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WEAK T-FIBRATIONS AND POSTNIKOV SYSTEMS

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Abstract. In this paper, we define a concept of weak T-fibration which is a generalization of weak H-fibration, and study some properties of weak T-fibration and relations between the weak T-fibration and the Postnikov system for a fibration.

1. Introduction

F. P. Peterson and E. Thomas [5] introduced a concept of a principal fibration. Meyer [4] introduced, as a generalization of a principal fibration, a concept of an H-fibration and obtained some properties of H-fibrations. In this paper, we define a concept of a weak H-fibration which is a generalization of an H-fibration, also define a concept of weak T-fibration which is a generalization of a weak H-fibration and study some properties of weak T-fibrations and relations between the weak T-fibration and the Postnikov system for a fibration. We obtain that if a fibration $F \stackrel{i}{\to} X \stackrel{p}{\to} B$ is a weak T-fibration and Y is a T-space with T-structure $G : Y \times \Sigma\Omega Y \to Y$ and $k : X \to Y$ is primitive with respect to G, then $E_{ki} \stackrel{\bar{i}}{\to} E_k \stackrel{p \circ p_k}{\to} B$ is a weak T-fibration, where $\Omega Y \stackrel{s}{\to} E_k \stackrel{p_k}{\to} X$ is the induced fibration by $k : X \to Y$ from $PY \to Y$. Applying the above result for the Moore-Postnikov system, we obtain that if $F \stackrel{i}{\to} X \stackrel{p}{\to} B$ is a weak T-fibration, and for each $n \geq 1$, the map $k_n : X_n \to K(\pi_{n+1}(F), n+2)$ is primitive with respect to T-structure $G = m(1 \times e)$, where $m : K(\pi_{n+1}(F), n+2) \times K(\pi_{n+1}(F), n+2) \to K(\pi_{n+1}(F), n+2)$ is the H-multiplication, then $f_n : X_n \to B$ is a weak T-fibration for all $n \geq 1$.

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Throughout this paper, space means a space of the homotopy type of 1-connected locally finite CW complex. We assume also that spaces have non-degenerate base points. All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by *. For simplicity, we use the same symbol for a map and its homotopy class. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. The diagonal map $\Delta\colon X \to X \times X$ is given by $\Delta(x) = (x, x)$ for each $x \in X$ and the folding map $\nabla\colon X \vee X \to X$ is given by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$. ΣX denote the reduced suspension of X and ΩX denote the based loop space of X. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by τ. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$and $\tau(1_{\Sigma X})$ respectively.

2. Weak T-fibratons

Let $F \xrightarrow{i} X \xrightarrow{p} B$ be a fibration. Let $k : X \to Y$ be a map and PY the space of paths in Y which begin at $*$. Let $\epsilon : PY \to Y$ be the fibration given by evaluating a path at its end point. Let $p_k : E_k \to X$ be the fibration induced by $k : X \to Y$ from ϵ. Let $i : F \to X$ be the inclusion. We can also consider the fibration $p_{ki} : E_{ki} \to F$ induced by $ki : F \to Y$ from ϵ. Then we have the following homotopy commutative diagram

$$\begin{array}{ccccc} E_{ki} & \xrightarrow{\bar{i}} & E_k & \xrightarrow{p_2} & PY \\ {\scriptstyle p_{ki}}\downarrow & & {\scriptstyle p_k}\downarrow & & {\scriptstyle \epsilon}\downarrow \\ F & \xrightarrow{i} & X & \xrightarrow{k} & Y, \end{array}$$

where $E_{ki} = \{(a, \eta) \in F \times PY | ki(a) = \epsilon(\eta)\}$, $E_k = \{(x, \eta) \in X \times PY | k(x) = \epsilon(\eta)\}$, $\bar{i}(a, \eta) = (i(a), \eta)$, $p_2(x, \eta) = \eta$, $p_k(x, \eta) = x$, $p_{ki}(a, \eta) = a$.

