. Then B is unitarily equivalent to $D + J(\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$. By Lemma 2.1, if $T_s(A)$ were reducible, then $T_s(B)$ would be reducible. It follows from Case 3.1 that $\frac{b}{d} = 0$ and $|\frac{a}{d}| = |\frac{c}{d}|$. Then $b = 0$ and $|a| = |c|$. Thus A is unitarily equivalent to $\alpha I + dD + J(a, 0, c)$. Since $|a| = |c|$, A is unitarily equivalent to $\alpha I + dD + aJ$, which is a contradiction. Therefore $T_s(A)$ is irreducible.

Next assume that the eigenvalues of A do not form a arithmetic sequence. By Lemma 2.1, we can assume that

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & \lambda \end{bmatrix}, \quad \text{where } A \text{ is irreducible and } \lambda \neq 0, 1, 2, -1, \frac{1}{2}.$$

Since A is irreducible, one of the following holds:

$$\text{(i) } ac \neq 0; \quad \text{(ii) } c = 0 \text{ and } ab \neq 0; \quad \text{(iii) } a = 0 \text{ and } bc \neq 0.$$

Note that

$$T_s = \begin{bmatrix} 0 & \sqrt{2}a & \sqrt{2}b & 0 & 0 & 0 \\ 0 & 1 & c & \sqrt{2}a & b & 0 \\ 0 & 0 & \lambda & 0 & a & \sqrt{2}b \\ 0 & 0 & 0 & 2 & \sqrt{2}c & 0 \\ 0 & 0 & 0 & 0 & 1+\lambda & \sqrt{2}c \\ 0 & 0 & 0 & 0 & 0 & 2\lambda \end{bmatrix},$$

$$T_s^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{a} & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{b} & \overline{c} & \overline{\lambda} & 0 & 0 & 0 \\ 0 & \sqrt{2}\overline{a} & 0 & 2 & 0 & 0 \\ 0 & \overline{b} & \overline{a} & \sqrt{2}\overline{c} & 1+\overline{\lambda} & 0 \\ 0 & 0 & \sqrt{2}\overline{b} & 0 & \sqrt{2}\overline{c} & 2\overline{\lambda} \end{bmatrix},$$

$$\begin{aligned}
\ker T_s &= \operatorname{Span}\{e_1\}, \\
\ker T_s^* &= \operatorname{Span}\{(1, -\sqrt{2}\overline{a}, \tfrac{\sqrt{2}(\overline{ac}-\overline{b})}{\overline{\lambda}}, \overline{a}^2, \tfrac{\sqrt{2}(\overline{b}-\overline{ac})\overline{a}}{\overline{\lambda}}, \tfrac{(\overline{b}-\overline{ac})^2}{\overline{\lambda}^2})\}, \\
\ker(T_s - I) &= \operatorname{Span}\{(\sqrt{2}a, 1, 0, 0, 0, 0)\}, \\
\ker(T_s^* - I) &= \operatorname{Span}\{(0, 1, \tfrac{\overline{c}}{1-\overline{\lambda}}, -\sqrt{2}\overline{a}, \tfrac{1-2\overline{\lambda}}{\overline{\lambda}}\tfrac{\overline{ac}}{1-\overline{\lambda}} - \tfrac{\overline{b}}{\overline{\lambda}}, \tfrac{\sqrt{2}\overline{c}(\overline{ac}-\overline{b})}{\overline{\lambda}(1-\overline{\lambda})})\}, \\
\ker(T_s - \lambda I) &= \operatorname{Span}\{(\sqrt{2}(\tfrac{b}{\lambda} - \tfrac{a}{\lambda}\tfrac{c}{1-\lambda}), \tfrac{-c}{1-\lambda}, 1, 0, 0, 0)\}, \\
\ker(T_s^* - \overline{\lambda} I) &= \operatorname{Span}\{(0, 0, 1, 0, -\overline{a}, \tfrac{\sqrt{2}(\overline{ac}-\overline{b})}{\overline{\lambda}})\}, \\
\ker(T_s - 2I) &= \operatorname{Span}\{(a^2, \sqrt{2}a, 0, 1, 0, 0)\}, \\
\ker(T_s^* - 2I) &= \operatorname{Span}\{(0, 0, 0, 1, \tfrac{\sqrt{2}\overline{c}}{1-\overline{\lambda}}, (\tfrac{\overline{c}}{1-\overline{\lambda}})^2)\}, \\
\ker(T_s - (1+\lambda)I) &= \operatorname{Span}\{(\tfrac{\sqrt{2}ab}{\lambda} - \tfrac{\sqrt{2}}{\lambda}\tfrac{a^2c}{1-\lambda}, \tfrac{b}{\lambda} - \tfrac{1+\lambda}{\lambda}\tfrac{ac}{1-\lambda}, a, \tfrac{-\sqrt{2}c}{1-\lambda}, 1, 0)\},
\end{aligned}$$