A fibration $F \xrightarrow{i} X \xrightarrow{p} B$ is called an *H-fibration* [4] if there exists a map $\mu : F \times X \to X$ and a homotopy $H_t : F \vee X \to X$ such that

$$(a) \qquad \begin{array}{ccc} F \times X & \xrightarrow{\mu} & X \\ {\scriptstyle 1\times p}\downarrow & & {\scriptstyle p}\downarrow \\ F \times B & \xrightarrow{p_2} & B, \end{array}$$

is commutative, where p_2 is the projection into the second factor.

(b) $H_0 = \mu j$, $H_1 = \nabla(i \vee 1)$, $pH_t(F \vee F) = *$, where $j : F \vee X \to F \times X$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map.

It is clear that $\mu_| F \times F$ and $H_{t|} F \vee F$ defines an H-space structure on F.

A fibration $F \xrightarrow{i} X \xrightarrow{p} B$ is called a *weak H-fibration* if there exists a map $\mu : F \times X \to X$ such that the diagrams

$$\begin{array}{ccc} F \vee X & \xrightarrow{i \vee 1} & X \vee X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \nabla} \\ F \times X & \xrightarrow{\mu} & X \\ {\scriptstyle 1 \times p}\downarrow & & \downarrow{\scriptstyle p} \\ F \times B & \xrightarrow{p_2} & B \end{array}$$

are homotopy commutative, where p_2 is the projection into the second factor and $j : F \vee X \to F \times X$ is the inclusion, and $\nabla : X \vee X \to X$ is the folding map. A map $f : A \to X$ is *cyclic* [9] if there is a map $F : X \times A \to X$ such that $F|_X \sim 1_X$ and $F|_A \sim f$. It is clear that a space X is an H-space if and only if the identity map of X is cyclic. A space X is called [10] an *H^f-space* if there is a cyclic map $f : A \to X$. It is clear, from the definition, that if a fibration $F \xrightarrow{i} X \xrightarrow{p} B$ is a *weak H-fibration*, then X is an H^i-space. We denote the set of all homotopy classes of f-cyclic maps from B to X by $G(B; A, f, X)$ which is called the *Gottlieb set for a map $f : A \to X$*. It is known [10] that X is an H^f-space for a map $f : A \to X$ if and only if $G(B; A, f, X) = [B, X]$ for any space B. Thus we have the following proposition.

Proposition 2.1. *If a fibration $F \xrightarrow{i} X \xrightarrow{p} B$ is a weak H-fibration, then $G(B; F, i, X) = [B, X]$ for any space B.*

Definition 2.2. *A fibration $F \xrightarrow{i} X \xrightarrow{p} B$ is called a weak T-fibration if there exists a map $\mu : F \times \Sigma\Omega X \to X$ such that the diagrams*

$$\begin{array}{ccc} F \vee \Sigma\Omega X & \xrightarrow{i \vee e} & X \vee X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \nabla} \\ F \times \Sigma\Omega X & \xrightarrow{\mu} & X \\ {\scriptstyle 1 \times e \circ \Sigma\Omega p}\downarrow & & \downarrow{\scriptstyle p} \\ F \times B & \xrightarrow{p_2} & B \end{array}$$

are homotopy commutative, where p_2 is the projection into the second factor and $j : F\vee\Sigma\Omega X \to F\times\Sigma\Omega X$ is the inclusion, and $\nabla : X\vee X \to X$ is the folding map.

In [1], Aguade introduced a *T-space* as a space X having the property that the evaluation fibration $\Omega X \to X^{S^1} \to X$ is fibre homotopically trivial. It is easy to show that any H-space is a T-space. However, there are many T-spaces which are not H-spaces in [8]. Aguade showed [1] that X is a T-space if and only if $e : \Sigma\Omega X \to X$ is cyclic. A space X is called [11] a *T^f-space for a map $f : A \to X$* if there is a map $F : \Sigma\Omega X\times A \to X$ such that $Fj \sim \nabla(e\vee f)$, where $j : \Sigma\Omega X\vee A \to \Sigma\Omega X\times A$ is the inclusion. It is clear, from the definition, that if a fibration $F \xrightarrow{i} X \xrightarrow{p} B$ is a *weak T-fibration*, then X is a T^i-space.