$$\begin{aligned}
\ker(T_s^* - (1+\overline{\lambda})I) &= \mathrm{Span}\{(0,0,0,0,1,\tfrac{\sqrt{2}\overline{c}}{1-\overline{\lambda}})\},\\
\ker(T_s - 2\lambda I) &= \mathrm{Span}\{((\tfrac{a}{\lambda}\tfrac{c}{1-\lambda} - \tfrac{b}{\lambda})^2, \tfrac{\sqrt{2}}{\lambda}(\tfrac{ac^2}{(1-\lambda)^2} - \tfrac{bc}{1-\lambda}),\\
&\qquad \tfrac{\sqrt{2}}{\lambda}(b - \tfrac{ac}{1-\lambda}), (\tfrac{c}{1-\lambda})^2, \tfrac{-\sqrt{2}c}{1-\lambda}, 1)\},\\
\ker(T_s^* - 2\overline{\lambda}I) &= \mathrm{Span}\{e_6\}.
\end{aligned}$$

Since every eigenspace is one-dimensional, it is easy to check that T_s and T_s^* share no eigenvectors. We will show that T_s is irreducible. Assume that T_s is reducible, that is, $T_s = T_1 \oplus T_2$ on $H_1 \oplus H_2$ where $\dim(H_i) \geq 2$, $i = 1, 2$. Without loss of generality, assume

$$0 \in \sigma(T_1) \quad \text{and} \quad \ker T_s = \mathrm{Span}\{e_1\} \subseteq H_1. \tag{2}$$

Case 1: $ac \neq 0$. Note that neither $\ker(T_s - I)$ nor $\ker(T_s - 2I)$ is orthogonal to $\ker T_s$ since $a \neq 0$. Therefore, by (2) and Lemma 2.3,

$$\ker T_s + \ker(T_s - I) + \ker(T_s - 2I) \subseteq H_1, \quad \text{and} \quad \mathrm{Span}\{e_1, e_2, e_4\} \subseteq H_1.$$

It follows from $T_s^* e_4 = (0,0,0,2,\sqrt{2}\overline{c},0)$ and $c \neq 0$ that $e_5 \in H_1$. Since $c \neq 0$, it is easy to see that

$$\dim H_1 \geq \dim \mathrm{Span}\{e_1, e_2, e_4, e_5, T_s^* e_5\} = 5,$$

which is a contradiction to $\dim H_2 \geq 2$.

Case 2: $c = 0$ and $ab \neq 0$. Since $a \neq 0$, as in the previous case we have $e_1, e_2, e_4 \in H_1$. It follows from $T_s^* e_2 = (0,1,0,\sqrt{2}\overline{a},\overline{b},0)$ and $b \neq 0$ that $e_5 \in H_1$. Since $a \neq 0$, it is easy to see that $\dim H_1 \geq \dim \mathrm{Span}\{e_1, e_2, e_4, e_5, T_s e_5\} = 5$, which is a contradiction to $\dim H_2 \geq 2$.

Case 3: $a = 0$ and $bc \neq 0$. Since $e_1 \in H_1$, it follows that $T_s^* e_1 = (0,0,\sqrt{2}\overline{b},0,0,0) \in H_1$. Since $b \neq 0$, we have $e_3 \in H_1$. Then $T_s e_3 = (\sqrt{2}b, c, \lambda, 0,0,0) \in H_1$ and $T_s^* e_3 = (0,0,\overline{\lambda},0,0,\sqrt{2}\overline{b}) \in H_1$. Since $bc \neq 0$, we have $e_2, e_6 \in H_1$. Since $T_s e_6 = (0,0,\sqrt{2}b,0,\sqrt{2}c,2\lambda) \in H_1$ and $c \neq 0$, we have $e_5 \in H_1$. Thus $\dim H_1 \geq \dim \mathrm{Span}\{e_1, e_2, e_3, e_5, e_6\} = 5$, which is a contradiction to $\dim H_2 \geq 2$.

Therefore T_s is irreducible. The proof is complete. □

3.2. The case when A has two distinct eigenvalues

We now deal with the case that A has two distinct eigenvalues. Then A cannot be unitarily equivalent to $\alpha I + dD + aJ$ for any α, d, a. To prove Main Theorem, we will show that $T_s(A)$ is irreducible.

Proof of Main Theorem when A has two distinct eigenvalues. Suppose that A is a 3×3 irreducible matrix with two distinct eigenvalues. By Schur's unitary triangularization and Lemma 2.1, we can assume that

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$T_s = \begin{bmatrix} 0 & \sqrt{2}a & \sqrt{2}b & 0 & 0 & 0 \\ 0 & 0 & c & \sqrt{2}a & b & 0 \\ 0 & 0 & 1 & 0 & a & \sqrt{2}b \\ 0 & 0 & 0 & 0 & \sqrt{2}c & 0 \\ 0 & 0 & 0 & 0 & 1 & \sqrt{2}c \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

$$T_s^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{a} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{b} & \overline{c} & 1 & 0 & 0 & 0 \\ 0 & \sqrt{2}\overline{a} & 0 & 0 & 0 & 0 \\ 0 & \overline{b} & \overline{a} & \sqrt{2}\overline{c} & 1 & 0 \\ 0 & 0 & \sqrt{2}\overline{b} & 0 & \sqrt{2}\overline{c} & 2 \end{bmatrix},$$

and

$$\begin{aligned} \ker T_s = \operatorname{Span}\{e_1\}, \quad \ker T_s^* &= \operatorname{Span}\{(0,0,0,1,-\sqrt{2}\overline{c},\overline{c}^2)\}, \\ \ker(T_s - I) &= \operatorname{Span}\{(\sqrt{2}(ac+b), c, 1, 0, 0, 0)\}, \\ \ker(T_s^* - I) &= \operatorname{Span}\{(0,0,0,0,1,-\sqrt{2}\overline{c})\}, \\ \ker(T_s - 2I) &= \operatorname{Span}\{((ac+b)^2, \sqrt{2}c(ac+b), \sqrt{2}(ac+b), c^2, \sqrt{2}c, 1)\}, \\ \ker(T_s^* - 2I) &= \operatorname{Span}\{e_6\}. \end{aligned}$$

We will show that T_s is irreducible by a contradiction. Assume that T_s is reducible. Since every eigenspace is one-dimensional, it is easy to check that T_s and T_s^* share no eigenvectors. Thus $T_s = T_1 \oplus T_2$ on $H_1 \oplus H_2$, where $\dim H_i \geq 2$, $i = 1, 2$. Without loss of generality, assume

$$2 \in \sigma(T_1) \quad \text{and} \quad \ker(T_s^* - 2I) = \operatorname{Span}\{e_6\} \subseteq H_1. \tag{3}$$

Since A is irreducible, Lemma 2.5(a) shows that one of the following holds:

(i) $ac \neq 0$; (ii) $c = 0$ and $ab \neq 0$.

Case 1: $ac \neq 0$. Since $c \neq 0$, neither $\ker T_s^*$ nor $\ker(T_s^* - I)$ is orthogonal to $\ker(T_s^* - 2I)$. Therefore, by (3) and Lemma 2.3,

$$\ker T_s^* + \ker(T_s^* - I) + \ker(T_s^* - 2I) \subseteq H_1, \quad \text{and} \quad \{e_4, e_5, e_6\} \subseteq H_1.$$

Then $T_s e_4 = (0, \sqrt{2}a, 0, 0, 0, 0) \in H_1$. Since $a \neq 0$, we have $e_2 \in H_1$, and $e_1 = \frac{1}{\sqrt{2}a} T_s e_2 \in H_1$. It follows that $\dim H_1 \geq \dim \operatorname{Span}\{e_1, e_2, e_4, e_5, e_6\} = 5$, which is a contradiction to $\dim H_2 \geq 2$.

Case 2: $c = 0$ and $ab \neq 0$. Since $\ker(T_s - 2I) = \operatorname{Span}\{(b^2, 0, \sqrt{2}b, 0, 0, 1)\}$ is not orthogonal to $\ker(T_s^* - 2I)$, we have $\ker(T_s - 2I) \subseteq H_1$, by (3) and Lemma 2.3. Since $b \neq 0$, neither $\ker T_s$ nor $\ker(T_s - I)$ is orthogonal to $\ker(T_s - 2I)$. It follows that

$$\ker T_s + \ker(T_s - I) + \ker(T_s^* - 2I) \subseteq H_1, \quad \text{and} \quad \{e_1, e_3, e_6\} \subseteq H_1.$$