PROPOSITION 2.3. *If a fibration $F \xrightarrow{i} X \xrightarrow{p} B$ is a weak H-fibration, then $F \xrightarrow{i} X \xrightarrow{p} B$ is a weak T-fibration.*

Proof. Since a fibration $F \xrightarrow{i} X \xrightarrow{p} B$ is a weak H-fibration, there exists a map $\mu : F\times X \to X$ such that $\mu j \sim \nabla(i\vee 1) : F\vee X \to X$ and $p\mu \sim p_2(1\times p) : F\times X \to B$, where p_2 is the projection into the second factor and $j : F\vee X \to F\times X$ is the inclusion, and $\nabla : X\vee X \to X$ is the folding map. Let $\mu' = \mu(1\times e) : F\times\Sigma\Omega X \to X$. Then we have, from the fact $p\circ e_X \sim e_B\circ\Sigma\Omega p : \Sigma\Omega X \to B$ and the above fact, that $\mu' j' \sim \nabla(i\vee e) : F\vee\Sigma\Omega X \to X$ and $p\mu' \sim p_2(1\times e\circ\Sigma\Omega p) : F\times\Sigma\Omega X \to B$, where p_2 is the projection into the second factor and $j' : F\vee\Sigma\Omega X \to F\times\Sigma\Omega X$ is the inclusion, and $\nabla : X\vee X \to X$ is the folding map. Thus we know that $F \xrightarrow{i} X \xrightarrow{p} B$ is a weak T-fibration. □

For a map $f : A \to X$, we denote $P(\Sigma B; A, f, X) = \{\alpha\in[\Sigma B, X] | [f_\#(\beta), \alpha] = 0$ for any space C and any map $\beta\in[\Sigma C, A]\}$. It is known [11] that $G(\Sigma B; A, f, X) \subset P(\Sigma B; A, f, X)$ for any map $f : A \to X$ and any space B. A space X is called a GW^f-space for a map $f : A \to X$ [11] if for any space B, $P(\Sigma B; A, f, X) = [\Sigma B, X]$. It is known [11] that X is a T^f-space for a map $f : A \to X$ if and only if $G(\Sigma B; A, f, X) = [\Sigma B, X]$ for any space B. Thus we have the following proposition.

PROPOSITION 2.4. *If a fibration $F \xrightarrow{i} X \xrightarrow{p} B$ is a weak T-fibration, then X is a GW^i-space for a map $i : F \to X$.*

Let Y be an H-space with multiplication m and $F \xrightarrow{i} X \xrightarrow{p} B$ an H-fibration with operation μ, and $k : X \to Y$ a map. Then $k : X \to Y$ is called *primitive* [4] if the following diagram is homotopy commutative;

$$\begin{array}{ccc} F\times X & \xrightarrow{\mu} & X \\ {\scriptstyle (ki\times k)}\downarrow & & {\scriptstyle k}\downarrow \\ Y\times Y & \xrightarrow{m} & Y. \end{array}$$

It is known [4] that if Y is an H-space with multiplication m and $F \xrightarrow{i} X \xrightarrow{p} B$ is an H-fibration with operation μ, and $k : X \to Y$ is primitive, then $E_{ki} \xrightarrow{\bar{i}} E_k \xrightarrow{p_k} B$ is an H-fibration, where $p_k : E_k \to X$ is the induced fibration by $k : X \to Y$.

We would like to extend the above fact for the case of T-spaces and weak T-fibrations.

Definition 2.5. *Let Y be a T-space with T-structure $G : \Sigma\Omega Y \times Y \to Y$ and $F \to X \xrightarrow{p} B$ a weak T-fibration with operation μ, and $k : X \to Y$ a map. Then $k : X \to Y$ is called primitive with respect to T-structure G if the following diagram is homotopy commutative;*

$$\begin{array}{ccc} F\times \Sigma\Omega X & \xrightarrow{\mu} & X \\ {\scriptstyle (ki\times \Sigma\Omega k)}\downarrow & & {\scriptstyle k}\downarrow \\ Y\times \Sigma\Omega Y & \xrightarrow{G} & Y. \end{array}$$

Let $k : X \to Y$ be a map and PY the space of paths in Y which begin at $*$. Let $\epsilon : PY \to Y$ be the fibration given by evaluating a path at its end point. Let $p_k : E_k \to X$ be the fibration induced by $k : X \to Y$ from ϵ. Since the induced fibration $\Omega Y \to E_k \xrightarrow{p_k} X$ is principal, there is an operation $\mu : \Omega Y \times E_k \to E_k$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} \Omega Y\times E_k & \xrightarrow{\mu} & E_k \\ {\scriptstyle (1\times p_k)}\downarrow & & {\scriptstyle p_k}\downarrow \\ \Omega Y\times X & \xrightarrow{p_2} & X, \end{array}$$

where p_2 is the projection into the second factor.

The following lemmas are standard.

Lemma 2.6. *A map $l : C \to X$ can be lifted to a map $C \to E_k$ if and only if $kl \sim *$.*

Lemma 2.7. [5] *If $f, g : C \to E_k$ are such that $p_k f \sim p_k g$, then there is a map $d : C \to \Omega Y$ such that $g \sim \mu(d \times f)\Delta$, where $\Delta : C \to C \times C$ is the diagonal map.*

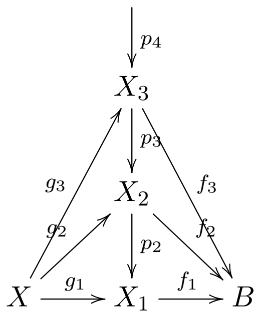
Theorem 2.8. *If Y is a T-space with T-structure $G : \Sigma\Omega Y \times Y \to Y$ and $F \xrightarrow{i} X \xrightarrow{p} B$ is a weak T-fibration with operation μ, and $k : X \to Y$ is primitive with respect to T-structure G, then $E_{ki} \xrightarrow{\bar{i}} E_k \xrightarrow{p \circ p_k} B$ is a weak T-fibration, where $p_k : E_k \to X$ is the induced fibration by $k : X \to Y$.*