Then $T_s^* e_1 = (0, \sqrt{2}\overline{a}, \sqrt{2}\overline{b}, 0, 0, 0) \in H_1$ and $T_s^* e_3 = (0, 0, 1, 0, \overline{a}, \sqrt{2}\overline{b}) \in H_1$. Since $a \neq 0$, we have $e_2, e_5 \in H_1$. It follows that

$$\dim H_1 \geq \dim \operatorname{Span}\{e_1, e_2, e_3, e_5, e_6\} = 5,$$

which is a contradiction to $\dim H_2 \geq 2$. □

3.3. The case when A has one distinct eigenvalue

Finally, we deal with the case when A has only one distinct eigenvalue. By Lemma 2.1, we can assume $\sigma(A) = \{0\}$ and we can also scale one of the nonzero off-diagonal entry of A to be 1. Let

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

We note that if A is irreducible, then $ac \neq 0$ by Lemma 2.5(c). We will assume $a = 1$. Then

$$T_s = \begin{bmatrix} 0 & \sqrt{2} & \sqrt{2}b & 0 & 0 & 0 \\ 0 & 0 & c & \sqrt{2} & b & 0 \\ 0 & 0 & 0 & 0 & 1 & \sqrt{2}b \\ 0 & 0 & 0 & 0 & \sqrt{2}c & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}c \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T_s^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\overline{b} & \overline{c} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \overline{b} & 1 & \sqrt{2}\overline{c} & 0 & 0 \\ 0 & 0 & \sqrt{2}\overline{b} & 0 & \sqrt{2}\overline{c} & 0 \end{bmatrix},$$

and

$$\begin{aligned} \ker T_s &= \operatorname{Span}\{(1,0,0,0,0,0), (0, \sqrt{2}b, -\sqrt{2}, c, 0, 0)\}, \\ \ker T_s^* &= \operatorname{Span}\{(0,0,-\sqrt{2}\overline{c}, 1, \sqrt{2}\overline{b}, 0), (0,0,0,0,0,1)\}. \end{aligned}$$

It is easy to check that T_s and T_s^* have a common eigenvector if and only if

$$(0, \sqrt{2}b, -\sqrt{2}, c, 0, 0) = c(0, 0, -\sqrt{2}\overline{c}, 1, \sqrt{2}\overline{b}, 0)$$

if and only if $b = 0$ and $|c| = 1$. We will show that T_s is reducible if and only if $b = 0$ and $|c| = 1$. The above two-dimensional kernels make the proof more difficult. The following lemma reveals a structure of those kernels.

Lemma 3.3. *Let B be an $n \times n$ matrix with $n \geq 2$ such that $\sigma(B) = \{\lambda\}$. Then there exist nonzero $v \in \ker(B - \lambda I)$ and nonzero $u \in \ker(B^* - \overline{\lambda} I)$ such that $v \perp u$.*

Proof. By Schur's unitary triangularization, there exists a unitary matrix U such that $U^*(B-\lambda I)U$ is a strictly upper triangular $n\times n$ matrix. It is easy to see that $U^*(B-\lambda I)Ue_1=0$ and $U^*(B^*-\overline{\lambda}I)Ue_n=0$. Then $v=Ue_1$ and $u=Ue_n$ satisfy the desired properties. $\square$

Case 3.4. *Suppose that*

$$A=\begin{bmatrix}0&1&b\\0&0&c\\0&0&0\end{bmatrix},\quad \textit{where } c\neq 0.$$

Then T_s is reducible if and only if $b=0$ and $|c|=1$, in which case T_s has two minimal reducing subspaces H_1 and H_2, where $\dim H_1=5$ *and* $\dim H_2=1$.

Proof. Assume $b=0$ and $|c|=1$. Let $v=(0,0,-\sqrt{2},c,0,0)$. Then $\ker T_s\cap\ker T_s^*=\operatorname{Span}\{v\}$, and

$$\ker T_s=\operatorname{Span}\{v,e_1\}\quad\text{and}\quad \ker T_s^*=\operatorname{Span}\{v,e_6\}. \tag{4}$$

Let $T_1=T_s|\operatorname{Span}\{v\}^\perp$ and $T_2=T_s|\operatorname{Span}\{v\}$. By (4), $\ker T_1=\ker T_s\cap\operatorname{Span}\{v\}^\perp=\operatorname{Span}\{e_1\}$. Assume that T_1 is reducible so that $T_1=T_3\oplus T_4$ on $H_3\oplus H_4$ with $\dim H_i\geq 1$, $i=3,4$. Then $\sigma(T_3)=\sigma(T_4)=\{0\}$, and so $\ker T_1=\ker T_3\oplus\ker T_4$. Hence $\dim\ker T_1\geq 2$, which is a contradiction. Therefore T_1 is irreducible. We conclude that T_s has two minimal reducing subspaces $H_1=\operatorname{Span}\{v\}^\perp$ and $H_2=\operatorname{Span}\{v\}$.

Assume $b\neq 0$ or $|c|\neq 1$. We will show that T_s is irreducible by a contradiction. Assume that T_s is reducible, that is, $T_s=T_1\oplus T_2$ on $H_1\oplus H_2$ with $\dim H_i\geq 2$ for $i=1,2$ since T_s and T_s^* have no common eigenvector. It is clear that

$$\sigma(T_1)=\sigma(T_2)=\{0\}. \tag{5}$$

Note that

$$\ker T_s=[\ker T_s\cap H_1]\oplus[\ker T_s\cap H_2],$$
$$\ker T_s^*=[\ker T_s^*\cap H_1]\oplus[\ker T_s^*\cap H_2].$$

Therefore,

$$\ker T_s\cap H_1=\operatorname{Span}\{v_1\},\quad \ker T_s\cap H_2=\operatorname{Span}\{v_2\},$$
$$\ker T_s^*\cap H_1=\operatorname{Span}\{v_3\},\quad \ker T_s^*\cap H_2=\operatorname{Span}\{v_4\},$$

for some $v_1,v_3\in H_1$ and $v_2,v_4\in H_2$. However, by (5) and Lemma 3.3, there exist $v\in\ker T_1=\ker T_s\cap H_1$ and $u\in\ker T_1^*=\ker T_s^*\cap H_1$ such that $v\perp u$. So, up to scalar multiples, $v_1=v$, $v_3=u$, and $v_1\perp v_3$. Similarly $v_2\perp v_4$. Thus $\{v_1,v_2,v_3,v_4\}$ is a set of orthogonal vectors. In particular, $\ker T_s\perp\ker T_s^*$, which is a contradiction since $(0,\sqrt{2}b,-\sqrt{2},c,0,0)$ is not orthogonal to $(0,0,-\sqrt{2}\overline{c},1,\sqrt{2}\overline{b},0)$. Therefore T_s is irreducible. $\square$

Now we are ready to prove Main Theorem.

Proof of Main Theorem when A has one distinct eigenvalue. We first prove (a). Suppose that A is a 3×3 irreducible matrix which is unitarily equivalent to $\alpha I + dD + aJ$ for some α, d, a. Since A has one distinct eigenvalue, we have $d = 0$. Put $B := A - \alpha I$. Then B is unitarily equivalent to aJ. By Case 3.4, $T_s(B)$ is reducible and has two minimal reducing subspaces whose dimensions are 5 and 1. It follows from Lemma 2.1 that $T_s(A)$ is reducible and has two minimal reducing subspaces whose dimensions are 5 and 1. This complete the proof of (a).

Next we prove (b).

Suppose that A is a 3×3 irreducible matrix which is not unitarily equivalent to $\alpha I + dD + aJ$ for any α, d, a. If we let $\sigma(A) = \{\alpha\}$, then $B := A - \alpha I$ is unitarily equivalent to $J(a, b, c)$ as in (1). If $T_s(A)$ were reducible, then $T_s(B)$ would be reducible, by Lemma 2.1. It follows from Case 3.4 that $b = 0$ and $|a| = |c|$. Thus $A - \alpha I$ is unitarily equivalent to $J(a, 0, c)$. Since $|a| = |c|$, A is unitarily equivalent to $\alpha I + aJ$, which is a contradiction. Therefore $T_s(A)$ is irreducible. $\square$

4. The case of $A \otimes I + I \otimes B$ and an open question

In this section we would like to mention a problem for which a preliminary work is done.

Suppose that A and B are $n \times n$ irreducible matrices. Let

$$T = T_{A,B} = A \otimes I + I \otimes B.$$

We use T to denote $T_{A,B}$ when the context is clear. There seem no obvious reducing subspaces of $T_{A,B}$. But if B is unitarily equivalent to $A + \lambda I$ for some $\lambda \in \mathbb{C}$, that is, $U^*BU = A + \lambda I$ for some unitary matrix U, then

$$\begin{aligned}(I \otimes U)^* T_{A,B}(I \otimes U) &= A \otimes U^*U + I \otimes U^*BU \\ &= A \otimes I + I \otimes (A + \lambda I) = T(A) + \lambda(I \otimes I).\end{aligned}$$

Hence, $T_{A,B}$ is unitarily equivalent to $T(A) + \lambda(I \otimes I)$. In particular, $T_{A,B}$ is reducible since $T(A)$ is reducible. We would like to verify whether $T_{A,B}$ is reducible when B is not unitarily equivalent to $A + \lambda I$ for any λ. When A and B are 2×2 irreducible matrices and if B is not unitarily equivalent to any $A + \lambda I$, then we can show that $T_{A,B}$ is irreducible.