Proof. Since $p \circ p_k \bar{i} \sim p \circ i \circ p_{ki} : E_{ki} \to B$, $E_{ki} \xrightarrow{\bar{i}} E_k \xrightarrow{p \circ p_k} B$ is a fibration. Since $F \xrightarrow{i} X \xrightarrow{p} B$ is a weak T-fibration with operation μ, there is a map $\mu : F \times \Sigma\Omega X \to X$ such that $\nabla(i \vee e_X) \sim \mu j$ and $p\mu \sim p_2(1 \times e_B \circ \Sigma\Omega p)$, where $j : F \vee \Sigma\Omega X \to F \times \Sigma\Omega X$ is the inclusion. Let $\phi = \mu(p_{ki} \times \Sigma\Omega p_k) : E_{ki} \times \Sigma\Omega E_k \to X$. Since $k : X \to Y$ is primitive with respect to T-structure G and $ki \circ P_{ki} \sim *$, and $k \circ p_k \sim *$, we know that $k\phi = k\mu(p_{ki} \times \Sigma\Omega P_k) \sim G(ki \times \Sigma\Omega k)(p_{ki} \times \Sigma\Omega p_k) = G(ki \circ p_{ki} \times \Sigma\Omega(k \circ p_k)) \sim *$. From Lemma 2.6, there is a map $\lambda : E_{ki} \times \Sigma\Omega E_k$ such that $p_k \circ \lambda = \phi$. Since the induced fibration $\Omega Y \to E_k \xrightarrow{p_k} X$ is principal, there is a map $\bar{\mu} : \Omega Y \times E_k \to E_k$ such that $p_k\bar{\mu} \sim p_2(1 \times p_k)$, where $p_2 : \Omega Y \times X \to X$ is the projection into the second factor. Now we have that $p_k \circ \lambda j' = \phi j' = \mu(p_{ki} \times \Sigma\Omega p_k)j' \sim \mu j(p_{ki} \vee \Sigma\Omega p_k) \sim \nabla(i \vee e_X)(p_{ki} \vee \Sigma\Omega p_k) = \nabla(i \circ p_{ki} \vee e_X \circ \Sigma\Omega p_k) \sim \nabla(p_k \circ \bar{i} \vee p_k \circ e_{E_k}) \sim p_k \circ \nabla(\bar{i} \vee e_{E_k}) : E_{ki} \vee \Sigma\Omega E_k \to X$, where $j : F \vee \Sigma\Omega X \to F \times \Sigma\Omega X$ is the inclusion and $j' : E_{ki} \vee \Sigma\Omega E_k \to E_{ki} \times \Sigma\Omega E_k$ is the inclusion. From Lemma 2.7, there is a map $d : E_{ki} \vee \Sigma\Omega E_k \to \Omega Y$ such that $\nabla(\bar{i} \vee e) \sim \bar{\mu}(d \times \lambda j')\Delta : E_{ki} \vee \Sigma\Omega E_k \to X$, where $j' : E_{ki} \vee \Sigma\Omega E_k \to E_{ki} \times \Sigma\Omega E_k$ is the inclusion. As ΩY is an H-space with multiplication $\bar{m}$, the map $\delta = \bar{m} \circ (d_{|E_{ki}} \times d_{|\Sigma\Omega E_k}) : E_{ki} \times \Sigma\Omega E_k \to \Omega Y$ has the property $\delta j' \sim d$, where $j' : E_{ki} \vee \Sigma\Omega E_k \to E_{ki} \times \Sigma\Omega E_k$ is the inclusion. Thus we have that $\nabla(\bar{i} \vee e) \sim \bar{\mu}(d \times \lambda j')\Delta \sim \bar{\mu}(\delta j' \times \lambda j')\Delta = \bar{\mu}(\delta \times \lambda)\Delta j' : E_{ki} \vee \Sigma\Omega E_k \to E_k$. Consider $\mu' = \bar{\mu}(\delta \times \lambda)\Delta : E_{ki} \times \Sigma\Omega E_k \to E_k$. Then we know that $\mu' j = \bar{\mu}(\delta \times \lambda)\Delta j \sim \nabla(\bar{i} \vee e)$ and $p \circ p_k \circ \mu' = p \circ p_k \circ \bar{\mu}(\delta \times \lambda)\Delta \sim p \circ p_2(1 \times p_k)(\delta \times \lambda)\Delta = p \circ p_2(\delta \times p_k \circ \lambda)\Delta \sim p \circ p_k\lambda = p \circ \phi = p \circ \mu(p_{ki} \times \Sigma\Omega p_k) \sim p_2(1 \times e \circ \Sigma\Omega p)(p_{ki} \times \Sigma\Omega p_k) = p_2(p_{ki} \times e \circ \Sigma\Omega(p \circ p_k)) \sim p_2(1 \times e \circ \Sigma\Omega(p \circ p_k)) : E_{ki} \times \Sigma\Omega E_k \to B$, where $j : E_{ki} \vee \Sigma\Omega E_k \to E_{ki} \times \Sigma\Omega E_k$ is the inclusion and $p_2 : E_{ki} \times B \to B$ is the projection into the second factor. Thus we have $E_{ki} \xrightarrow{\bar{i}} E_k \xrightarrow{p \circ p_k} B$ is a weak T-fibration. □

In 1951, Postnikov [6] introduced the notion of the Postnikov system as follows; *A Postnikov system for X(or homotopy decomposition of X)* $\{X_n, i_n, p_n\}$ consists of a sequence of spaces and maps satisfying (1) $i_n : X \to X_n$ induces an isomorphism $(i_n)_\# : \pi_i(X) \to \pi_i(X_n)$ for

$i \leq n$. (2) $p_n : X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$. (3) $p_n i_n \sim i_{n-1}$. It is well known fact [3] that if X is a 1-connected space having a homotopy type of CW-complex, then there is a Postnikov system $\{X_n, i_n, p_n\}$ for X such that $p_{n+1} : X_{n+1} \to X_n$ is the fibration induced from the path space fibration over $K(\pi_{n+1}(X), n+2)$ by a map $k^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$.