Proposition 4.1. *Let A, B be 2×2 irreducible matrices. Then B is unitarily equivalent to $A + \lambda I$ for some $\lambda \in \mathbb{C}$ if and only if $T_{A,B}$ is reducible.*

Proof. It follows from the above discussion, we need to show that if B is not unitarily equivalent to any $A + \lambda I$, then $T_{A,B}$ is irreducible. Assume now that B is not unitarily equivalent to any $A + \lambda I$. Since

$$(\gamma A + \alpha I) \otimes I + I \otimes (\gamma B + \beta I) = \gamma T_{A,B} + (\alpha + \beta)(I \otimes I),$$

we can assume that ones of the eigenvalues of A and B are 0, and another eigenvalue of A is 1 if A has two one distinct eigenvalues. We can also assume that $(1,2)$-entries of A and B are positive by unitary equivalence. We will show that $T_{A,B}$ is irreducible by a contradiction. Assume that $T_{A,B}$ is reducible, that is, $T_{A,B} = T_1 \oplus T_2$ on $H_1 \oplus H_2$ with $\dim H_i \geq 1$, $i = 1, 2$.

There are several cases according to the numbers of eigenvalues of A and B. Here we just prove the case that both A and B have one distinct eigenvalue, and the proofs for other cases are omitted since they are similar to the approach in previous section.

Suppose that $\sigma(A) = \{0\}$ and $\sigma(B) = \{0\}$. In this case we may assume that

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad \text{where } a > 0,\ a \neq 1.$$

Then

$$T = T_{A,B} = \begin{bmatrix} 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad T^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \end{bmatrix}.$$

Note that

$$\begin{aligned} \ker T &= \operatorname{Span}\{(1,0,0,0), (0,1,-a,0)\}, \\ \ker T^* &= \operatorname{Span}\{(0,-a,1,0), (0,0,0,1)\}. \end{aligned}$$

Since $a \neq 1$, T and T^* have no common eigenvector. We can assume that $\dim H_i = 2$ for $i = 1, 2$. It is clear that

$$\sigma(T_1) = \sigma(T_2) = \{0\}.$$

Note that

$$\begin{aligned} \ker T &= [\ker T \cap H_1] \oplus [\ker T \cap H_2], \\ \ker T^* &= [\ker T^* \cap H_1] \oplus [\ker T^* \cap H_2]. \end{aligned}$$

Therefore,

$$\begin{aligned} \ker T \cap H_1 &= \operatorname{Span}\{v_1\}, \quad \ker T \cap H_2 = \operatorname{Span}\{v_2\}, \\ \ker T^* \cap H_1 &= \operatorname{Span}\{v_3\}, \quad \ker T^* \cap H_2 = \operatorname{Span}\{v_4\}, \end{aligned}$$

for some $v_1, v_3 \in H_1$ and $v_2, v_4 \in H_2$. However, by (5) and Lemma 3.3, there exist $v \in \ker T_1 = \ker T_s \cap H_1$ and $u \in \ker T_1^* = \ker T_s^* \cap H_1$ such that $v \perp u$. So, up to scalar multiples, $v_1 = v$, $v_3 = u$, and $v_1 \perp v_3$. Similarly $v_2 \perp v_4$. Thus $\{v_1, v_2, v_3, v_4\}$ is a set of orthogonal vectors. In particular, $\ker T \perp \ker T^*$, which is a contradiction since $a > 0$. This proves that T is irreducible. $\square$

Remark 4.2. When A and B are unitarily equivalent, after translation and scaling, we may assume that

$$A = B = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix}.$$

Then $T_{A,B}$ is unitarily equivalent to the matrix

$$\left[\begin{array}{ccc|c} 0 & \sqrt{2} & 0 & 0 \\ 0 & \alpha & \sqrt{2} & 0 \\ 0 & 0 & 2\alpha & 0 \\ \hline 0 & 0 & 0 & \alpha \end{array}\right]$$

with respect to the orthonormal basis

$$\left\{ e_1 \otimes e_1, \frac{e_1 \otimes e_2 + e_2 \otimes e_1}{\sqrt{2}}, e_2 \otimes e_2, \frac{e_1 \otimes e_2 - e_2 \otimes e_1}{\sqrt{2}} \right\}.$$

Hence, $T_{A,B}$ has two minimal subspaces H_s and H_{as}.

We would like to mention the following deep result of Arveson for unitary similarity [1], [8]. For a matrix F, let $\|F\|$ denote the spectral norm of F which is defined to be the largest singular value of F.

Theorem 4.3 ([1]). *Let A and B be two irreducible $n \times n$ matrices. Then B is unitarily equivalent to A if and only if*

$$\|A \otimes E + I \otimes F\| = \|B \otimes E + I \otimes F\| \quad \textit{for all} \quad E, F \in M_n.$$

Inspired by Arveson's theorem and Theorem 4.1, we ask the following question:

Problem 4.4. *Does Proposition 4.1 hold when A and B are two non-equivalent irreducible $n \times n$ matrices?*

We conjecture Proposition 4.1 holds when A and B are two non-equivalent irreducible 3×3 matrices. We present a couple of examples to support this conjecture. We computed several examples using the 9×9 matrix $T_{A,B}$. The proofs are omitted since they are similar to the approach in previous section, nevertheless they are not short.

Example 4.5. Let $0, \beta, \gamma$ be three distinct numbers. Consider the following 3×3 irreducible matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{bmatrix}, \quad \text{where two of } a, b, c \text{ are nonzero.}$$

Then $T_{A,B}$ is irreducible.

Example 4.6. Let a and c be positive numbers with $a \neq 1$. Consider the following 3×3 irreducible matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $T_{A,B}$ is irreducible.

We would like to make a few final remarks. Even though what we did is just for A and B being 3×3 matrices, a pattern emerges which raises the hope that some pattern may exist in higher dimension, and it begs the question of if and how our result extends. It is foreseeable that our approach can be extended to the 4×4 matrices, possibly with considerably more complicated algebra. But a quest for 5×5 will likely require innovative new ideas and a less computational approach or a more extensive symbolic computational approach using software such as Mathematica.

References

[1] W. B. Arveson, *Unitary invariants for compact operators*, Bull. Amer. Math. Soc. **76** (1970), 88–91. https://doi.org/10.1090/S0002-9904-1970-12374-6

[2] R. Bellman, *Introduction to Matrix Analysis*, Second edition, McGraw-Hill Book Co., New York, 1970.

[3] J. W. Brewer, *Kronecker products and matrix calculus in system theory*, IEEE Trans. Circuits and Systems **25** (1978), no. 9, 772–781. https://doi.org/10.1109/TCS.1978.1084534

[4] R. A. Brualdi, *Combinatorial verification of the elementary divisors of tensor products*, Linear Algebra Appl. **71** (1985), 31–47. https://doi.org/10.1016/0024-3795(85)90233-2

[5] J. A. Dias da Silva and Y. O. Hamidoune, *A note on the minimal polynomial of the Kronecker sum of two linear operators*, Linear Algebra Appl. **141** (1990), 283–287. https://doi.org/10.1016/0024-3795(90)90326-8

[6] X. Fang and C. Gu, *Abundance of irreducible tensor product of linear operators*, in preparation.

[7] J. S. Fang, C.-L. Jiang, and P. Y. Wu, *Direct sums of irreducible operators*, Studia Math. **155** (2003), no. 1, 37–49. https://doi.org/10.4064/sm155-1-3

[8] D. Farenick, *Arveson's criterion for unitary similarity*, Linear Algebra Appl. **435** (2011), no. 4, 769–777. https://doi.org/10.1016/j.laa.2011.01.039

[9] D. Farenick, V. Futorny, T. G. Gerasimova, V. V. Sergeichuk, and N. Shvai, *A criterion for unitary similarity of upper triangular matrices in general position*, Linear Algebra Appl. **435** (2011), no. 6, 1356–1369. https://doi.org/10.1016/j.laa.2011.03.021

[10] P. A. Fillmore and D. M. Topping, *Sums of irreducible operators*, Proc. Amer. Math. Soc. **20** (1969), 131–133. https://doi.org/10.2307/2035974

[11] P. A. Fuhrmann and U. Helmke, *On the elementary divisors of the Sylvester and Lyapunov maps*, Linear Algebra Appl. **432** (2010), no. 10, 2572–2588. https://doi.org/10.1016/j.laa.2009.12.003

[12] V. Futorny, R. A. Horn, and V. V. Sergeichuk, *A canonical form for nonderogatory matrices under unitary similarity*, Linear Algebra Appl. **435** (2011), no. 4, 830–841. https://doi.org/10.1016/j.laa.2011.01.042

[13] C. Gu, *Reducing subspaces of non-analytic Toeplitz operators on weighted Hardy and Dirichlet spaces of the bidisk*, J. Math. Anal. Appl. **459** (2018), no. 2, 980–996. https://doi.org/10.1016/j.jmaa.2017.11.004

[14] ______, *Common reducing subspaces of several weighted shifts with operator weights*, J. Math. Soc. Japan **70** (2018), no. 3, 1185–1225. https://doi.org/10.2969/jmsj/74677467

[15] C. Gu, A. Mendes, and J. Park, *Reducing subspaces of tensor products of operators and representation of permutation group*, in preparation.