Let $p : X \to B$ be a map. *A Moore-Postnikov system for* $p : X \to B$ [2] is commutative diagram as shown below, with each composition $X \overset{g_n}{\to} X_n \overset{f_n}{\to} B$ homotopic to p, and such that;



(1) The map $X \overset{g_n}{\to} X_n$ induces an isomorphism on π_i for $i < n$ and a surjection for $i = n$.

(2) The map $X_n \overset{f_n}{\to} B$ induces an isomorphism on π_i for $i > n$ and an injection for $i = n$.

(3) The map $X_{n+1} \overset{p_{n+1}}{\to} X_n$ is an induced fibration with fibre $K(\pi_n(F), n)$.

It is known [2] that every map $p : X \to B$ between CW complexes has a Moore-Postnikov system which is unique up to homotopy equivalence. Moreover, a Moore-Postnikov system of principal fibrations exists, that is, each X_{n+1} is the homotopy fibre of the map $k_n : X_n \to K(\pi_{n+1}(F), n+2)$ if and only if $\pi_1(X)$ acts trivially on $\pi_n(M_p, X)$ for all $n > 1$, where M_p is the mapping cylinder of p. We may apply Theorem 2.8 to study the successive fibrations in a Moore-Postnikov system for a fibration $F \overset{i}{\to} X \overset{p}{\to} B$. Let $F \overset{i}{\to} X \overset{p}{\to} B$ be a weak T-fibration. Since X is 1-connected, there is a Moore-Postnikov system of principal fibrations with each composition $X \overset{g_n}{\to} X_n \overset{f_n}{\to} B$ homotopic to p, $X_1 = X, X_0 = B, g_1 = 1, f_1 = p_1 = p$, $f_n = f_{n-1}p_n : X_n \to B$, and such that the map $X_{n+1} \overset{p_{n+1}}{\to} X_n$ is an induced fibration with fibre $K(\pi_n(F), n)$ and each X_{n+1} is, up to weak homotopy equivalence, the homotopy fibre of the map $k_n : X_n \to K(\pi_{n+1}(F), n+2)$. Since for each $n \geq 1, K(\pi_{n+1}(F), n+2)$ is an H-space, it is a T-space. Thus we have a

T-structure $G = m(1\times e) : K(\pi_{n+1}(F), n+2)\times \Sigma\Omega K(\pi_{n+1}(F), n+2) \to K(\pi_{n+1}(F), n+2)$, where $m : K(\pi_{n+1}(F), n+2)\times K(\pi_{n+1}(F), n+2) \to K(\pi_{n+1}(F), n+2)$ is the H-multiplication.

Corollary 2.9. *Let $F \xrightarrow{i} X \xrightarrow{p} B$ be a weak T-fibration. If for each $n \geq 1$, the map $k_n : X_n \to K(\pi_{n+1}(F), n+2)$ is primitive with respect to T-structure $G = m(1\times e)$, where $m : K(\pi_{n+1}(F), n+2) \times K(\pi_{n+1}(F), n+2) \to K(\pi_{n+1}(F), n+2)$ is the H-multiplication, then $f_n : X_n \to B$ is a weak T-fibration for all $n \geq 1$.*

Proof. Since $F \xrightarrow{i} X \xrightarrow{p} B$ be a weak T-fibration, there is a map $\mu : F\times \Sigma\Omega X \to X$ such that $\mu j \sim \nabla(i\vee e) : F\vee \Sigma\Omega X \to X$. Since $k_1 : X = X_1 \to K(\pi_2(F), 3)$ is primitive with respect to T-structure G, we have, from Theorem 2.8, that $f_2 = p\circ p_2 : X_2 \to B$ is a weak T-fibration. Continuing in this manner, we have, applying Theorem 2.8 for a weak T-fibration $f_{n-1} : X_{n-1} \to B$, that $f_n = p_1\circ p_2\circ\cdots\circ p_n : X_n \to B$ is a weak T-fibration for all $n \geq 1$. □

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