[16] K. Guo and X. Wang, *Reducing subspaces of tensor products of weighted shifts*, Sci. China Math. **59** (2016), no. 4, 715–730. https://doi.org/10.1007/s11425-015-5089-y

[17] P. R. Halmos, *Irreducible operators*, Michigan Math. J. **15** (1968), 215–223. http://projecteuclid.org/euclid.mmj/1028999975
[18] C. Jiang and Z. Wang, *Strongly Irreducible Operators on Hilbert Space*, Pitman Research Notes in Mathematics Series, 389, Longman, Harlow, 1998.
[19] C. S. Kubrusly, *Regular lattices of tensor products*, Linear Algebra Appl. **438** (2013), no. 1, 428–435. https://doi.org/10.1016/j.laa.2012.07.013
[20] C. S. Kubrusly and N. Levan, *Preservation of tensor sum and tensor product*, Acta Math. Univ. Comenian. (N.S.) **80** (2011), no. 1, 133–142.
[21] W. S. Li and E. Strouse, *Reflexivity of tensor products of linear transformations*, Proc. Amer. Math. Soc. **123** (1995), no. 7, 2021–2029. https://doi.org/10.2307/2160937
[22] D. E. Littlewood, *On unitary equivalence*, J. London Math. Soc. **28** (1953), 314–322. https://doi.org/10.1112/jlms/s1-28.3.314
[23] C. Pearcy, *A complete set of unitary invariants for operators generating finite W^*-algebras of type I*, Pacific J. Math. **12** (1962), 1405–1416. http://projecteuclid.org/euclid.pjm/1103036140
[24] H. Radjavi, *Every operator is the sum of two irreducible ones*, Proc. Amer. Math. Soc. **21** (1969), 251–252. https://doi.org/10.2307/2036905
[25] H. Radjavi and P. Rosenthal, *Matrices for operators and generators of $B(\mathcal{H})$*, J. London Math. Soc. (2) **2** (1970), 557–560. https://doi.org/10.1112/jlms/2.Part_3.557
[26] ———, *Invariant Subspaces*, Second edition, Dover Publications, Inc., Mineola, New York, 2003.
[27] M. Rosenblum, *On the operator equation $BX - XA = Q$*, Duke Math. J. **23** (1956), 263–269. http://projecteuclid.org/euclid.dmj/1077466818
[28] H. Shapiro, *A survey of canonical forms and invariants for unitary similarity*, Linear Algebra Appl. **147** (1991), 101–167. https://doi.org/10.1016/0024-3795(91)90232-L
[29] W. Specht, *Zur Theorie der Matrizen. II*, Jber. Deutsch. Math.-Verein. **50** (1940), 19–23.
[30] A. Trampus, *A canonical basis for the matrix transformation $X \to AXB$*, J. Math. Anal. Appl. **14** (1966), 153–160. https://doi.org/10.1016/0022-247X(66)90068-0
[31] ———, *A canonical basis for the matrix transformation $X \to AX + XB$*, J. Math. Anal. Appl. **14** (1966), 242–252. https://doi.org/10.1016/0022-247X(66)90024-2
[32] W. Tung, *Group Theory in Physics*, World Scientific, Singapore, 1985.
[33] D. Voiculescu, *A non-commutative Weyl-von Neumann theorem*, Rev. Roumaine Math. Pures Appl. **21** (1976), no. 1, 97–113.

Caixing Gu
Department of Mathematics
California Polytechnic State University
San Luis Obispo, CA 93407, USA
Email address: cgu@calpoly.edu

Jaehui Park
Research Institute of Mathematics
Seoul National University
Seoul 08826, Korea
Email address: nephenjia@snu.ac.kr

Chase Peak
Department of Mathematics
California Polytechnic State University
San Luis Obispo, CA 93407, USA
Email address: cpeak@calpoly.edu

Jordan Rowley
Department of Mathematics
California Polytechnic State University
San Luis Obispo, CA 93407, USA
Email address: mackayrowley@gmail.com